Ordinary and Partial Differential Equations and Applications
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Lecture - 24
Stability of Linear Systems- III

Hello friends welcome to my next lecture on stability of linear systems. Let us consider the linear system as we said.

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Let us consider linear system

$$\frac{dx}{dt} = ax + by,$$
$$\frac{dy}{dt} = cx + dy. \qquad (1)$$

Dx/dt=ax+by dy/dt=cx+dy

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Case IV:

Theorem: The roots λ_1 *and* λ_2 of the characteristic equation

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0 \qquad (*)$$

are conjugate complex with real part not zero (i.e. not purely imaginary). Then the critical point (0,0) of the linear system (1) is a spiral point.

Proof: Let $\lambda_1 = \alpha + i\beta$ *and* $\lambda_2 = \alpha - i\beta$, where $\alpha, \beta \in \Re - \{0\}$. Then, the general solution of the system (1) is given by

$$\begin{aligned} x &= e^{\alpha t}\left[c_1\left(A_1 \cos\beta t - A_2 \sin\beta t\right) + c_2\left(A_2 \cos\beta t + A_1 \sin\beta t\right)\right], \\ y &= e^{\alpha t}\left[c_1\left(B_1 \cos\beta t - B_2 \sin\beta t\right) + c_2\left(B_2 \cos\beta t + B_1 \sin\beta t\right)\right], \end{aligned} \qquad (2)$$

where A_1, A_2, B_1 *and* B_2 are definite constants and c_1 *and* c_2 are arbitrary constants.

Now we are going to discuss the fourth here the roots lambda 1 and lambda 2 of the characteristic equation are conjugate complex with real paths not=0 that means the roots are not purely imaginary. So let us consider this case here we shall see that the critical point 0,0 of the linear system is a spiral point. So, let us see how we get this situation suppose lambda 1=alpha+I beta and lambda=alpha-I beta.

Where alpha and beta are both non-Euclidean numbers that is alpha, beta belongs to r-simulant 0 and then the general solution of the linear system 1 is given by x=e to the power alpha t c1 times A1 cos beta t-A2 sin beta t+c2 times A2 cos beta t+A1 sin beta t y= e to the power at times c1 B1 cos beta t-B2 sin beta t+c2 times B2 cos beta t+B1 sin beta t and A1, A2, B1 and B2 are definite constants and c1 and c2 are arbitrary constants.

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Let $\alpha < 0$. Then from (4) we have $\lim_{t \to +\infty} x = 0$, $\lim_{t \to +\infty} y = 0$. Hence, all paths defined by (2) approach (0,0) as $t \to +\infty$. Now, (2) can be expressed as

$$x = e^{\alpha t}\left[c_3 \cos \beta t + c_4 \sin \beta t\right],$$
$$y = e^{\alpha t}\left[c_5 \cos \beta t + c_6 \sin \beta t\right], \qquad (3)$$

where $c_3 = c_1 A_1 + c_2 A_2$, $c_4 = c_2 A_1 - c_1 A_2$, $c_5 = c_1 B_1 + c_2 B_2$, *and* $c_6 = c_2 B_1 - c_1 B_2$.

Let c_1, c_2 be real. Then, we may write the solution (3) as

$$x = K_1 e^{\alpha t} \cos(\beta t + \phi_1),$$
$$y = K_2 e^{\alpha t} \cos(\beta t + \phi_2),$$

First, we consider the case when the real paths alpha of the roots lambda 1 and lambda 2 is negative then we can see from the equation then we see from here that when alpha is negative x and by both go to 0 as t goes to +infinity. So, then from 2 we see that the limit of x as t goes to +infinity 0 and limit of y as t goes to +infinity 0 and therefore all paths defined by 2 approach 0,0 as t goes to +infinity.

Now we can write the equations 2 by collecting the co efficient of cos beta t and sin beta t x and by can be expressed in this form x=e to the power alpha t *c3 cos beta t+c4 sin beta t and y=e to the power alpha t times c5 cos beta t+c6 sin beta t and when you do this that is you collect the co efficient of cos beta t and sin beta t it turns out that the values of c3 c4 c5 and c6.

Are given by c1 A1+c2 A2 c4 is c2 A1-c1 A2 c5 is c1 B1+c2 B2 and c6 is c2 B1-c1B2. Now let us assume that c1 and c2 be real numbers then this representation of x and y can be further written in this form x=k1 times e to the power alpha t cos Beta t+ phi 1 and y=K2 times e to the power alpha t cos beta t+phi 2.

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where $K_1 = \sqrt{c_3^2 + c_4^2}, K_2 = \sqrt{c_5^2 + c_6^2}$ and ϕ_1 *and* ϕ_2 are defined by

$$\cos\phi_1 = \frac{c_3}{K_1},\ \cos\phi_2 = \frac{c_5}{K_2},$$

$$\sin\phi_1 = -\frac{c_4}{K_1}\ and\ \sin\phi_2 = -\frac{c_6}{K_2}.$$

And while doing this it is easy to see that K1 is under root c3 square+c4 square and k2 is under root c5 square+c6 square and phi1 and phi 2 are defined by the equations cos phi 1=c3/K1 cos phi2=c6/K2 sin phi1=-c4/K1 and sin phi2=-c6/K2

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Now, let us consider

$$\frac{y}{x} = \frac{K_2 e^{at}\cos(\beta t + \phi_2)}{K_1 e^{at}\cos(\beta t + \phi_1)}$$

Suppose $K = {K_2}/{K_1}$ and $\phi_3 = \phi_1 - \phi_2$ then

$$\frac{y}{x} = \frac{K\cos(\beta t + \phi_1 - \phi_3)}{\cos(\beta t + \phi_1)}$$

$$= K\left\{\frac{\cos(\beta t + \phi_1)\cos\phi_3 + \sin(\beta t + \phi_1)\sin\phi_3}{\cos(\beta t + \phi_1)}\right\}$$

$$= K\{\cos\phi_3 + \tan(\beta t + \phi_1)\sin\phi_3\}, \qquad (4)$$

Provided $\cos(\beta t + \phi_1) \neq 0$.

Now let us consider the ratio of y and x so let us consider y/x y is K2 times e to the power at cos*betat+phi2 and x is K1 times e to the power alpha t *cos beta t+phi1 so when we divide y/x e to the power alpha t will get cancelled and so k will let us if you note K2/K1=K then y/x will be K times cos beta t+ let us take difference of phi 1 and phi 2 to be phi 3.

So, then phi2=phi1-phi3 you will have K cos beta t+phi1-phi 3 over cos beta t+phi 1. Let us expand cos beta t+phi 1 -phi 3 we will have cos beta t+phi 1*cos phi 3 and+sin beta t+phi 1 *sin phi 3 and/ cos beta t+phi1. So, when we divide by this assuming that cos beta t+phi s is not 0 what do we get y/x we will get as k times cos phi 3+tan beta t+ phi1 *sin phi 3.

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Since the trigonometric functions occurring in (4) are periodic, it follows that $\lim_{t\to\infty}\frac{y}{x}$ does not exist and hence the paths do not enter (0,0). Instead, the paths approach (0,0) in a spiral-like manner, winding around (0,0) an infinite number of times as $t \to +\infty$.

Hence, the critical point (0,0) is a spiral-point and it is asymptotically stable as shown in fig.1.

If $\alpha > 0$, the solution is the same except that the path approach (0,0) as $t \to -\infty$, the spiral point (0,0) is unstable and the arrows in fig.1 are reversed.

Now the trigonometric functions occurring in 4 you can see here the functions are trigonometric functions they are periodic cos phi 3 is periodic, sin phi 3 is periodic 10 beta t+ phi 1 is also periodic. So, since the trigonometric functions occurring in 4 are periodic followed that limit t tends to infinity y/x does not exist okay so here as t tends to infinity does not exist.

And so what we get hence the paths do not enter 0,0 instead the paths approach 0,0 in a spiral manner winding around 0,0 an infinite number of times as t goes to+ infinity and hence the critical point 0,0 is a spiral point and it is asymptotically stable as shown in the figure 1

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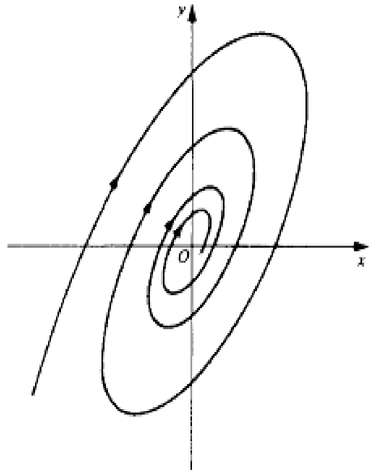

$Fig.1$

You can see here they are binding around 0,0 in a spiral like manner and approaching 0,0 in a spiral like manner by binding around 0,0 infinite number of times. Now 0,0 is a spiral point and it is asymptotically stable if alpha is >0 if the real paths of the roots of lambda 1 and lambda 2 which is alpha is>0 then the solution we get the same solution that means we again have 0,0 a spiral point.

That is the path approach 0,0 as t goes to - infinity and the spiral point 0,0 is unstable because the arrows now are reversed. So, when alpha >0 again we get the same situation but the arrows will now be in the opposite direction so we have this.

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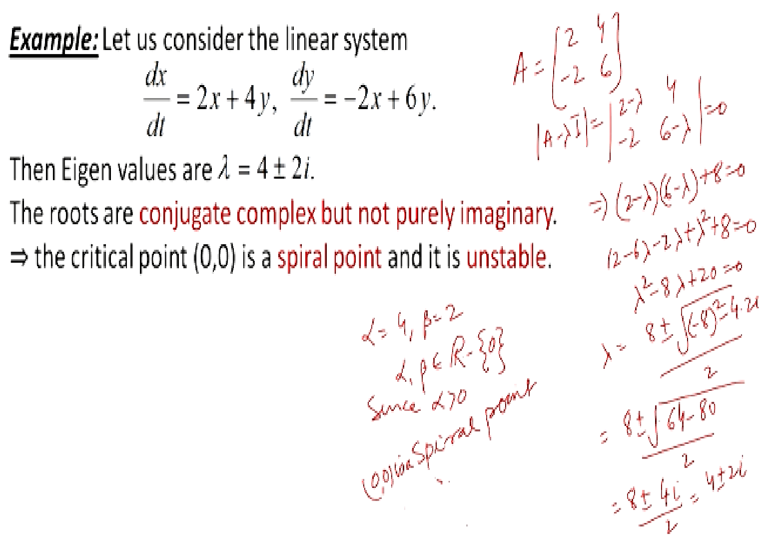

And now consider the linear system an example over this dx/dy=2x+4y dy/dt=-2x+6y so here A=2 4 -2 6 and so the characteristic equation Determinant A - lambda I= 2 - lambda 4 – 2 6-lambda=0 and this gives you 2- lambda* 6 - lambda+8=0 or I can say this is 12-6 lambda -2lambda+lambda square+8=0. So, I can say this is lambda square-8 lambda+20=0.

So, what do we get lambda=-b and let us apply that formula lambda=-b so 8=+ - under root so -8 whole square -4*20/2 so we can get 8+- under root 64-80/2 which is 8+- this is -16 so 4i/2 so you will get 4+-2i so eigen values are lambda-4+-2I. so they are conjugate complex where alpha is 4 beta is 2. So, alpha and beta both belong to R-0 they are non-0-real numbers.

Since alpha is >0 the critical point 0,0 is a spiral point and it is unstable. So, we get spiral point 0,0 is a spiral point and unstable.

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Case V:

Theorem: The roots λ_1 *and* λ_2 of the characteristic equation are purely imaginary. Then the critical point of system (1) is a center.

Proof: Let $\lambda_1 = \alpha + i\beta$ *and* $\lambda_2 = \alpha - i\beta$, where $\alpha = 0$ *and* β is a non zero real number. Then, the general solution of (1) is given by

$$x = K_1 \cos(\beta t + \phi_1),$$
$$y = K_2 \cos(\beta t + \phi_2), \quad (5)$$

where K_1, K_2, ϕ_1 *and* ϕ_2 are as defined in case (4).

$\lambda_1 = \alpha + i\beta$
$\lambda_2 = \alpha - i\beta$
$\alpha = 0$, Here $\lambda_1 = i\beta$, $\lambda_2 = -i\beta$, $\beta \neq 0$

The roots of lambda 1 and lambda2 of the characteristic equation are purely imaginary here we will assume that lambda 1 and lambda 2 are purely imaginary that is their real path is 0. We had assumed lambda 1 as lambda+I beta and lambda2 to be alpha -I beta. So, here in this case in the case 5 we will have alpha=0. So, here lambda 1=I beta and lambda 2=-I beta here beta is non-0.

So, let us say lambda=alpha+I beta lambda2=alpha-i beta where alpha is 0 and beta is a non-zero-real number. Then the general solution of 1 we had written as if you can go to the previous case this is the final form of the solution x=K1 e to the power alpha t cos Beta t+phi 1 y= K2 e to the power alpha t cos beta t+phi 2. So, in this case 5 since alpha=0 x will be K1 cos beta t+phi 1 and y will be K2 cos beta t+phi 2.

We have here X=K1 cos beta t+phi 1 and y=K2 cos beta+phi2 where K1 K2 Phi1 and Phi 2 are as defined in case 4.

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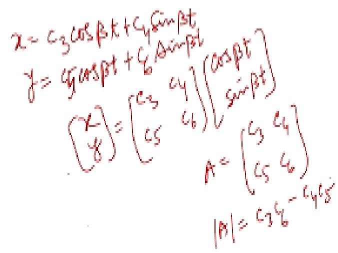
Since the trigonometric functions in (5) oscillate indefinitely between ± 1 as $t \to +\infty$ and as $t \to -\infty$, the path do not approach (0,0) as $t \to +\infty$ or as $t \to -\infty$. Clearly, x and y are periodic functions of t hence the paths are closed curves surrounding (0,0). In fact they are an infinite family of ellipses.



Okay now let us see since the trigonometric functions in 5 trigonometric function let us look at this because when you open this what you have cos beta t then you have cos phi 1-sin beta t sin phi 1 okay so cos beta t and sin beta y they are trigonometric functions and therefore they oscillate indefinitely between+1 and -1 as t goes to+ infinity and as t goes to - infinity the paths do not approach 0,0.

As t goes to +infinity or as t goes to -infinity x and y are periodic functions of t because cos beta t sin beta t they are periodic functions so clearly x and y are periodic functions of t and the paths are closed curves surrounding 0,0 in fact that they are infinite family of ellipses. Now how they are they are in infinite family ellipses let us see that you see x and y we had come to this form of x and y from this situation.

From this situation okay from here we came here now alpha=0 we have x= c2 cos beta t+c4 sin beta t y= c5 cos beta t+c6 sin beta t so let us go there. so here x= c3 cos beta t+c4 sin beta t and y=c5 cos beta t+c6 sin beta t. I can write the systems of equations as xy in the form of a matrix c3 c4 c5 c6 and we have cos beta t sin beta t okay so we have a system of equations.

And let me call this as the co efficient matrix of the system so A=c3 c4 c5 c6 determinant of A=c3 c6 -c4 c5.

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Let $\alpha < 0$. Then from (4) we have $\lim_{t \to +\infty} x = 0$, $\lim_{t \to +\infty} y = 0$. Hence, all paths defined by (2) approach (0,0) as $t \to +\infty$. Now, (2) can be expressed as

$$x = e^{\alpha t}\left[c_3 \cos \beta t + c_4 \sin \beta t\right],$$
$$y = e^{\alpha t}\left[c_5 \cos \beta t + c_6 \sin \beta t\right], \quad (3)$$

where $c_3 = c_1 A_1 + c_2 A_2$, $c_4 = c_2 A_1 - c_1 A_2$, $c_5 = c_1 B_1 + c_2 B_2$, *and* $c_6 = c_2 B_1 - c_1 B_2$.

Let c_1, c_2 be real. Then, we may write the solution (3) as

$$x = K_1 e^{\alpha t} \cos(\beta t + \phi_1),$$
$$y = K_2 e^{\alpha t} \cos(\beta t + \phi_2),$$

$A_2B_1 - A_1B_2 \neq 0$, $A_2B_1 \neq A_1B_2$

$A = \begin{pmatrix} c_3 & c_4 \\ c_5 & c_6 \end{pmatrix}$, $|A| = c_3c_6 - c_4c_5$

$|A| = (c_1A_1 + c_2A_2)(c_2B_1 - c_1B_2) - (c_2A_1 - c_1A_2)(c_1B_1 + c_2B_2)$

$A_2B_1(c_2^2 + c_1^2) - A_1B_2(c_1^2 + c_2^2) = (c_1^2 + c_2^2)(A_2B_1 - A_1B_2)$

Let us put the values of c3 c4 c5 c6 from here okay c3 c1 A1+c2 A2 or I can find here itself we have the matrix c3 c4 c5 see we have this matrix we have A=c3 c4 c5 c6 okay. So, determinant of AB found as c3 c6 -c4 c5 if you put the values of c3 c6 c4 c5. What do we get c3=c1 A1+c2 A2 c6=c2 B1-c1B2- C4 is c2 A1-C1 A2 and c5 is c1 B1+c2 B2.

Let us multiply what do we get c1 c2 A1 B1 C2 square A2 B1 then we have C1 square A1 B2-C1 C2 A2 B2 and then we have c1 c2 A2 B2 we have- c1 square A2 B1 and then we have c2 square A1 B2 and we have -c1 c2 A2 B2 okay now what do we notice c1 c2 A1 b1 this is positive here it is negative -c1 c2 A1 B1. So, this will cancel with this okay.

And her we have -c1 c2 A2 B2 here we have+c1 c2 A2 B2 this gets cancelled with this what do we get then then A2 B1 times c2 square+c1 square okay we get A2 B1 times c2 square+c1 square and here A1 B2 times C1 square+c2square. So determinant if A is actually c1 square+c2 square*A2 B1-A1 B2 okay now let us look at the situation where determinant of A can be 0.

Determinant of A can be 0 if c1 square+c2 square=0 since c1 and c2 are real quantities we are taking okay if c1 square+c2 square=0 then c1=0 c2=0. Since c1 and c2 belong to R and then c1 and c2 are 0 what will happen c1 and c2 are 0 we will get c3=0 c4=0 c5=0 c6=0 okay so c1 c2=0 implies c3 c4 c5 and c6 all are 0 and which implies that x=0 and y=0 we get a trivial situation.

So, c1 square and c2 square cannot be 0 and – A1 B2 cannot be 0 okay this cannot be 0 because we know that R=A2 B1 is not=A1 B2 this is the condition of we have on A1 B1 A2 B2 okay. So, this is also not 0 so determinant of A cannot be=0 that is what we get that from here okay now let us go there.

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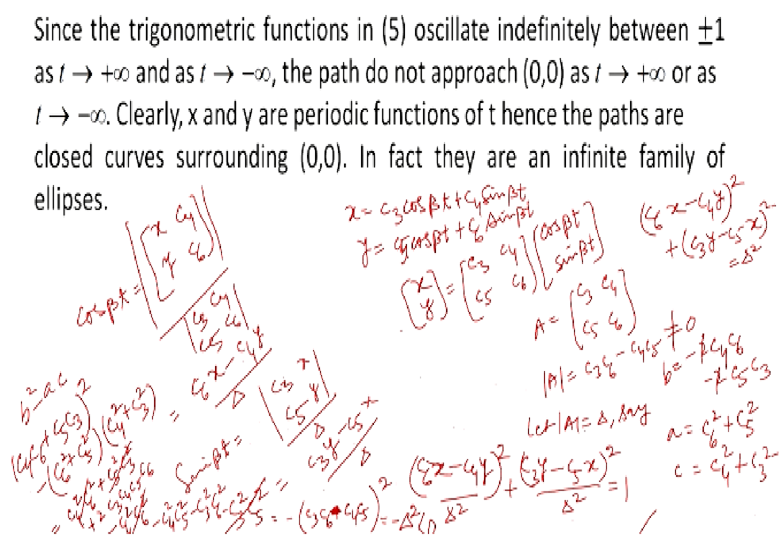

Okay so determinant of A cannot be 0 this is what we have found okay. We can solve this equation by (()) (18:32). So we get cos beta t we have determinant of the matrix xy we replace first column of the function matrix column of constant so xy c4 or c6/determinant of the co efficient matrix c3 c4 c5 c6 okay this is cos beta t=c6 x-c4y/ let me call the determinant of the co efficient matrix determinant of A= let us call as delta.

So, this is delta and sin beta t=similarly we will have determinant c3 c5 c4 c6. Second column will be replaced by xy/delta and you will get c3y-c5x/delta. Now we have the values cos beta t sin beta t we want to determine the quotient of second degree in x and y that results on elimination of t from these two quotients so sin square t+cos square t+1.

So, we have c6x -c4y whole square/delta square+c3y-c5x whole square/delta square=1 okay or we can get c6x or -c4y whole square+c3y-c5x whole square=delta square when you square these expressions you will get a second-degree quotient in x and y. Now this second-degree equation x and y will result in ellipse the co efficient of x and y say that is b co efficient of x square is A.

Coefficient of y square is A then B square- Ac if it is <0 then it will represent the ellipse. Let us look at the co efficient of xy that we get from here so A-B whole square so A square+B square -2A-B whole square so 2AB we have the co efficient of xy in the first expression is - 2 c4 c6 okay here the co efficient of xy is -2c5 c3 okay so b=-2c5 c3 okay the co efficient of xy.

We are denoting by b so this is co efficient of b so co efficient of x square is what let us denote so c6 square+c5 square and the co efficient of y square we are denoting by c so c will be= c4square+c3 square. Now let us find b square -ac so then what will you get if we have equation 2 and xy I think we need to take 2 objects we need not take these two that is b is the co-efficient of 2xy.

I think we need not take these two so we have this one c4 c6+c5 c3 whole square b square means c4 c6 c5 c3 whole square-c6 square*+c5 square*c4 square+c3 square and this will come out to be you can see c4 square c6 square+c5 square+c3 square and then 2 times c3 square c5 c4 c6 and here what we will get c6 square+c5 square so do we get okay c4 square okay c4 square c6 square.

And then c5 square c3 square and then 2c3c4c5c6 and here what we will get c4 square c6 square this will cancel with this and we will get c4 square c5 square okay and here we will get c3 square c6 square and here we will get -c3 square c5 square so this will cancel with this okay and what we do get now so this is -c3 c6+c4 c5 I think we will get it here.

Okay this can be written as -c3 c5-c4 c5 whole square and c3 c6-c4 c5 is delta okay -delta square and delta is non-0 and this is <0 so this represents x=c3 cos beta t+c4 sin beta t and y=c5 cos beta t+c6 sin beta t which is the summation of the linear system they represent infinite family of ellipses

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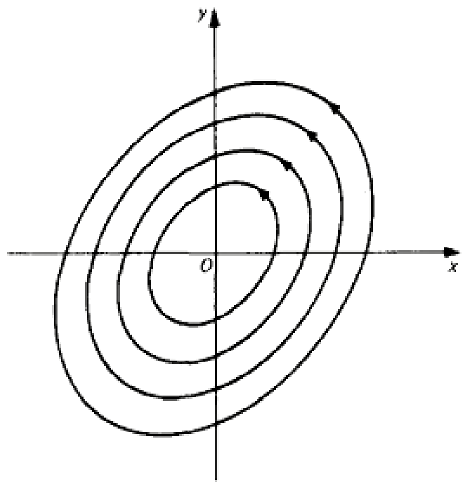

Fig.2

So here we have a node but we have unstable node

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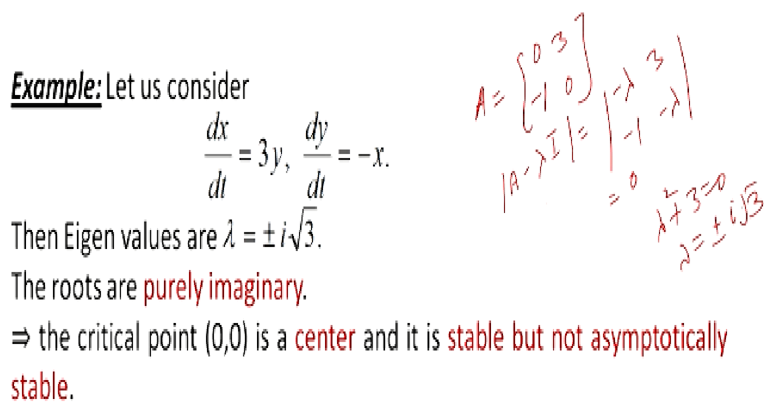

Okay we have a center we have a center out of this situation we have a center but it is stable but not asymptotically stable so we have this situation let us consider dx/dt=3y dy/dt=-x so here we can see the matrix A is 0 3 and-1 -1 0 a is 0 b is 3 c is -1 d is 0 so we get determinant of A-lambda I=-Lambda 3-1-lambda so we get this is=0 so we get lambda square+3=0 so lambda=+-I root 3.

So, the roots are purely imaginary so the critical point 0,0 as we have already just not discussed the critical point 0,0 in the center it is stable but not asymptotically stable.

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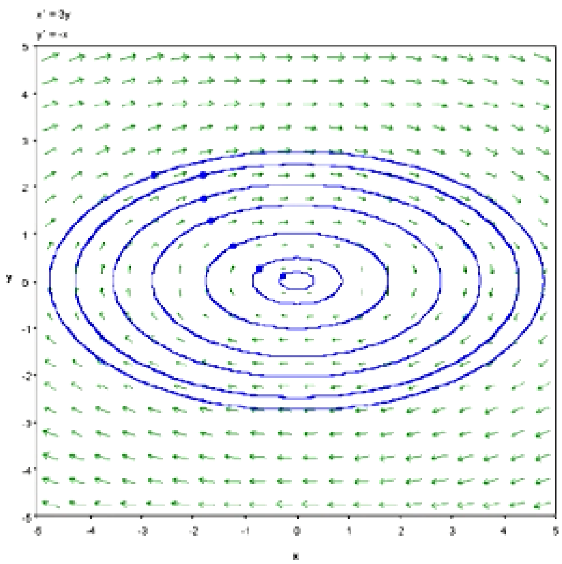

And this is the figure here okay you can see 0,0 center and the it is stable but not asymptotically stable.

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Nature of roots λ_1 and λ_2 of characteristic equation (*)	*Nature of critical point (0,0) of linear system (1)*	*Stability of critical point (0,0)*
conjugate complex but not purely imaginary	Spiral point	Asymptotically stable if real part of roots is negative; unstable if real part of roots is positive
purely imaginary	Center	Stable but not asymptotically stable

So, let us consider the first the case where lambda 1 and lambda 2 are both conjugate complex roots but they are not purely imaginary that is alpha is not=0. We have seen that in this case 0,0 is a spiral of point and we have seen that if alpha is negative then we had the stability point 0,0 is asymptotically stable but if alpha is positive then point 0,0 is unstable.

And just now we discussed the case that lambda 1 and lambda 2 are purely imaginary that is+-I beta in that case we had the critical point 0,0 as a center it is table but not asymptotically stable. Thank you very much for your attention.