Ordinary and Partial Differential Equations and Applications

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Lecture - 23
Stability of Linear Systems -II

Hello friends welcome to my lecture on stability of linear systems. We begin with an example on a system of linear linear system dx/dt=2x-7y, dy/dt=3x-8y.

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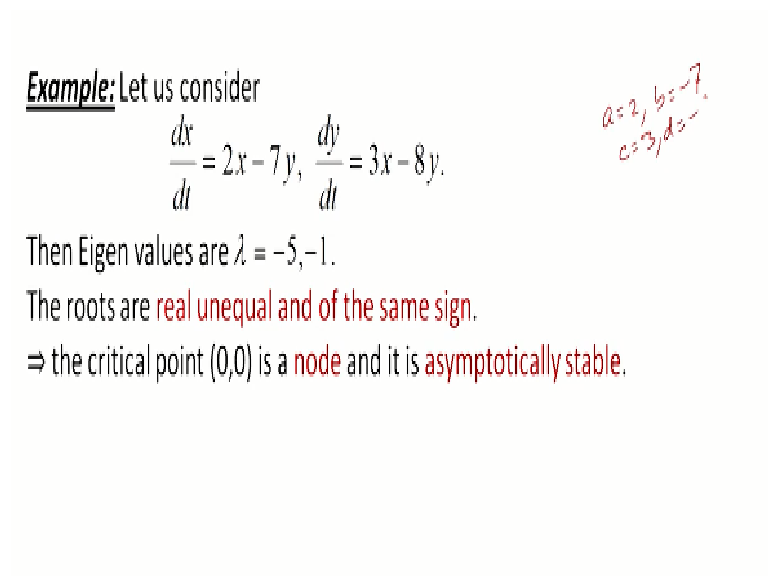

Here if a=2b=-7, c=3 and d=-8. So the characteristic equation is lambda square-a+d* lambda+ad-bc=0. When we put the values of abcd here okay so what do we get lambda square-a+d, so 2-8 means-6 so we get lambda square+6x and then ad, ad is-16and then we have bc , bc=-21 so+21=0. So we get lambda square+6x+5=0. The factors here are lambda+5 and lambda+1.

So the eigen values are lambda=-5 and lambda=-1. Thus we can see that the roots of the equation are the roots of the characteristic equation are real unequal and they are of the same sign. And therefore why the case 1 which we have considered in the previous lecture. The critical point 00 here is a node and it is asymptotically stable.

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Case II:

Theorem: The roots λ_1 *and* λ_2 of the characteristic equation

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0 \qquad (*)$$

are real, unequal and of opposite sign. Then the critical point (0,0) of linear system

$$\frac{dx}{dt} = ax + by,$$
$$\frac{dy}{dt} = cx + dy, \qquad (1)$$

Is a saddle point.

Now let us go to the case 2 here we will consider the case where lambda 1 and lambda 2 are real, equal and are of opposite signs, so let us say the root lambda 1 and lambda 2 of the characteristic equation lambda square-a+dlambda+ad-bc=0 are real unequal and of opposite signs. So, then the critical point 00 of the linear system we shall see is a saddle point.

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Proof: Suppose $\lambda_1 < 0 < \lambda_2$. The general solution of the system is given by

$$x = c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 t},$$
$$y = c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t},$$

where A_1, B_1, A_2 *and* B_2 are definite constants and $A_1 B_2 \neq A_2 B_1$, and where c_1 *and* c_2 are arbitrary constants.

Choosing $c_2 = 0$, we obtain the solutions

$$x = c_1 A_1 e^{\lambda_1 t},$$
$$y = c_1 B_1 e^{\lambda_1 t}.$$

$c_1 > 0$, $B_1 x = A_1 y$, $c_1 < 0$, $y = \frac{B_1 x}{A_1}$

Now let us see how do we get a saddle point here, suppose lambda 1 is <0 and lambda 2 is >0, so that means lambda 1 is negative and lambda 2 is positive. This we do for definiteness. The general solution of the system 1 here is then given by x=c1A1e to the power lambda t+c2 A2e to the power lambda 2t and y=c1B1 e to the power lambda 1t+c2 B2 e to the power lambda 2 t. And A1 B1 A2 and B2 are definite constants.

And y=c1B1 e to the power lambda 1 t+c2B2 e to the power lambda 2t and A1 B1 A2 and B2 are definite constants and satisfy the condition that A1 B2 is != A1 B1 c1 and c2 are arbitrary constants. Now we will analyze the situation here that was analyzed in case 1 okay so choosing c2=0 what do we obtain, we obtain x=c1 A1e to the power lambda 1t and y=c1B1e to the power lambda 1 t.

Now let us assume that c1 is >0then what will happen when we assume c1>0 then these two equations B1x=A1y. So, in this solution c1>0 in this case B1x=A1y this gives you half line path okay and since lambda 1 is negative and t goes to infinity okay x goes to 0 and y goes to 0. So, that means the half line path B1x=A1y enters 00 as t goes to infinity, and when we take c1<0 we get the other half line path given by the line B1x=A1y.

So, this solution x=c1A1 e to the power lambda 1t y=c1 B1 e to the power lambda 1t represents two half line paths okay which approach an enter 00 as t goes to +infinity, because lambda 1 is negative and the line is given by B1x=A1y. SO they enter 00 with the slope y= B1//A1x the slope of the line is B1/A1.

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If we choose $c_1 = 0$, we obtain the solutions

$$x = c_2 A_2 e^{\lambda_2 t},$$
$$y = c_2 B_2 e^{\lambda_2 t}. \qquad (4)$$

There are two half line paths, which approach and enter (0,0) as $t \to +\infty$ and two other half-line paths which approach and enter (0,0) as $t \to -\infty$. All other paths are non-rectilinear paths which do not approach (0,0) as $t \to +\infty$ or as $t \to -\infty$ but which become asymptotic to one or another of the four half-line paths as $t \to +\infty$ and as $t \to -\infty$. Thus, the critical point (0,0) is a saddle point and it is unstable as shown in the fig.1. $c_1 \neq 0, c_2 \neq 0$

Now when we take c1=0 x=c1 A2 e to the power lambda 2 t and y=c2B2 e to the power lambda 2 t, this also gives two half line paths which are given by B2x=A2y, if you take c2=0 then the half line path is given by B2x=A2y when c2<0 then we get the other half line path given by the equation B2x=A2y. But since lambda 2 is >0 here as t goes to-infinity x and y goes to 0 x goes to 0 y goes to 0 and t goes to-infinity.

So these two half line paths enter 00 when t goes to-infinity. Now let us take c1is !=0 c2 is !=0 when c1 is !=0, c2 !=0 then what do we get.

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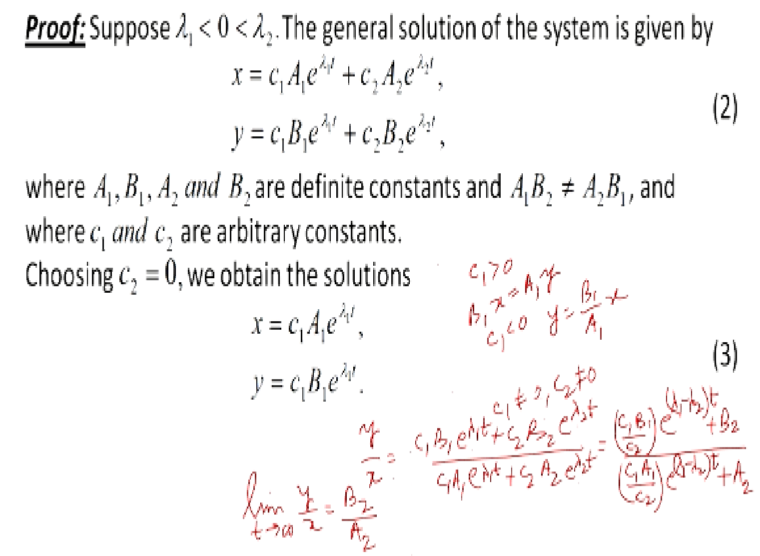

We have x= c1 A1 when c1 is !=0 and then c1 is mot=0 then these two parts this solution 2 represents rectilinear paths because as t goes to +infinity this part e to the power lambda 1 t goes to0 but e to the power lambda 2t goes to infinity. As lambda 2 is positive. Similarly, for the case solution y e to the power lambda 1 t goes to 0 as t goes to infinity while e to the power lambda 2 t goes to infinity as t goes to infinity so one part goes to 0 while the other part goes to infinity.

And same is the situation when t goes to-infinity, when t goes to-infinity e to the power lambda 1t goes to infinity while e to the power lambda 2t goes to 0. So, the solution 2 represent rectilinear paths and do not approach to 00 or as t goes to +infinity or-infinity. Let us look at y/x so y/x in this case is c1B1 e to the power lambda 1t+c2 B2 e to the power lambda 2t/c1 A1 e to the power lambda 1t+c2 A2e to the power lambda 2t.

Let us divide this numerator and denominator value to the power lambda 2 t so that we get C1

B1 first let us divide /c2 and then we will divide by e to the power lambda 2t. So, what do we get, c1 B1 e to the power lambda 1-lambda 2 t+B2 and we divide by here we take also c1 A1/c2 and then e to the power lambda 1-lambda 2*t+A2.So, as 2 goes to infinity since lambda 1 <lambda 2 lambda1-lambda2 is negative.

So as t goes to infinity e to the power lambda1-lambda2 goes to 0 and therefore y/x=B2/A2 and therefore what we can say is that the rectilinear paths enter 00 with slope B2/A2.

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If we choose $c_1 = 0$, we obtain the solutions

$$x = c_2 A_2 e^{\lambda_2 t},$$
$$y = c_2 B_2 e^{\lambda_2 t}. \qquad (4)$$

There are two half line paths, which approach and enter (0,0) as $t \to +\infty$ and two other half-line paths which approach and enter (0,0) as $t \to -\infty$. All other paths are non-rectilinear paths which do not approach (0,0) as $t \to +\infty$ or as $t \to -\infty$ but which become asymptotic to one or another of the four half-line paths as $t \to +\infty$ and as $t \to -\infty$. Thus, the critical point (0,0) is a saddle point and it is unstable as shown in the fig.1. $c_1 \neq 0, c_2 \neq 0$

So, we have there are two half line paths we can conclude that there are two half line paths which approach and enter 00 as t goes to infinity and two other half line paths which approach and enter 00 as t goes-infinity. This is in the case where we have taken c2=0 and this is +infinity. Two other half line path which enter 00 as t goes to-infinity. This is the case where we have taken c2=0 and this is the case where we can take c1=0.

All other paths are non-rectilinear paths that means when we have taken c1 !=0 and c2 !=0 which do not approach 00 as t goes to +infinity or t goes to-infinity. But which become asymptotic to one or another of the four half line paths. So, this means that these are non-rectilinear paths which do not approach to 00 as goes to infinity. This is what we have y/x y/x is the slope of any point xy on the rectilinear paths this is B2/A2.

Okay, so this means that the slope of the tangent to the rectilinear path approaches to B2/A2 that means the slope of the line B2x=A2y B2x=A2y this line has the slope B2/A2. So, the non-rectilinear paths approach or become asymptotic to the rectilinear path B2x= A2y. So, they do not approach 00 as t goes to +infinity or t goes to-infinity but which asymptotic to one or another o the four half line paths as t goes to +infinity or t goes to-infinity.

Thus the critical point 00 is a saddle point and it is unstable as shown in the figure.

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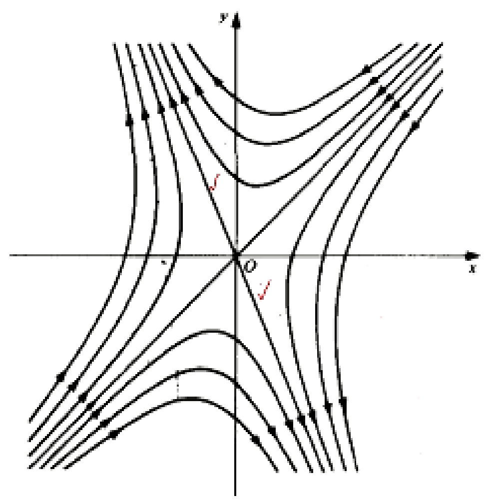

Fig 1

As shown in this figure this is your half line path as t goes to +infinity and it goes to 00 so it enters 00 as t goes to +infinity. Now this is another half line paths those one as t goes to-infinity okay it enters 00 and the directions are opposite. The directions are towards point 00 now here they are above from the origin and above from 00. Now these paths are non-rectilinear paths and you can see that they become asymptotic to one of the two half line oaths this one.

These are asymptotic to half line paths and these are asymptotic too this half line path. So, they become asymptotic to one of the half line paths therefore this point at origin is a saddle point and these non-rectilinear paths do not approach 00 as t goes to +infinity or-infinity.

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Example: Let us consider

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = -y.$$

Then Eigen values are $\lambda = 1, -1$.

The roots are real unequal and of opposite sign.

$\Rightarrow$ the critical point (0,0) is a saddle point and it is unstable.

Now let us consider the example of dx/dt=x dy/dt=-y so a=1 b=0 c=0 and d=-1 so what do we have lambda square-a+d a+d means 1-0 so 0 *lambda and then+ad ad is-1-bc okay. So, lambda square-1=0 and therefore lambda=+-1 so Eigen values are lambda=1, -1 they are real unequal and of opposite sign. So, we have case two here and just now we have seen eigen values are real, unequal and of opposite sign. Then the critical point 00 is a saddle point and it is unstable.

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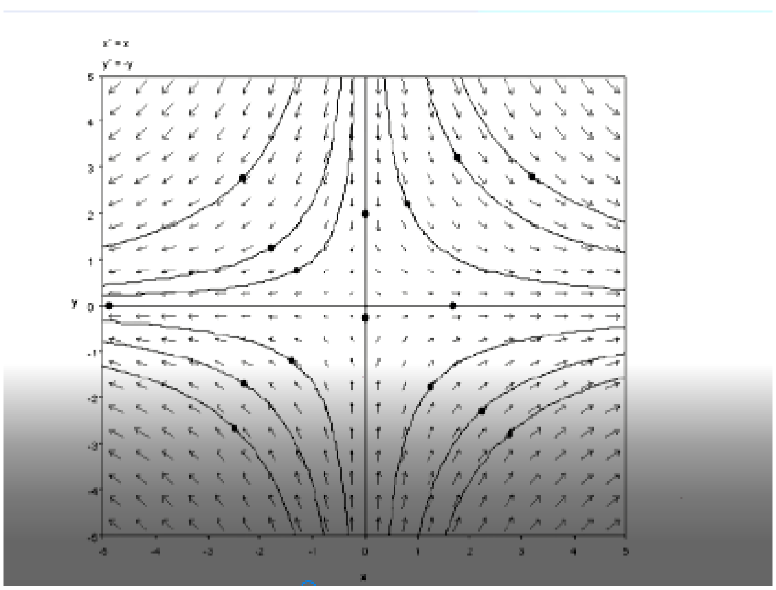

This is the figure here we can see here the two half line paths are this is an half line path this is another half line path. This is other side of the half line path and this is one half line path this one and this is another half line path and these are non-rectilinear paths which become asymptotic to the two half line paths this one and this one goes to +infinity or-infinity. Now here 00 is a saddle point and it is unstable.

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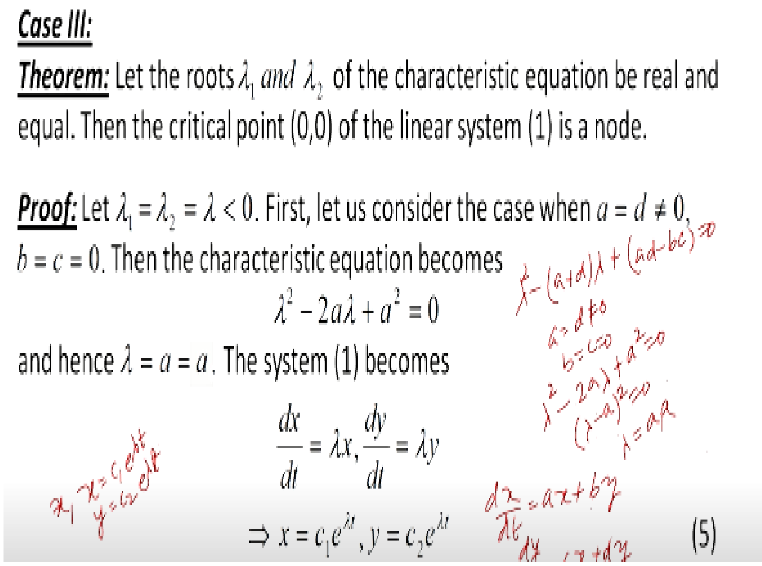

Now let us consider the case 3 where we will assume lambda 1 and lambda 2 of the system are real and equal. So, let us say the roots of lambda 1 and lambda 2 of the characteristic equation be real and equal then we are going to show that the critical point 00 of the linear system is a node. So, let us begin with the situation where lambda 1 and lambda 2 are both equal to lambda and lambda is negative.

So, we are assuming that both roots of the characteristic equation are equal, real, equal and are negative. So, first let us consider the case when a=d now let us look at the characteristic equation characteristic equation is lambda square-a+d*lambda+ad-bc=0 okay for the equal roots we must have B2=4ac of the quadratic equation. So, that we can achieve in this case that is one case is a= d !=0 suppose if I write a=d is !=0 and bc=0.

Then what I will get I will get lambda square-2 a lambda+a square=0 moreover we will have ad-bc which is=a square a is non-0 so ad-bc is non-0. Now these two roots of this equation are a each okay. So, the characteristic equation lambda square-a+d lambda+ad-bc=0 becomes lambda square-2a lambda+a square=0 and therefore lambda=a a=d. SO, the system of the co efficient are dx/dt=ax+by dy/dt=cx+dy.

So a=lambda and b=0 and I will get dx/dt=lambda x and here c=0 d=lambda so we have dy/dt=lambda y and we know how to find system the general solution of an equation of the type dx/dt=lambda x, so the solution is you can write dx/x=lambda dt and you can integrate you will get x=some constant c1 e to the power lambda t, and similarly dy/dt=lambda will give you other solution y=c2e to the power lambda t.

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Since $c_1 y = c_2 x$, the paths defined by (5) for the various values of c_1 *and* c_2 are half-line paths of all possible slopes. Further, since $\lambda < 0$, each of these half-lines approaches and enters (0,0) as $t \to +\infty$. Hence the critical point is a node and it is asymptotically stable.

If $\lambda > 0$, we have the same situation except that the paths enter (0,0) as $t \to -\infty$, the node (0,0) is unstable and the arrows in fig.2 are all reversed.

This type of node is called ***a star-shaped node***.

Now you can see from here we have x=c1e to the power lambda t x=c1e to the power lambda t and y=c2 e raised to the power lambda t so what will you get you will get this equation c1y=c2x okay c1y=c1c2 e raised to the power lambda t which is=c2x okay we will get this equation c1y=c2x now the paths defined by 5 defined by this 5 x=c1 e to the power lambda t y=c2 e to the power lambda t for the various values of c1 c2 are half line paths of all possible slopes.

And slopes will be you can see slope will be you can take any point xy on the path then the slope of the line joining the point xy to the origin will be y/x which is=c2/c1 c2/c1 and so when c1 and c2 are arbitrary you can say that you will get half line paths of all possible slopes. Further since lambda<0 see here lambda<0 so as t goes to infinity of these half line paths approaches and enters 00as t goes to +infinity.

And therefore we can say that the critical point Is a node and also it is asymptotically stable. If lambda is positive, we will have the same situation we will have the same equation c1y=c2x. So we will have for various values of c1 and c2 we will have half line paths but because lambda is positive these half line paths will enter 00 as t goes to-infinity and therefore point 00 is unstable.

The node at 00 is unstable and the arrows will be having opposite directions will all be reversed. Such type of node we will call as star shaped node.

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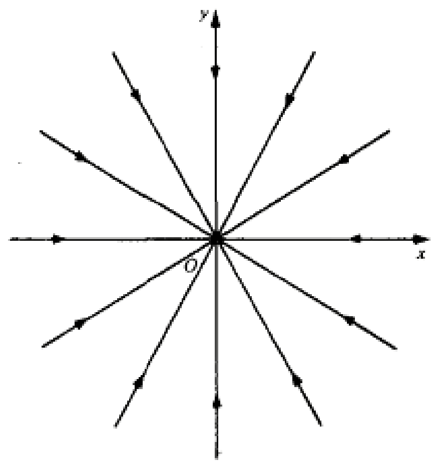

Fig .2

So you can see the situation here this is the case where we are taking lambda to be negative. So all the it is a star shaped node here the directions are towards the center if you take lambda to be positive these directions of arrows will be opposite to the directions here. So it is a star shaped figure so we call it as star shaped node.

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Now, let us consider all other possibilities which lead to a double root.
Then the general solution of system (1) is given by

$$x = c_1 A e^{\lambda t} + c_2 (A_1 t + A_2) e^{\lambda t},$$
$$y = c_1 B e^{\lambda t} + c_2 (B_1 t + B_2) e^{\lambda t}, \qquad (2)$$

where the A's and B's are definite constants and such that $\frac{B_1}{A_1} = \frac{B}{A}$ and

c_1 *and* c_2 are arbitrary constants. If we choose $c_2 = 0$,

$$x = c_1 A e^{\lambda t},$$
$$y = c_1 B e^{\lambda t}. \qquad (3)$$

Then there are two half-line paths which approach and enter (0,0) as $t \to +\infty$ with slope B/A.

Now let us consider the other possibilities which will lead us to a double root that means where the quadratic equation the characteristic equation lambda square-a+d*lambda+ad-bc=0 gives us double root in those possibilities let us consider now. So, in those possibilities the general solution of the system we can write as x=c1 A e to the power lambda t+c2 A1+t+A2* e to the power lambda t.

y=c1B e to the power lambda t+c2 B1t+B2*e to the power lambda t. Now As and Bs here are definite constants and they are such thatB1/A1=B/A c1 and c2 are arbitrary constants. Now if you take c2=0 here what do you get from here x=c1 A e to the power lambda t y=c1 B e to the power lambda t okay now then there are two half line paths which approach and enter 00 as t goes to infinity.

We are taking lambda to be negative, we have taken that lambda1=lambda2 and=lambda which is negative, so x and y both approach to 0 as t goes to +infinity if you take c1=0we will get one half line path if you take c1<0 we will get the other half line path both the half line path will approach and enter 00 as t goes to +infinity and when you consider the other case. So there are two half line paths which approach and enter 00 as t goes to infinity.

And the slope is by over x means B/A.

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If $c_2 \neq 0$, the general solution (2) represents non-rectilinear paths. Since $\lambda < 0$ we have $\lim_{t \to +\infty} x = 0, \ \lim_{t \to +\infty} y = 0$.

Thus, all the non-rectilinear paths approach (0,0) as $t \to +\infty$. Further,

$$\frac{y}{x} = \frac{\frac{c_1 B}{c_2} + (B_2 + B_1 t)}{\frac{c_1 A}{c_2} + (A_2 + A_1 t)} \Rightarrow \lim_{t \to +\infty} \frac{y}{x} = \frac{B_1}{A_1} = \frac{B}{A}.$$

Hence, all the rectilinear paths enter (0,0) with limiting slope B/A.

Thus all the paths (both rectilinear and non-rectilinear) enter (0,0) as $t \to +\infty$ with slope B/A. Hence, the critical point (0,0) is a node and it is asymptotically stable as shown in fig.3.

So, with the slope B/A if c2 is !=0 now let us consider this equation c2 is !=0 then the general solution 2 represents non-rectilinear paths if c2 is !=0 then the general solution is non-rectilinear paths and what we have since lambda is negative all these non-rectilinear paths satisfy the equation limited as x t towards infinity is 0 and limit of y t tends to infinity.

So, all these non-rectilinear paths enter 00 as t goes to infinity further if you find y/x. Since c2 is non-0 we can div ide the numerator and denominator by c2 and we get c1b/c2+B2+B1t=c1A/c2+A2+A1T and when t goes to infinity we can see here we can divide the numerator and denominator further by t so that c1B/c2t+B2/t+B1 and here we will have c1A/c2t.

I mean to say this when you divide when you want to take the limit as t goes to infinity you write y/x as c1B/C2t+B2yt+B1 and similarly in the denominator we write c1A/c2t+A2/t+A1 so as t goes to infinity we will have the limit as B1/A1 which is=B/A. Hence all the rectilinear paths enter 00 with slope B/A. Thus all the paths both rectilinear and non-rectilinear enter 00 as t goes to infinity with slope B/A.

In the case of rectilinear paths also we have seen that the slope is B/A. Hence the critical point B/A is unknown and it is asymptotically stable as shown in figure 3.

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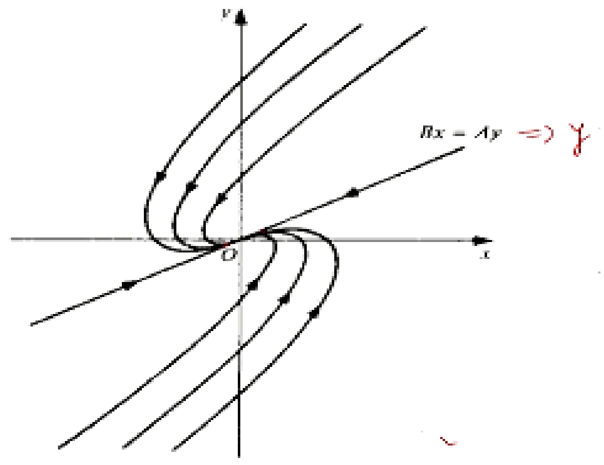

Fig .3

As you can see these are non-rectilinear paths okay they are entering 00 with the slope of the line Bx=Ay slope of the line Bx=Ay is B/A okay this gives you y= B/A*x and you can see y= B/A becomes the tangent to the non-rectilinear paths while passing through the origin. So, these non-rectilinear paths approach and enter 00 with slope B/A and the slope of the half line path has B/A. So, this point 00 is a node and it is asymptotically stable.

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Example: Let us consider

$$\frac{dx}{dt} = -x, \quad \frac{dy}{dt} = -y.$$

Then Eigen values are $\lambda = -1, -1$.

The roots are real and equal.

⇒ the critical point (0,0) is a node and it is asymptotically stable.

Now let us take an example on this. Suppose we consider dx/dt=-x dy/dt=-y then the matrix here is A=abcd so this is=-1 0 0-1 okay the equivalent of A-lambda=0 will give you-1-lambda 0 0-1-lambda so lambda=-1-1 okay so the eigen values are real and equal they have negative sign, the critical point 00 we have discussed right now the critical point 00 is a node and it is asymptotically stable.

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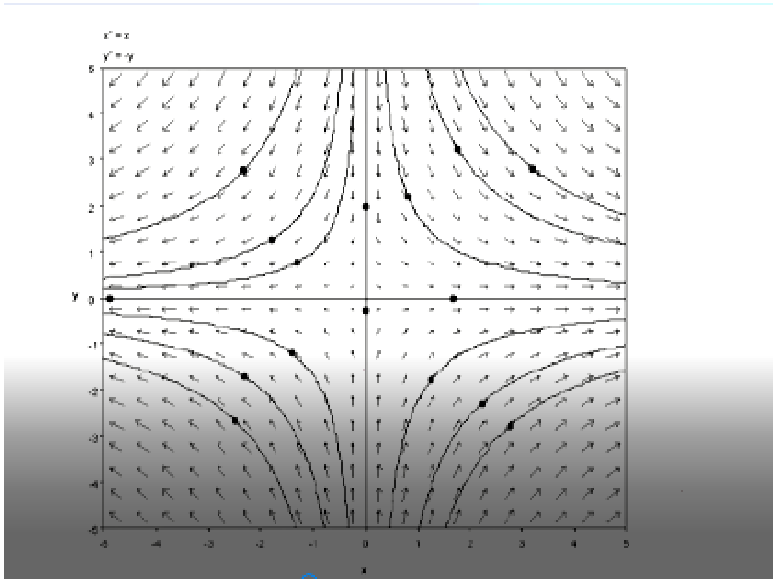

And this is the figure you can see you get the point 00 is a node and it is asymptotically stable this is the case of s star shaped node here we have a sat shaped node like in the case this one, since we have star shaped node we have this assumption a=d d is !=0 b=c=0. This is a special situation where we get a star shaped node root in this case are equal okay root in the other cases we discussed are also equal.

In this case we get a star shaped node and there we get a node which is not star shaped. SO, here you can see the example which we have taken here a=d a= -1 okay and b=c=0 till we get a star shaped node okay. So, this node is a star shaped node and this is asymptotically stable the figure is this one.

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Example: Let us consider

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = y.$$

Then Eigen values are $\lambda = 1, 1$.

The roots are real and equal.

$\Rightarrow$ the critical point (0,0) is a node and it is unstable.

Now let us consider another example dx/dt=x and dy/dt=y so here again we will get A=1 0 01 so this is unique matrix the eigen values are 11 okay so the eigen values are 1.1 they are real eigen values are equal okay. In this case you can see again a=d != 0a=d=1 != 0 and b and c are 0 okay so we get the critical point is a node but remember here lambda is positive lambda is not negative lambda is positive the roots are real and equal. So, it is a node but it is unstable.

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Nature of roots λ_1 and λ_2 of characteristic equation (*)	*Nature of critical point (0,0) of linear system (1)*	*Stability of critical point (0,0)*
Real, unequal, and of same sign	Node	Asymptotically stable if roots are negative; unstable if roots are positive
Real, unequal, and of opposite sign	Saddle point	Unstable
Real and equal	Node	Asymptotically stable if roots are negative; unstable if roots are positive

Now we have complied the summations that we have so far regarding the Eigen values lambda 1 lambda 2 the nature of the critical point depending of the Eigen values and the stability of the critical point which can be unsolved from the nature of the root lambda 1 and lambda 2. So, let us summaries what we have done so far so if the eigen values lambda 1 and lambda 2 of the characteristic equation of the matrix A are real unequal and of the same sign.

Then the nature of the critical point of linear system is a node and it is asymptotically stable if roots are negative, unstable if roots are positive> now here if the eigen values of lambda 1 and lambda 2 are real and unequal and opposite sign then we have a saddle point and stability of the critical point is unstable. Here we have lambda 1 and lambda 2 are real and equal you will have a node in the case where lambda is positive.

Okay we will have a star shaped node while in the case where lambda is positive we will have unstable node so it is asymptotically stable if the roots are stable star shaped node if asymptotically stable if roots are negative again if the roots are positive they will again have a node but it will be unstable. So, with this I will conclude my lecture thank you very much for your attention.