Ordinary and Partial Differential Equations and Applications
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Lecture - 26
Boundary Value Problems for Second Order Differential Equations

Hello friends, welcome to this lecture. In this lecture will start of a decision of boundary value problems. So, boundary value problems means equation is given and the conditions are defined on the boundary of the domain.

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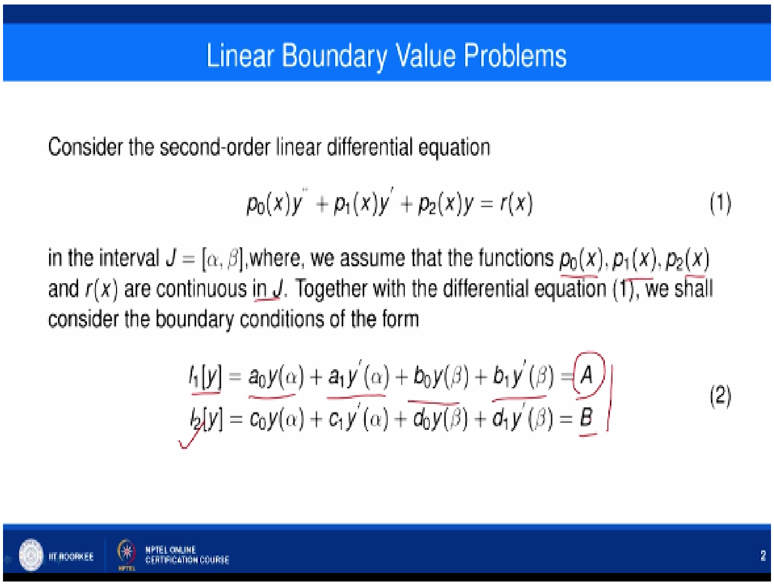

So let us see, what is the boundary value problem in a precise manner. So, consider the second order linear differential equation p0(x) y double dash + p1(x) y dash + p2(x) y = r(x) here. So, here coefficient p0, p1 and p2 are continuous in an interval J which is alpha 2 beta. Alpha beta are any real numbers and alpha is < beta we will say. Then we assume that function p0, p1 and p2 and r(x) are continuous in J.

So, by existent and unique, we simply say that this solution exist if we are prescribe the initial condition along with this. But now let us consider the differential equation we with the boundary condition like this. So, here we define boundary condition of the form l1(y) = l1(y), where l1(y) is given as a0y (alpha) + a1y dash (alpha) + b0y (beta) = b1y dash (beta) and it is taken as some value that is A.

And second boundary condition is c0y (alpha) + c1y dash (alpha) + d0y (beta) + d1y dash (beta) = B. Here and A and beta are again some real constant.

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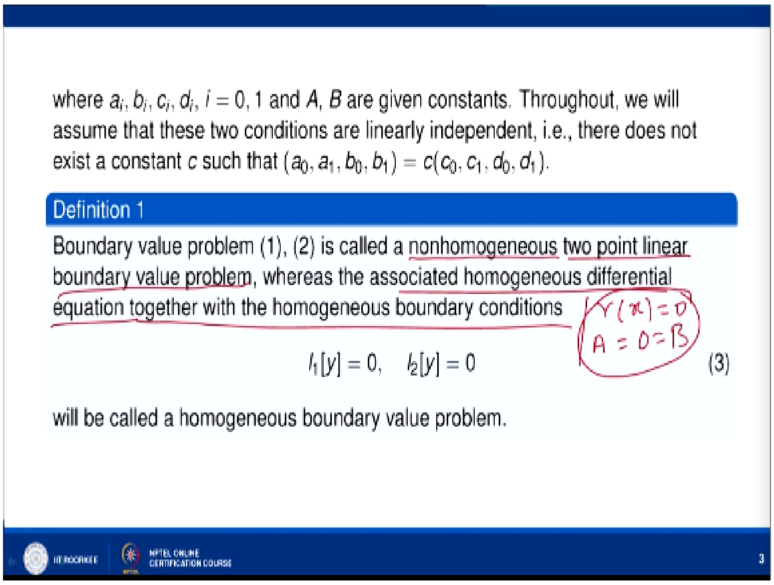

So, here ai, bi, ci, di for i = 0 to 1 and A and B are given constant, real constant and throughout we assume that these 2 conditions, the previous 2 conditions are linearly independent. So, we simply say that these are not the precisely the same condition. So, here we are assuming that the vector a0, a1, b0, b1 are not scale of multiple of c0, c1, d0 and d1. So, it means that these 2 vectors are linearly independent vectors.

So, it means that we are precisely working with 2 boundary conditions rather than one boundary condition. So, here I am assuming that there exists no K for which this is true. So, it means that we are assuming that the condition given in 2 are linearly independent boundary conditions. So, it means that now we define the boundary value problems.

So, boundary value problem 1, 2 is called a nonhomogeneous 2-point linear boundary value problem. Whereas, the associated homogeneous differential equation together with the homogeneous boundary condition will be called a homogeneous boundary value problem. So first you just look at here is an equation and a boundary conditions. Because these condition, the define or a 2-point alpha one end point and beta is another end point.

So, we simply say that these l1(y) and l2(y) are conditions defined at 2 different point that is alpha and beta. So, we say that l1(y) and l2(y) are boundary condition rather than initial condition because it is defined at 2 different point, in fact at the end point of this. So, we call it boundary conditions or 2 point boundary conditions. It may happen that we have some more point on which we may define condition in terms of more points.

But right now it is a 2 point boundary value problem. Now, we define nonhomogeneous equation and nonhomogeneous boundary condition. So, if this r(x) is non-zero for non-zero in the whole interval then we say, call it nonhomogeneous differential equation. And if A and B are non-zero real constant then we call these boundary conditions are nonhomogeneous boundary conditions.

So, it means that if we consider nonhomogeneous differential equation with nonhomogeneous boundary condition we call this as a nonhomogeneous 2 point linear boundary value problem. But if we consider that r(x) = 0 and A = 0 = B, we call it that this is a homogeneous boundary value problem. So, it means that associated homogenous differential equation together with the homogeneous boundary condition that is ly l1(y) = 0, l2(y) = 0 will be called a homogenous boundary value problem.

So, it means that when r(x) = 0, A = 0, A = B then in this case we call this as homogenous boundary value problem. But if r(x) is non-zero A and B are all non-zero then we call it nonhomogeneous boundary value problem, okay. In fact, one of A and B has to be non-zero in nonhomogeneous boundary value problem.

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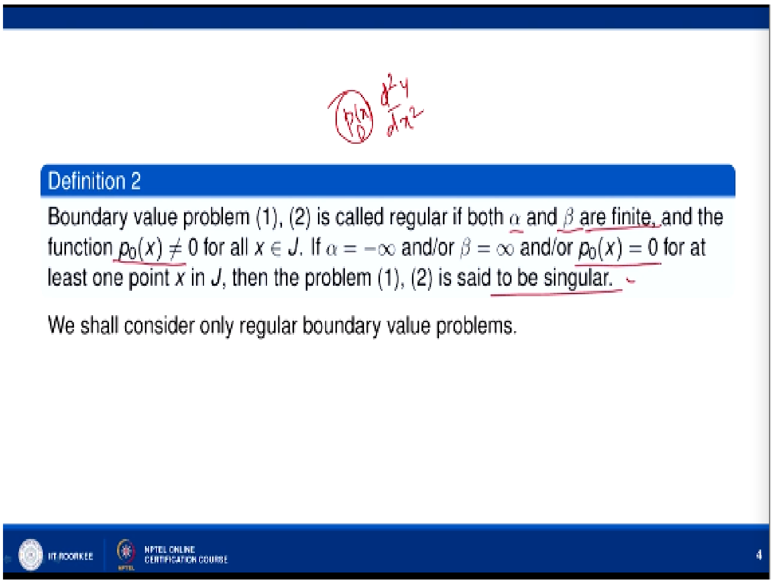

Now, we define a regular boundary value problem and similar boundary value problem. So, we simply say that boundary value problems 1 and 2 is called regular, if both alpha and beta are finite and the coefficient p0(x), coefficient of d2y/dx square is p0(x), this is non-zero in the entire interval J. So, if interval is finite and the coefficients are non-zero then we call our boundary value problem as regular boundary value problem.

Otherwise we call our boundary value problem as single boundary value problem. It means that if your alpha is – infinity n or beta and or r beta = infinity and or p0(x) = 0 for at least one point in J, then the problem 1 and 2 is set to be a singular boundary value problem and for this particular course we will consider only the regular boundary value problem, it means that we consider that our end points are finite and the coefficient of d2y/dx square is never 0 in the said interval in finite interval.

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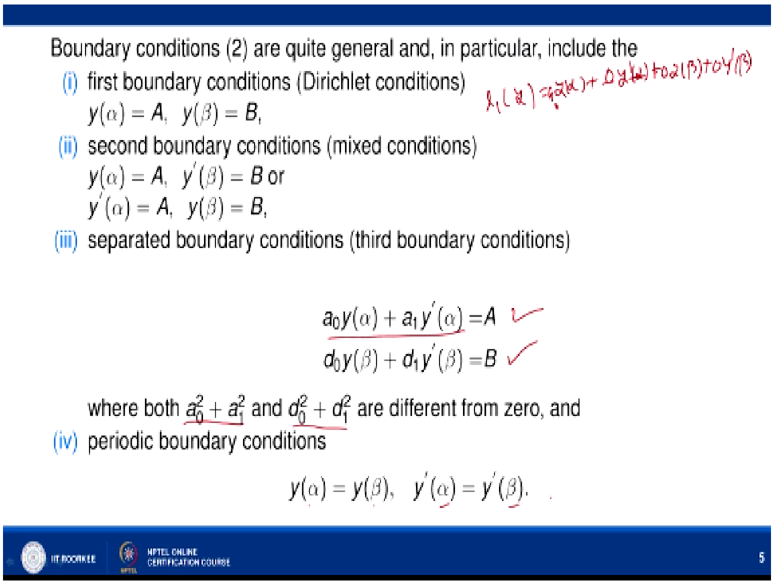

Now, the boundary condition which we have consider l1(y) = A and l2(y) = B are very general boundary conditions and we can summarize and we can write many more boundary condition as a special case of l1(y) and l2(y). So, first let us take a special case where l1(y) is given as y alpha = A. So, here l1(y) is y only. So here l1(y) which I am writing as here, we are simply taking l1(y) is basically, taking only a0 as 1, rest all 0.

So, we say that the simplest form of l1(y) is that a0 = 1 rest are all simply 0. So, l1(y) = simply y only. So, here we simply say a0 = 1, so y alpha and all are 0, 0y dash (alpha) + 0y (beta) + 0y dash (beta). So, it is a particular case of your l1(y). So, here we simply assume that A0 = 1, rest are all 0. Similarly, you assume that c0 = 1 rest are all 0.

So, we have boundary condition y(alpha) = A and y(beta) = B. So, we say that it is first boundary condition and sometimes we call it a Dirichlet condition. So, here in first boundary condition or Dirichlet condition your function is given, function is given some value at the endpoint. And then second boundary condition or which we call us mix boundary conditions where function as well as its define at the endpoint.

So here, one kind of mix boundary condition is y alpha is given as A and y dash beta is = B. So, it means that one point your function is given and the other point your derivative is given. So, y alpha = A, y dash (beta) = B or the other way that is y dash (alpha) = A means at the end of x = alpha your derivative is given a particular value and the other end the function is define are getting a particular value.

So, this is your second boundary condition or we say mix condition. Now, third that is separated boundary condition. So, it means that here one condition is given in entirely at one point, one end point and other boundary condition is given at the other endpoint. So, here your l1(y) is define only, say alpha, x = alpha. So, we say a0y (alpha) + a1y dash (alpha) = A and d0y (beta) + d1y dash = B.

So, here your a0 a1 is some non-zero and b0 and b1 are 0. Similarly, in l2(y) your c0 and c1 is given as 0 and d0 and d1 are non-zero value. So, here we say that your boundary condition, first boundary condition you find only at x = alpha and the second boundary condition is defined only at the other point that is x = beta. So, your boundary conditions are separated. So, we call these boundary conditions are separated boundary conditions.

And here we are assuming that not all a0 a1 is 0. So, we assume that a0 square + a1 is square is different from 0 and similarly d0 is square + d1 is square is different from 0 here. Then there is one more very important kind of boundary conditions that is periodic boundary condition. So, it means that the initial point is the same as final point. So, it means that the value are y given at alpha is same as given at beta.

So, y alpha is = y beta and same similarly the derivative. So, derivative is the value of derivative at alpha is same as the value given at the point x = beta. So, here y (alpha) = y (beta) and y dash (alpha) = y dash (beta) is these kind of conditions are known as periodic boundary condition. So, the initial point the value given at initial point is same as the value given at the final point.

So, it means that y (alpha) = y (beta) and y dash (alpha) = y dash (beta). So, there are some important boundary conditions which we need to discuss very often.

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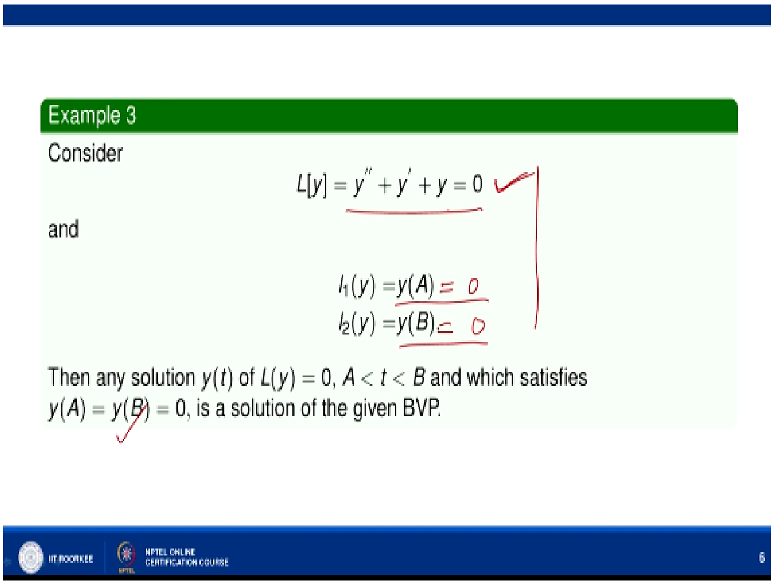

Now, let us consider one example. So, let us take differential equations say y double dash + y dash + y = 0 and here boundary conditions are separated boundary condition and it is a Dirichlet kind of boundary condition that is ly(y) = y(A) and l2(y) = y(B). So, it is the first kind of boundary condition and it is also a separated boundary condition. And so any solution of this we satisfy the conditions at A and B are simply a solution of this.

So, in particular if I take your y(A) = 0 and y(B) = 0 we can say the solution of this satisfying this is a solution of the given. So, we simply right it 0. So, it means that any solution we satisfy this and these conditions are known as the solution of this boundary value problem and in fact this is a homogenous boundary value problem.

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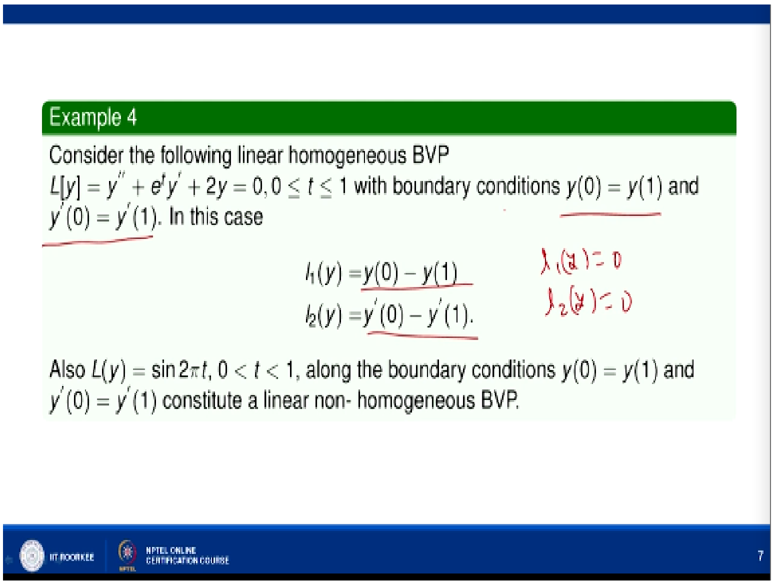

Then consider the next example. Here, we have a homogenous boundary value problem. So, y double dash + et y dash + 2y = 0 and t is lying between 0 to 1. Boundary conditions are this, y(0) = y(1) and y dash (0) = y dash (1). If you recall this is a periodic boundary conditions and here let us write this y(0), y(1) this is a l1(y) = y0-y1, l2y = y dash (0) – y dash (1). So, here l1(y) = 0 and l2(y) = 0 give you these periodic boundary condition which is like y(0)=y(1) and y dash (0) y dash (1).

So, we can simply say that if you take any say nonhomogeneous boundary value problem that is L(y) = sin 2 pi t or you can say y double dash + et y dash + 2y = sin 2 pi t along with the boundary condition that is y(0) = y(1) and y dash (0) = y dash (1). This will constitute a linear nonhomogeneous boundary value problem.

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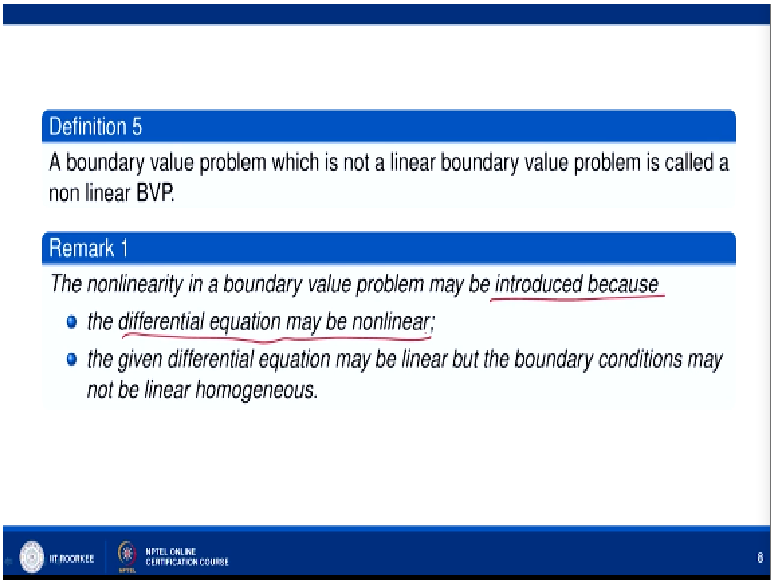

So, these are example of some boundary value problem and some boundary conditions. Now, we again define the classification of boundary value problem in terms of linear and nonlinear. So, a boundary value problem which is not a linear boundary value problem is called a nonlinear boundary value problem and this is happening because this nonlinearity in a boundary value problem may be introduced because of the following thing.

The first thing is, the differential equation may be nonlinear, right? So here we have 2 components, one is differential equation other component is your boundary conditions. So, your nonlinearity a boundary value problem is said to be linear if both the equation as well as the conditions are linear and if any of these 2 fail then we call this as a nonlinear boundary value problem.

So, it may happen due the differential equation may be nonlinear and the given differential equation may be linear but the boundary condition may not be linear. So, we simply say that nonlinear boundary value problem is maybe because of equation maybe nonlinear or the condition may be nonlinear. So, let us consider some example of nonlinear boundary value problem.

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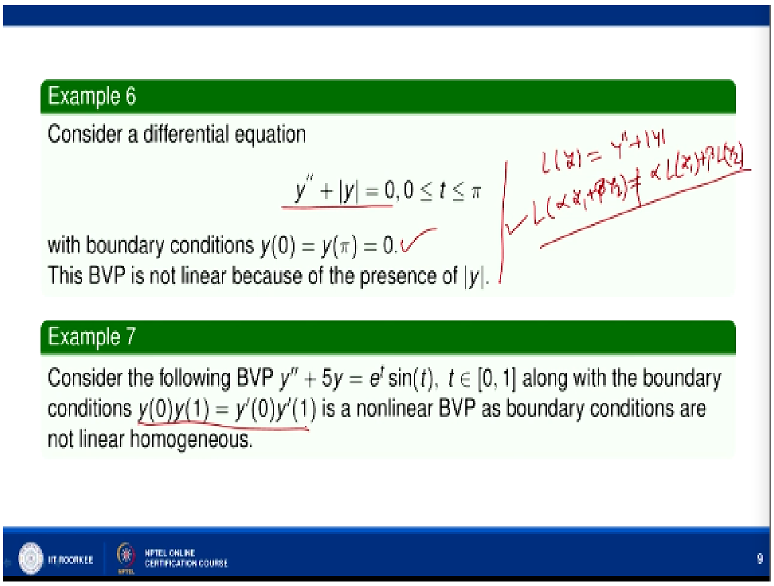

First problem is y double dash + = 0. Here t is define from 0 to pi here. Our boundary condition is y(0)=y(pi) = 0. So, these are condition which are defined on the boundaries and if you look at these are very simple boundary condition, separated boundary conditions. But if you look at the equation, equation content your modulus y terms. So, it is a nonlinear differential equation.

Your boundary conditions are linear but equation is nonlinear. So, we simply say that it is a nonlinear boundary value problem and I hope that you already know how to check whether it is a linear or nonlinear boundary value problem. For that you define ly, y double dash + y and you check that if L (alpha y1+ beta y2) if it is = alpha L(y1) + beta L(y2).

If this holds then we call it linear and if this does not hold we call it nonlinear and if you check that here it is not holding. So, you can check that differential equation is not a linear differential equation. So, we call this problem as nonlinear boundary value problem. Now, look at the next example here boundary value problem is y double dash + 5y = e to power t sin(t).

t is again in some interval let us say 0 to 1 along with the boundary condition y(0) * y(1) = y dash(0) = y dash(1). So, here if you look at your boundary conditions are not linear. So, we simply say that your equation is linear, differential equation is linear but boundary condition is not linear. So, we again call this as a nonlinear boundary value problem.

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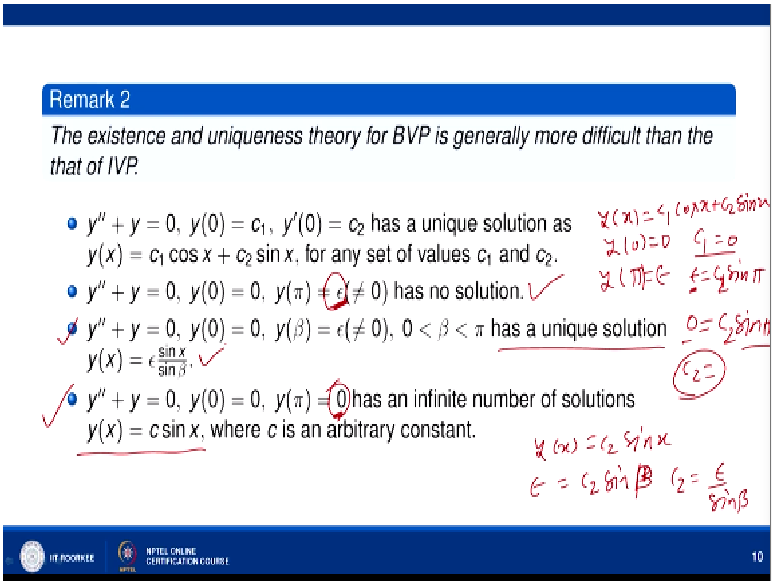

Now, why again we are discussing the boundary value problem different from initial value problem because we have already discussed the initial value problem, we have discussed the existence and uniqueness theorem and consider several methods to find out the solution of initial value problem. But why we are bothering about boundary value problem?

So, we can say that the existence and uniqueness results of boundary value problem is in general quite difficult than the theory of initial value problem. For example, take very simple example and try to understand why this theory of existence and uniqueness for boundary value problem is say difficult than the initial value problem. So, first let us take simple example y double dash + y = 0 and take the initial condition y(0) = c1 and y dash (0) = c2.

This is a simple initial value problem and we know the solution is given as y(x) = c1 cos of x + c2 sin of x and you can take any c1, c2 and you can get your solution in a very nice way and solution is given. But now let us in place of initial condition now let us define boundary condition and see that how this nice solution is change in a difficult solution. So, equation remains the same that is y double dash + y = 0.

But now your boundary conditions are changed. Your boundary conditions are y0 = 0 and y pi = some epsilon. So some value, epsilon may not be that. And we can see that it has no solution at all. Though how we can check that it has no solution. So, what we try to do we try to solve this equation for a general initial condition. So we simply see that your yx = general solution of this differential equation is c1 cos of x + c2 sin of x.

And we want to find out this c1 and c2 but not with the help of initial condition but with the help of boundary condition. Now, we look at y(0) = 0, this force has to choose c1 = 0, right? Because if you put x = 0 then sin x is 0, so and cos x is 1. So c1 has to be 0. And but if you look at y (pi) = epsilon then look at here what you have. This is epsilon = c1 and c1 is 0. So, it is c2 sin of pi. So, sin of pi is simply 0 but here we have a non-zero value.

So, it means that this has no solution, right. So, simply say that when y (pi) is some non-zero value then it has no solution, right. But suppose you look at this case that here, y double dash + y = 0, y = 0, y (pi) = 0. Now, here we have taken the value epsilon. But if you take epsilon may be any value maybe very small but it is still it will have no solution. But if you take epsilon as 0 then we can easily check that it has an infinitely many solutions.

That is quite say very, say strange thing. Because here we just differ our boundary condition by that. So, this epsilon may be very small quantity but it is still in one case we have no solution but in other case we have infinitely many solution and you can easily find out the infinitely many solution as you here if you look at in place of epsilon if you put 0 then 0 = c2 sin pi of course this is 0, this is 0, so I can take any value of c2, right?

So, it means that we have infinitely many choices for this c2 and we can say that our solution is given as y(x) = c sin x and you can take any value of c and you take different value we have a different solution of this. So, here we say that we have just change the boundary condition a little, but we have in place of no solution now we have infinitely many solutions. So that is very strange for this boundary value problem.

Now, look at next example that is y double dash + y = 0, y(0) = 0 and y beta = epsilon. Here, beta is any number between 0 to pi and you can say that it has a unique solution and solution is given as y(x) = epsilon sin x sin beta, that also you can get it from this. So, you can say that y(0) = 0 means c1 = 0. So, it means that now we have y(x) = c2 sin of x and y beta = epsilon is given. So, epsilon = c2 sin of beta.

So, c2 sin of beta you can get, so c2 is what? epsilon sin of beta. So, solution is given as y(x) = epsilon sin x sin beta and this is a unique solution. So, it means that if you change your boundary condition a bit and you have all the cases that no solution unique solution and infinitely many solutions. So, it means that here we have to look at our existence and uniqueness theory in a very careful manner.

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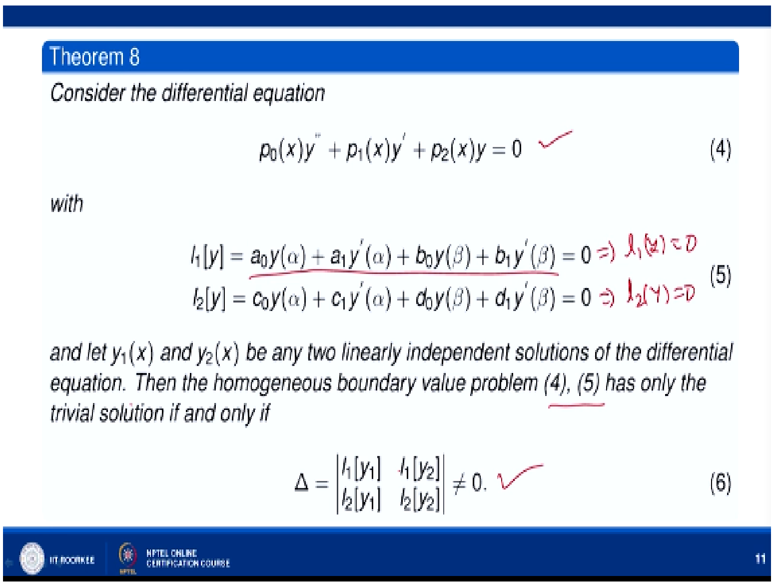

So, now let us start with the simplest possible cases, the simplest possible is the homogenous equation with homogenous boundary condition and we know that a homogenous boundary value problem has a trivial solution that is 0 solution. So, we say that consider the differential equation this p0(x) y double dash + p1(x) y dash + p2(x) y = 0 with the boundary condition l1(y) which is define like this.

So, l1(y) is define as a0y (alpha) + a1(y) dash alpha + b0y (beta) + b1y dash (beta) = 0. So, we can summarize it to right l1(y) = 0. Similarly, this as l2(y) = 0. So, we have a homogenous differential equation and homogenous boundary condition and you already know that 0 solutions satisfy this. But we are interested in finding the non-trivial solution. In trivial solution already exists that is 0 solution.

How to find out non-trivial solution? So, we try to first find out the condition under which we have a non-trivial solution for this homogenous boundary value problem. So for that let y1 and y2(b) any 2 linearly independent solution of the differential equation then the homogenous boundary value problem 4 and 5 has only the trivial solution if and only if this term is, this quantity is different from 0.

If this quantity is = 0 then we may have more than one solution but in case when this quantity delta is non-zero then we have only and only trivial solution. So, let us prove this.

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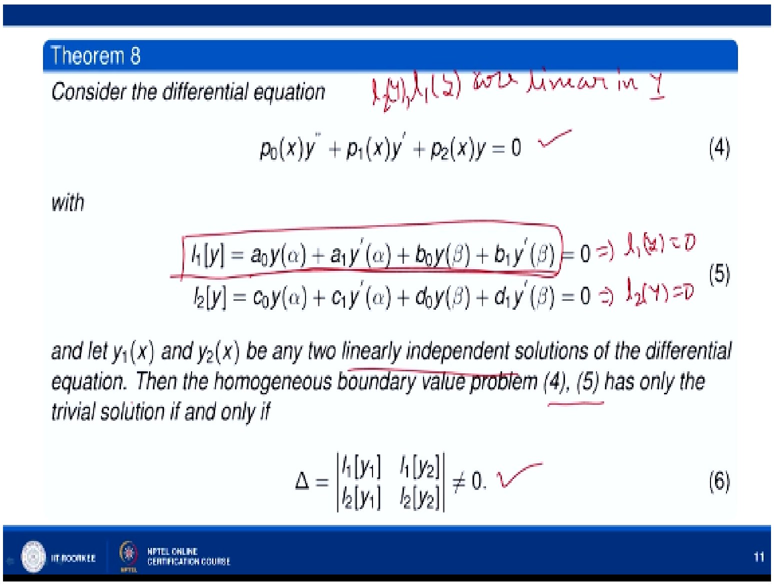

So, proof is quite simple. You simply say that your general solution is written as y(x) = c1y1(x) + c2y2(x) because it is already given that y1 and y2 are 2 linearly independent solutions. So, general solution you can easily find out, so general solution given then. Now, we can find out c1, c2 using the boundary condition. So, if you use boundary condition then l1[c1y1 + c2y2] is c1l1[y1] + c2l1[y2] = 0.

Please look at here your boundary condition which is define as l1(y) and l2(y) this is a linear form, I will say. It means that this is a linear, in terms of your y alpha, y dash alpha, y beta and y dash beta. So, it means that I can easily check, in fact I request you to check that l1(y) and l2(y) are linear, are linear in argument that is y here. How you can check? You look at here.

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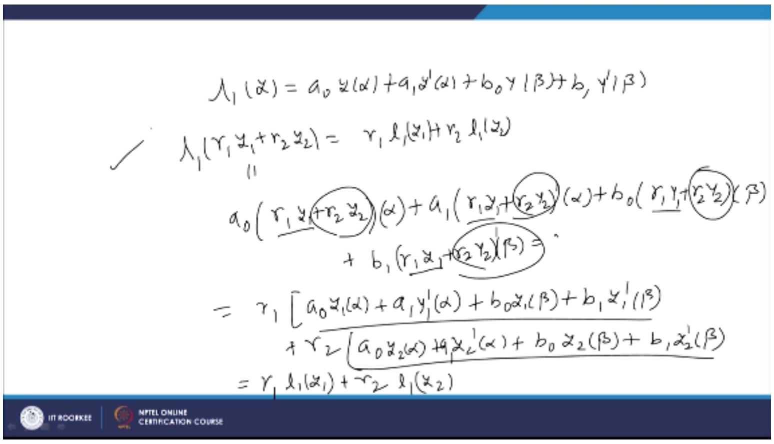

So, let me write it 1, l1(y) = a0 y (alpha) + a1 y dash (alpha) + b0 y (beta) + b1 y dash (beta) and we want to check whether it is linear in terms of y or not. For that you write l1(y) alpha y + beta, alpha is we have already taken, so let us use some other, so let us say you say r1 and r2. So, let us say r1y 1 + r2 y2. And we want to check whether it is r1 l1 of (pi 1) + r2 l1(y2). If it is then we say that l1 is a linear in terms of y.

And that you can easily check that it is actually satisfying this. Let me write it here. So, this is what this I am writing as a0 y alpha is (r1y1 + r2y2) define at (alpha) + a1 (r1y1 + r2y2) dash (alpha) + b0 (r1y1+r2y2) define at (beta) + b1(r1y1+r2y2) dash (beta) =, now this if you simplify it is what? Here, r1y1 + r2y2 given at alpha is what? r1y1 (alpha) + r2y2 (alpha).

So, we can simplify and we can write it r1 and here we have a0 y1(alpha) that you can easily check. So, this we are taking and here we are taking. So, here a1y1 dash (alpha) + look at this term. Here we have b0y1 (beta) + here we have b1y1 dash (beta) + look at r2. So, here r2 look at the last term. So, here if you look at this then we have a0y2 (alpha) + look at this term, so it is y2 dash (alpha) * a1 + this term that is b0y2 (beta) + this term, that is b1y2 dash (beta).

So, this is what r1 and if you look at this is what l1 in terms of y1 + r2. Now, look at this, this is what l1(y2). So, we have proved or we have shown that l1r1y1 + r2y2 = r1l1y1 + r2l1y2. So, it means that l1 similarly we can prove that l2 also is a linear in terms of y. So, we are going to utilize this.

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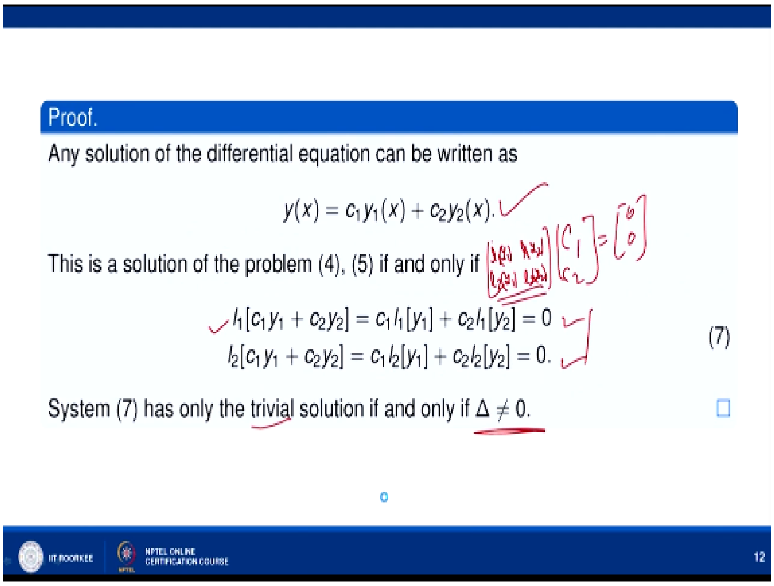

So, here we simply say that l1 (c1y1 + c2y2) now using linear what c1l1y1 + c2l1y2 = 0 and l2 (c1y1 + c2y2) = c1l2 (y1) + c2l2 (y2) = 0. Now, to find out a non-trivial solution we must have that we should have a non-trivial value of c1 and c2. Non-trivial values of c1 c2 means non-zero values of c1 and c2. So, here we have this and l1, so we simply say that we will get a non-zero value provided the c1 let me write this equation, right.

So, this is what ax = b kind of thing. So, here we have c1 c2 and here we have what l1(y1) l1(y2) l2(y1) l2(y2) and here we have 0 0. So, we have ax = 0 kind of thing. So, here this will have an only a trivial solution provided that determined of this coefficient is basically non-zero. So, we say that this system has an only a trivial solution provided that determinant is non-zero, right.

So, that is the contain of this theorem that this homogenous boundary value problem will have a trivial solution if and only if this quantity delta is non-zero. Determinant of this value is non-zero. So that is one, so if it is 0 then we may have more than one solution.

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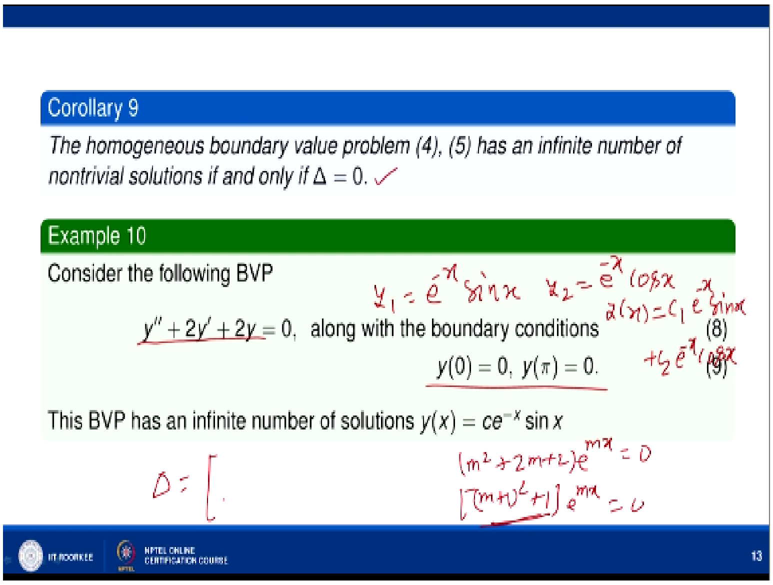

In fact, we may have infinitely many solutions. So, that is the contain of this that the homogenous boundary value problem has an infinite number of non-trivial solution if and only if delta is = 0. So, we have just taken the half way, you have to prove the remaining part. So, we say that if delta that quantity is 0 then we have infinitely many solutions if delta is non-zero then we have only at this. But it will always have a solution.

So, let us consider one example, that is y double dash + 2y dash + 2y = 0. Boundary condition is given as y(0) = 0 and y(pi) = 0 and to find out the solution here we simply first find out the general solution of this for finding the general solution of this we can use, we can find out the solution by given method. So, here you can simply say that here solution is given as e to power –x sin of x, you can call it y1 and y2 as e to the power –x cos of x.

And that you can simply check that it is what you can write it. Your auxiliary equation is (m square + 2m + 2) e to power mx = 0 and it is m + 1 whole square + 1 e to power mx = 0 when you put y as e to power mx as a solution here then you will get this as a solution. So, this quantity has to be 0 and using the boundary condition you can fix it. But if you look at here I just want to go aback again.

Because here, if you look at this is depending on the boundary condition basically, it is not depending on say solutions. you take y1 and y2 are any 2 linearly independent solutions but this delta is depending on the boundary condition rather than its specific initial, its specific solution of this boundary value problem. So, it is very much centric about the boundary condition rather than your solutions.

So, here you can take any 2 linearly independent solutions and you can easily check this. So, let me take solutions as e to power –x sin x and e to power –x cos of x and that you can easily check. So, general solution you can find out y(x) = c1 e to power –x sin(x) + c2 e to power – x cos(x). Now, we have already condition, l1(y) = 0. So, you can look at the delta here, delta is what? l1. So let me write it, what is l1 here?

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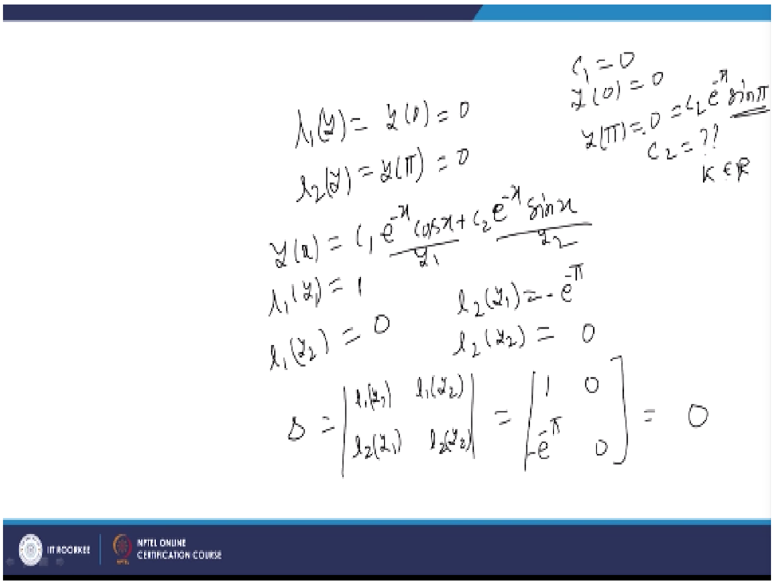

So here l1(y) is basically y(0), right and l2(y) is y(pi) and it is given as 0. And y(x) is already given c1 e to power –x cos of x + c2 e to power – x sin of x. So, we can easily check l1 (y1) is basically what y1 is this y2 is this. Let me see whether it where I have defined, okay. So, you can easily check your what is l1 (y1). So, let me write it y1 is this y2 is this. So, l1(y1) is basically what? e to power –x cos x defines at 0. So, that is going to be what?

It is going to be one only. Is that okay? Now, l1 (y2) is what? l1 (y2) is going to bought define this at 0. So, this is going to be 0. Then l2 (y1) means this define at pi. So, it is e to power – pi cos of pi is -1, so it is -. l2 (y2) is going to be again 0. So, now look at the determinant, determinant is l1 (y1) l1 (y2) l2 (y1) l2 (y2) that determinant that this is your delta.

So, this l1 (y1) is basically 1, l1 (y2) is 0 and this is what e to power -, - of e to power pi and this is 0 and you can see that it is having what? It is 0 basically. So, it means that this delta is coming out to be 0. So, it means that our theory says that it has infinitely many number of solution and you can easily check that your infinitely many solution is given by y(x) = ce * e to power –x sin of x. How can you check that?

So, this theorem simply says that it has infinite number of solution. How to find out? That is up to you, means you have a general solution and you have boundary condition you can easily find out the solution. Let me say that here your condition, your general solution is this then y0 = 0 means here you simply say that c1 has to be 0, using y0 = 0. Now, y (pi) = 0 let us use this and how to see then this is what c1 is 0 means only c2 is there.

So, c2 e to power – pi here and sin of pi is simply 0. So, 0 = c2 * 0. So, c2 you can take any value. So, let us say K belongs to R. So, here we simply say that your solution is c e to power –x sin of x and c is coming from R. So, we can say that here we have infinitely many solution and given by this and this we have checked using delta. Here delta is coming out to be 0 and so we have infinitely many solutions.

So, we have seen that in the homogenous boundary value problem when we have a one solution, trivial solution or a non-trivial solution, infinite number of non-trivial solution. Now, let us look at the nonhomogeneous boundary value problem with nonhomogeneous boundary condition.

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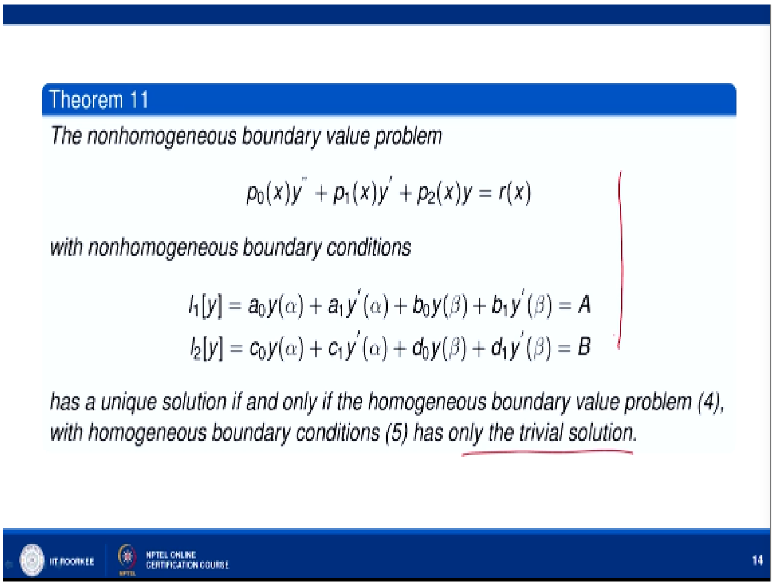

So, the nonhomogeneous boundary value problem p0(x) y double dash + p1(x) y dash + p2(x)y = r(x) and nonhomogeneous boundary condition is given l1(y) = a and l2(y) = b. This has a unique solution, if and only if the homogenous boundary value problem with homogenous boundary conditions has only the trivial solution.

So, that is why whatever we have discussed is going to be very useful in finding the unique solution for nonhomogeneous boundary value problem. So, if we have a homogenous boundary value problem has a trivial solution then nonhomogeneous boundary value problem will have a unique solution. So, let us see how the proof goes.

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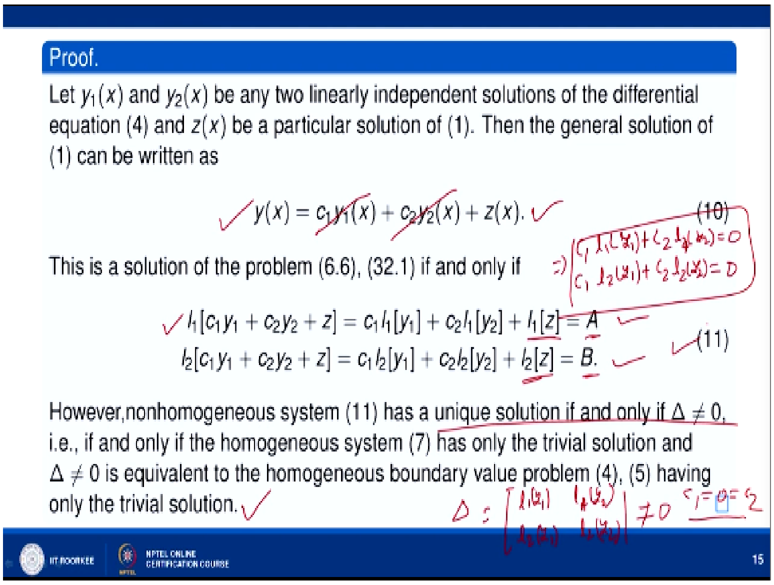

So, let us look at y1 and y2 be any 2 linear independent solution of the differential equation 4 means your homogeneous part and z(x) be a particular solution of nonhomogeneous equation that is this. So, z(x) is a particular solution of this nonhomogeneous boundary value problem, this and it will satisfy this. So, it means that then the general solution of 1 can be written as $y(x) = c1y1(x) + c2y2(x) + Z(x)$.

So, it means that this is the general solution of this along with boundary condition. Now, since it satisfy the boundary condition, so we write $l1(c1y1 + c2y2 + z) = A$ and $l2(c1y1 + c2y2 + z) = B$. Because if, okay so, it satisfy this condition because y is a general solution of the nonhomogeneous boundary value problem. So, if you simplify this you simply say that since $l1(z) = A$ and $l2(z) = B$.

So, this equation number 11 deal through this $c1\ l1(y1) + c2\ l2(y2) = 0$ and $c1\ l2(y1)$, sorry this is $a(l1) + c2\ l2(y2) = 0$ and remember this is precisely the term we say that this = 0 means this implies that the homogenous part has a trivial solution provided that this delta which is given as l1(y1) l1(y2), l2(y1) l2(y2) this determinant is non-zero. So, if this determinant is non-zero then you may have c1 = c2, we have a trivial solution for homogenous part.

When you put a trivial solution for homogeneous part then this will simply say that this will cancel and we have y(x) = z(x) as the only solution. So, we say that nonhomogeneous system (11) has a unique solution if and only if delta is non-zero. Because if delta is non-zero then we have c1 = c2 = 0 and we have only solution given as y(x) = z(x).

So, we can say that this nonhomogeneous system has unique solution if and only delta is non-zero and we can say that if and only if the homogeneous system has only the trivial solution and delta is not = 0 is the equivalent to the homogenous boundary value problem having only the trivial solution. That is what we want to conclude from this equation.

So from 11, you can conclude that trivial solution for homogeneous boundary value problem is equivalent to say that we have a unique solution for nonhomogeneous boundary value problem with nonhomogeneous boundary condition. So, let us take one example based on this.

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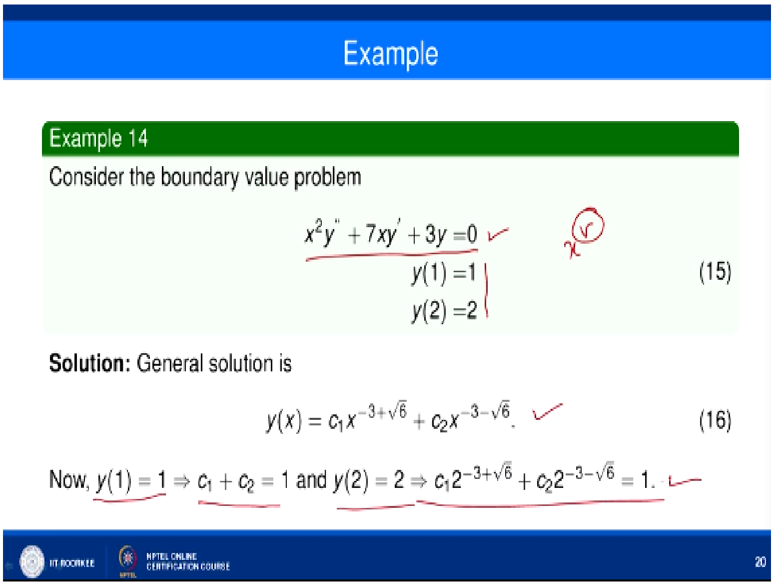

So let us consider this example, consider the boundary value problem x square y double dash + 7 xy dash + 3y = 0 with the nonhomogeneous boundary condition that is y(1) = 1 and y(2) = 2. And we can simply say that since it is an equation we can easily find out our solution is given in terms of x to power R kind of thing.

So, for which values of R, x to power R is a solution that we have already discussed and we can easily see that y(x) = c1 x to power – 3 + root 6 + 3 to x power – 3 – of root 6 is a general solution of this. So, this I am leaving it to you that how to find out a general solution of this. Now, let us proceed further and we want to find out y1 = 1 and if you use the condition y1 = 1 then c1 + c2 = 1 and y2 = 2 means c1 2 to power – 3 + root 6 + c2 2 to power – 3 – root of 6 = 1 and we want to find out the c1 and c2 for which it has a solution.

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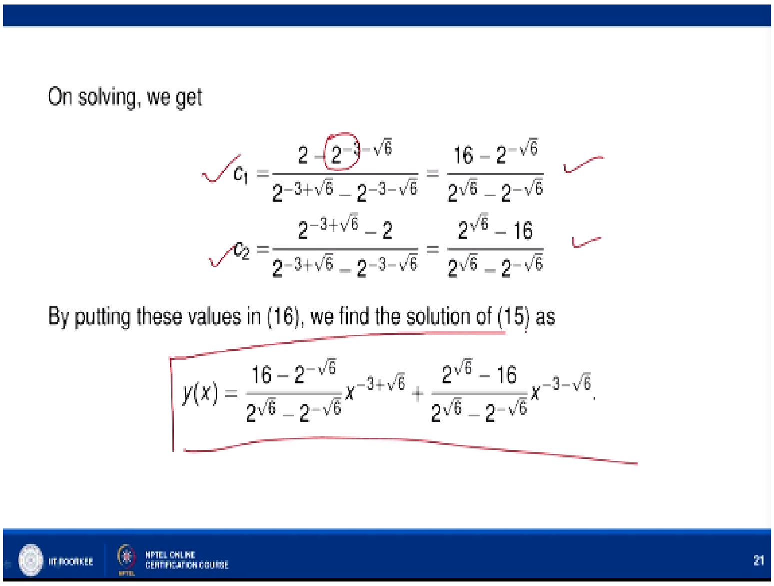

So, you just look at the value of c1. c1 is here you can simply say c1(x) 1 – 3 to put it back and you can simplify. So, here you can get c1 as 2-2 to power -3 – root 6/this quantity and if you simplify or you simply say that take this term out, 2 to power -3 common then you will get that c1 is 16 – 2 - root 6/2 power root 6 – 2 to power – of root 6.

Similarly, in c2 also we take 2 to power -3 out and you will get c2 as 2 to power root 6 – 16 upon 2 to power root 6 – 2 to power – of root 6. And once we have c1 and c2 you can write on the solution general solution as this y(x) = this and since now c1 and c2 are particular values then we can call this as a solution of the initial value problem and you can say that this is a unique solution of the boundary value problem here.

And this we simply say that since we are getting a unique solution because the homogenous boundary value problem has a trivial solution. How we can check? You put y1 = 0 and y2 = 0 and when you put y1 = 0 and y2 = 0 then we have c1 + c2 = 0 and c1 2 power – 3 root 6 + c2 2 to power – 3 – root 6 = 0. So, here we can easily check than this is what 1 1 and 2 to power -3 + root 6 and 2 to power – 3 – of root 6 this is your delta, right.

And if you look at the value is going to be what, this delta is not coming out to be, this is coming out to be non-zero value. So, we simply say that since homogeneous boundary value problem has a trivial solution then nonhomogeneous boundary value problem has a unique solution and we have not only prove that it has a unique solution we also have find out the unique solution here.

So, with this we conclude our lecture here. In next class will continue our study of boundary value problem. Thank you for listening us, thank you.