Ordinary and Partial Differential Equations and Applications

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Lecture - 22
Stability of Linear Systems- I

Hello friends, welcome to my lecture on stability of linear systems. Let us say 0,0 be an isolated critical point of the system dx/dt=Pxy and dy dt=Qxy. First, we will define a stable point.

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Let (0,0) be an isolated critical point of the system

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \qquad (1)$$

Stable point: Let 'C' be a path of (1). Let $x = f(t), y = g(t)$ be a solution of (1) defining parametrically. Let

$$D(t) = \sqrt{(f(t))^2 + (g(t))^2}$$

be distance between the critical point (0,0) and the point $R(f(t), g(t))$ on the path 'C'.

So, let C be a path of 1 this is term 1 let x=ft y=gt be a solution of the system1 defining parametrically the path c and let Dt=under root ft square +gt square be the distance between the critical point 0,0 and the point are that is having coordinates f t gt on the path c.

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The critical point (0,0) is called ***stable*** if for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that every path 'C', for which $D(t_0) < \delta$ for some value t_0, is defined for all $t \geq t_0$ and is such that $D(t) < \varepsilon$ for $t_0 \leq t < \infty$.

In other words, if (0,0) is stable then every path 'C', which is inside the circle K_1 of radius δ at $t = t_0$, will remain inside the circle K_2 of radius ε for all $t \geq t_0$ i.e. every path 'C' stays as close to (0,0) as we want it to (i.e. within distance ε) after it once gets close enough (i.e. within distance δ).

Then the critical point 0,0 is called a stable if for every epsilon > 0 do you know we can find a number delta >0 such that every path C Dt0< delta that is the distance of Ft0 gt0 from, origin is <delta for some value t=t0 is defined for all t>=t0 and such that Dt<epsilon for t0<=t<delta so what do we want for every epsilon <0 we should be able to find a number delta >0. Such that every path okay for which Dt<delta for some value t=t0 is defined for all t>=to.

And Dt should be <epsilon for all t0<=t<epsilon. 0,0 will be called a stable point if every path c which is inside the circle k1 of radius delta at t=t0 remain inside the circle K2 of radius epsilon for all t>=t0 that is every path C stays as close to 0,0 as we wanted to after it gets once close enough 20,0. so let us see within this delta.

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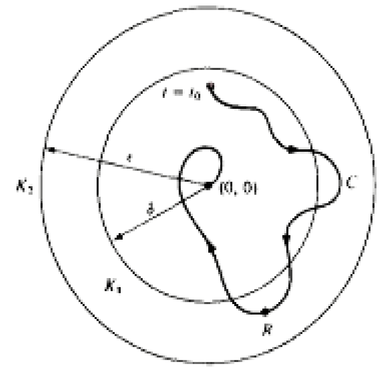

Fig.1

So let us look at this okay here you can see the K1, circle of radius delta with 0,0 center and K2 is circle with radius epsilon okay t0 is a point the distance of this point from the curve that is Dt0 is <delta and then you can see that for all t>=t0 okay Dt Is <epsilon okay so when this we have this condition that is Dt<epsilon for all t>=t0<epsilon. So, at the end you can see that the curve is defined for some value t=t0.

For all curve is defined C is defined for all t>=t0 and Dt is <epsilon. The path every path C which is inside the circle k1 of radius delta at t=t0 will remain inside the circle K2 of radius epsilon. So, it remains inside the circle K2 of radius epsilon it does not go out of the circle K2 okay. So, this 0,0 here is a stable point.

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Asymptotically stable point: The critical point (0,0) is called ***Asymptotically stable*** if

i. It is stable; and
ii. There exists a number $\delta_0 > 0$ such that if $D(t_0) < \delta_0$ for some value t_0, then $\lim_{t \to +\infty} f(t) = 0$, $\lim_{t \to +\infty} g(t) = 0$.

Thus, asymptotic stability is a stronger condition than mere stability because, in addition to stability, the condition (ii) requires that every path that gets sufficiently close to (0,0) ultimately approaches (0,0) as $t \to +\infty$. The path 'C' of fig.1 has this property.

Now let us see this asymptotically stable point is a stronger condition the point must be stable okay and satisfy one more condition. So, the critical point 0,0 will be called asymptotically stable first of all it is stable and then there should adjust a number delta 0>0 such that whenever Dt0 is <delta 0 for some value t0 then limit of ft and t goes to infinity at0 limit of gt t goes to infinity=0. So, that is the asymptotically stable stronger condition than near stability.

Because in addition to stability the condition II requires that every path that gets sufficiently close to 0,0 ultimately approaches 0,0 as t goes to infinity. The path C of fig. 1 ultimately approaches to 0,0 so this has this property this is stable path, and this is ultimately approaching to 0,0. So, this path has this second property also.

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Unstable point: A critical point is called unstable is it is not stable. For instance the center in fig.2, the spiral point in fig.4 and the node in fig.5 are all stable.

Of these three, the spiral point and the node are asymptotically stable. Further, if the directions of the paths in fig.4 and fig.5 are reversed, then the spiral point and the node of these figures will be unstable. The saddle point of fig.3 is unstable.

So, A critical point is called unstable if it is not stable.

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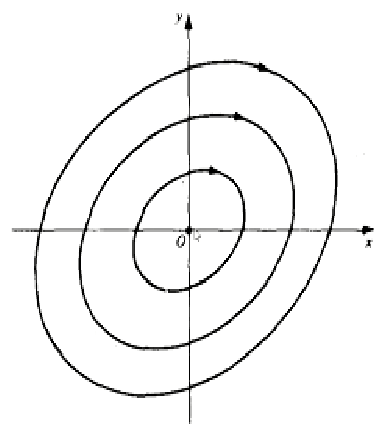

Fig.2

For example, the center in figure 2 here okay the center in figure 2.

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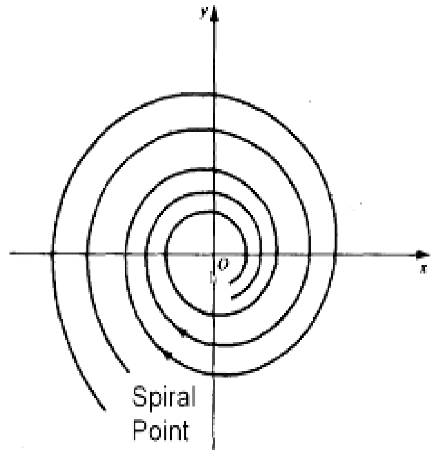

Fig.4

And this parallel point in figure 4 this one.

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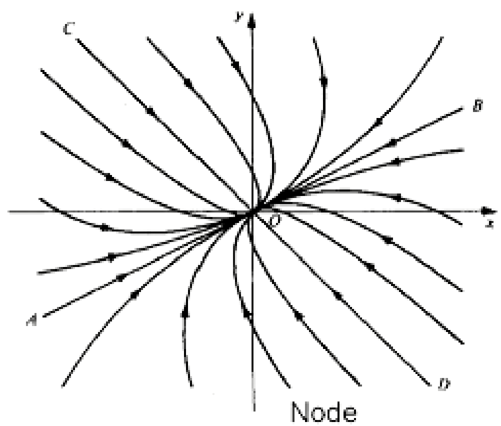

Fig.5

And the node in figure 5 they are all stable. Here if you go to the definition if the stable point then the condition of the stable point is satisfied. Again we will look at this 0,0 is called stable if every path C which is inside the circle K1of radius delta will remain inside the circle K2 radius delta. For all t>=t0 that every path C stays as close to 0,0 as we want it to after it get once close enough.

So, that is satisfied in the case of fig. 2 in the case of and in the case of fig. 4 here they are going as close to as we want them to be and then figure 5. So, they are all stable points. Now of these three figure 2 figure 4 figure 5 the spiral point okay and the node are asymptotically stable. So, let us see figure 4 the spiral point that is figure 4 and the node in figure 5 are asymptotically stable figure 4 these are asymptotically stable.

Ultimately it is approaching to 0,0 and here in figure 5 this is also ultimately approaching to 0,0. So, they are asymptotically stable if the directions of the paths in figure 4 and figure 5 okay in the direction of the paths in figure 4 and figure 5 are reversed, then the spiral point and the node of these figures will become unstable okay the saddle point of figure 3 the saddle point of figure 3 this is unstable.

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Critical points of a linear autonomous system: Let us consider the linear system

$$\frac{dx}{dt} = ax + by, \qquad \frac{dy}{dt} = cx + dy, \qquad (2)$$

where a, b, c and d are real constants.

Clearly, (0,0) is a critical point of (2). Let us assume that $ad - bc \neq 0$, then (0,0) is the only critical point of (2).

Solving the system (2), we get solution of the form

$$x = Ae^{\lambda t} \text{ and } y = Be^{\lambda t}. \qquad (3)$$

Okay let us consider critical points of a linear autonomous system let us consider the linear system dx/dt=ax+bydy/dt=cx+dy where a b c and d are real constants.so here Pxy=ax+by and Qxy=cx+dy. So, this critical points of the autonomous system pxy=0 and qxy=0 that means ax+by=0 and cx+dy=0. Now let us assume the condition that ad-bc is not=0. Because ax+by=0 cx+dy=0 the homogenous system of this form.

So, the determinant of the co efficient matrix is not=0 then xy will be 0,0 okay determinants of the co efficient matrix this is not=0 then we know that the only solution that is possible is the trivial solution that is x=0 and y=0 and therefore 0,0 will be the only solution of pxy=0 and Qxy=0 if we assume the condition ad-bc is not=0. So, 0,0 is the only critical point of this system

2.

Solving the system 2 when we solve the system homogeneous system of linear equations linear system then we have x=A e lambda t and y=B lambda the solution of the form we get. And so if X=A e lambda t and y = B raised to the power lambda t, solutions of 2.

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Critical points of a linear autonomous system: Let us consider the linear system

$$\frac{dx}{dt} = ax + by,$$

$$\frac{dy}{dt} = cx + dy, \qquad (2)$$

$P(x,y) = ax+by=0$

$Q(x,y) = cx+dy=0$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left|\begin{matrix} a & b \\ c & d \end{matrix}\right| \neq 0 \quad x=0=y$

where a, b, c and d are real constants.

Clearly, (0,0) is a critical point of (2). Let us assume that $ad - bc \neq 0$, then (0,0) is the only critical point of (2).

Solving the system (2), we get solution of the form

$$x = Ae^{\lambda t} \text{ and } y = Be^{\lambda t}. \qquad (3)$$

Then we have A lambda e to the power lambda t= A e to the power lambda t dx/dt x=A e raised to the power lambda t so dx/dt=A lambda e raised to the power lambda t and y=B e raised to the power lambda T dy/dt=B Lambda e to the power lambda t. So, putting the values as dx/dt dy/dt we have this equation. A lambda e raised to the power lambda t=a*A e raised to the power lambda t+b*B e raised to the power lambda t.

And b lambda e to the power lambda t =C A e raised to the power lambda t +dB e raised to the power lambda t E to the power lambda t is never 0 so you can cancel e to the power lambda t from this equation and this equation and we can write A times lambda-a=Bb=0 and B lambda-d-cA=0. From this we can write this as lambda-a-b and lambda -d a oh lambda -c A B okay. If the determinant of this co efficient matrix lambda-a-b lambda-d-c is not=0.

Then what will happen we will have then AB=0,0, when A=0 and B=0 we will get the trivial solution of the autonomous system. So, in order to have a non-trivial solution let us assume that determinant of lambda-a-b lambda-d-c=0 okay. So, this will give us a second order polynomial equation in lambda of degree 2 So, lambda square -a+d lambda +ad-bc=0.Now this equation is called characteristic equation.

Since we have assumed that ad-bc is not=0, lambda=0 can never be a root of this equation.

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Let λ_1 *and* λ_2 be the roots of the characteristic equation (4). We shall show that the nature of critical point (0,0) of the system (2) depends upon the nature of the roots λ_1 *and* λ_2. In this regard, there arise the following five cases:

i. λ_1 *and* λ_2 are real, unequal and of the same sign.
ii. λ_1 *and* λ_2 are real, unequal and of opposite sign.
iii. λ_1 *and* λ_2 are real and equal.
iv. λ_1 *and* λ_2 are conjugate complex but not purely imaginary.
v. λ_1 *and* λ_2 are purely imaginary.

Okay now let us say lambda 1 and lambda 2 be the 2 roots of this characteristics equation, we have a characteristic equation polynomial equation lambda in degree 2, so it will have 2 roots, so let us say lambda 1 and lambda 2 be the 2 roots of the characteristic equation, then we shall see that the nature of the critical point whether it is a center or it is a node okay the nature are stable or critically stable.

Or t is unstable it will depend upon the nature of roots lambda 1 and lambda 2. So in this regard now here we have 5 as such we have 3 cases when we have roots of the second order equation they may be real and distinct that is one case then there may be complex conjugate of each other and the third case is that they are real and they are equal but here we have 5 cases, So, lambda 1 and lambda 2 are real unequal and of the same sign.

So, we have 2 cases sub cases of the situation where lambda 1 and lambda 2 are real and unequal. One is that of the real and unequal and of the same sign and the other one is that they are real unequal and are of opposite sign and then in the conjugate complex again we have 2 cases that is lambda 1 and lambda 2 are conjugate complex of each of them and they are not purely imaginary that is their real part is not =0 and then the one more case that the real part is 0.

So, lambda 1 and lambda 2 are purely imaginary and the fifth case is lambda 1 and lambda 2 are really equal.

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Theorem: If the roots λ_1 *and* λ_2 of the characteristic equation (4) are real, unequal and of the same sign then the critical point (0,0) of the linear system (2) is a node.

Proof: Let λ_1 *and* λ_2 be both negative and $\lambda_1 < \lambda_2$. The general solution of the system (2) may be written as

$$x = c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 t},$$
$$y = c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t}, \qquad (5)$$

where A_1, B_1, A_2 *and* B_2 are definite constants and $A_1 B_2 \neq A_2 B_1$, and where c_1 *and* c_2 are arbitrary constants.

Okay so let us first we discuss the case when the characteristic equation have 2 roots lambda 1 and lambda 2 they are real unequal and are of the same sign then the critical point 0,0 of the linear system 2 is a node. Let us see we have lambda 1 and lambda 2 be the 2 roots of the characteristic equation and they may be both negative, So they are of the same size so let us take them both to be negative.

And suppose lambda 1 and lambda 2 satisfy the equation lambda 1 is <lambda 2. The general solution of the system 2 may be written as see how we write this system solution, see we have seen that this equation has two roots lambda 1 and lambda 2 and we are assuming that they are real unequal and of the same sign then what happens here that the values of lambda 1 and lambda 2 equations give us for lambda 1.

We get the value of a and b as A1 and B1 okay for the value lambda 2 we get the value A2 and

B2 okay then what will happen corresponding to lambda 1 x=A1 e raised to the power lambda 1 t, y will be =B1 e raised to the power lambda 1 t, so this is 1 solution and corresponding to the other root lambda 2 will have x=A2 e raised to the power lambda 2 t and y=B2 e raised to the power lambda 2 t okay so corresponding to 1 value lambda 1.

We have solution of this equation okay we take the values of A and B corresponding to lambda 1 as A1 B1 corresponding to lambda 2 we take the other values of A and B they are A2 and B2 So, then xy= A1 e to the power lambda 1 t and y=B e to the power lambda 1 t this is 1 set of solution and this is another set of solution, they are linearly independent of each other and in the case of homogenous system linear homogenous system.

We write the general solution as the linear combination of x=c1 A1+c2 b1 yeah x=c1 A1 e raised to the power lambda 1 t +c2 A2 e raised to the power lambda 2 t because corresponding lambda 1 the solution is A1 e raised to the power lambda 1 t and corresponding to lambda 2 the solution is A2 raised to the power lambda 2t. so, the general solution will be given by x=c A1 e raised to the power lambda 1 t +c2 A2.

E raised to the power lambda 2 t and y will be c1B1 e raised to the power lambda1 t+c2 B2 e raised to the power lambda 2 t. Now A1, B1,A2 and B2 are definite constants they are given by this equation, so they are definite constant and moreover that these are linearly independent solutions okay x= A e raised to the power lambda1 t y= B e raised to the power lambda t this set of solution is linearly independent to the other one xy x= A2 raised to the power lambda 2 t.

And y=B 2 raised to the power lambda 2 t. So, what we get is A1 B2 is not=A2 B1 and that condition comes from there c1 and c2 are arbitrary constants.

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Choosing $c_2 = 0$, we get

$$x = c_1 A_1 e^{\lambda_1 t}, \quad \Rightarrow B_1 x = A_1 y$$
$$y = c_1 B_1 e^{\lambda_1 t}. \qquad (6)$$

For any $c_1 > 0$, the solutions (6) represent a path consisting of "half" of the line $B_1 x = A_1 y$ of the slope $\frac{B_1}{A_1}$. For any $c_1 < 0$, they represent a path consisting of "other half" of this line. Since $\lambda_1 < 0$, both of these half lines paths approach (0,0) as $t \to +\infty$. Also, since

$$\frac{y}{x} = \frac{B_1}{A_1},$$

these two paths enter (0,0) with slope $\frac{B_1}{A_1}$.

Figure

Let us choose c2=0, okay let us choose c2=0 here x=c1 A1 e raised to the power lambda1 t y=c1 B1 e raised to the power lambda 1 t, so we get this when we choose c2=0 general solutions gives us this. Now here you can say C1 when c1<0 sorry when c1>0 these solutions 6 okay represent a path consisting of half of the line B1x=A1y. If you eliminate C1 here what do you get these 2 will you B1x=A1y okay when we eliminate this.

Now A1 and B1 are definite constants, so let us say, suppose A1 and B1 are both having same sign either positive or negative then what will happen then we will have assuming c2=0 we will get x=c1 A1e raised to the power lambda 1 t y=c1 B1 e raised to the power lambda t when you eliminate C 1 here you get B1x=A1 y `then we will take a case where C1 is positive then what will happen e raised to the power lambda t is always positive.

So A1 c1 e raised to the power lambda 1 t will be positive and c1 here what will happen is that let us take A1 and B1 both to be positive then.

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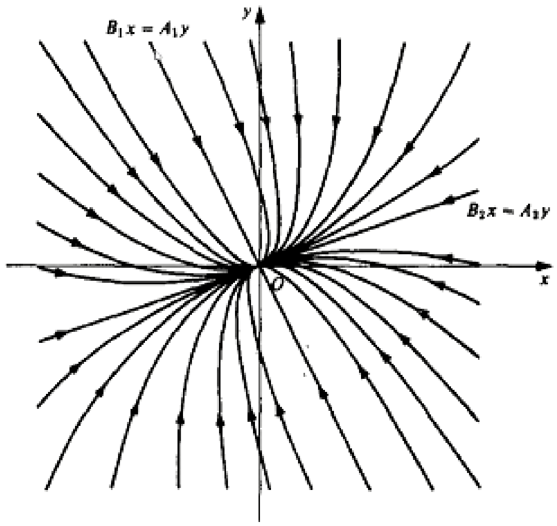

Fig .6

Then what will happen you see here if I take A1 B1 let us take this line B1 x=A1 here what we will do is that let us take say suppose A1 B1 have different signs then what will happen is that if A1 B1 have different signs. Suppose A1 is negative and B1 is positive then we will have in this quadrant okay this like this half line we will have okay. because we are taking C1 to be >0 okay A1 x=c1 A1 e raised to the power lambda 1 t c1 we are taking positive.

If I take A1 to be negative and B1 to be positive okay then what will happen is that we will have this half line, the other half line we will get here which comes from when you take c1 to be negative, For any c1>0 these represented path consisting of half of the line 1x=A1y of the slope of the line will be B1 upon A1, B1 upon slope of the line will be usually here B1 upon A1*x so these C1>0 this solution represent a path consisting of half of the line.

B1x=A1 y of the slope B1/A1 and then when you take c1 to be <0 we have chosen A1 to be negative b1 to be positive. If I take c1 to be <0 then what will happen x and y x will be positive and y will be negative. So, we will get the other half line okay this half line we will get and so when c1>0 these solutions represent the other half of this line, since lambda 1 is negative both of these half lines paths approaches 0,0 as t goes to infinity.

You see lambda 1 is positive <0. so, when t goes to + infinity x goes to 0 and y goes to 0 okay, so since lambda 1 <0, both of these half lines approach 0,0 as t goes to infinity also we notice that y over x =B1/xB1/A1, so these two paths enter 0,0 with slope B1/A1 you can see here these two half line paths enter 0,0 with slope B1/A1.

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Next, choosing $c_1 = 0$, we get

$$x = c_2 A_2 e^{\lambda_2 t},$$
$$y = c_2 B_2 e^{\lambda_2 t}. \qquad (7)$$

Similarly, for any $c_2 > 0$, the solutions (7) represent a path consisting of "half" of line $B_2 x = A_2 y$; while for $c_2 < 0$, the path so represented consists of the "other half" of this line. These two half line paths also approach (0,0) as $t \to +\infty$ and enter it with slope $\frac{B_2}{A_2}$.

Thus the solutions (6) and (7) provide us with four half line paths, which all approach and enter (0,0) as $t \to +\infty$.

Now let us choose c1=0 when we take c1=0 in the general solution we get x=c1 A2e power lambda 2t y=c2 B2 e raised to the power lambda 2t, so similarly for any c2>0 again when we choose c2>0 okay these solutions 7 represent a path consisting of half of line B2x=A2y so here again when you eliminate c2 hat we will get is B2x=A2y so when c2>0 okay when c2>0 the path so represented consists of see c2>0 gives us the half of the path.

You can see here this half of the path B2x=A2y this half path and another half path. So, these two half line paths also approach 0,0 now we will have to see again lambda 2 is negative because we have chosen lambda1, lambda 2 to be of the same sign and that is negative. So, when t goes to + infinity x and y will be 0 and therefore these two half lines paths also approach 0,0 as t goes to infinity and these two half line paths enter the slope B2/A2.

You can see in this these two-half line pass okay enter 0,0 with the slope given by y=B2/A2

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Now, let $\lambda_1 > \lambda_2 > 0$, then the general solution of (2) is still of the form

$$x = c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 t},$$
$$y = c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t},$$

Choosing $c_2 = 0$, we obtain the solutions

$$x = c_1 A_1 e^{\lambda_1 t},$$
$$y = c_1 B_1 e^{\lambda_1 t},$$

and

$$x = c_2 A_2 e^{\lambda_2 t},$$
$$y = c_2 B_2 e^{\lambda_2 t},$$

of the same forms.

The question is y=b2/a2*x so the 0,0 in the slope B2/A2, now thus the solutions 6 and 7provide us with 4 half-line paths, you see two half line paths we get from here and 2 half-line paths we get from here, so we get 2 half-line paths, four half line paths which all approach and enter 0,0 as t goes to infinity we can see here this is one half line path this is another half line path and this is third half line path this is fourth half line path.

They all approach 0,0 and this half line path approaches 0,0 by B2/A2 while these two half line paths approaches 0,0 with flow B1/A1, thus we have 4 half-line paths which all enter 0,0 as t goes to infinity.

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If $c_1 \neq 0, c_2 \neq 0$, then the general solution represents non-rectilinear paths.
Again, since $\lambda_1 < \lambda_2 < 0$, all of these paths approach (0,0) as $t \to +\infty$.
Further,

$$\frac{y}{x} = \frac{c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t}}{c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 t}}$$
$$= \frac{\frac{c_1 B_1}{c_2} e^{(\lambda_1 - \lambda_2)t} + B_2}{\frac{c_1 A_1}{c_2} e^{(\lambda_1 - \lambda_2)t} + A_2}$$
$$\Rightarrow \lim_{t \to \infty} \frac{y}{x} = \frac{B_2}{A_2}$$

and so all of these paths enter (0,0) with limiting slop $\frac{B_2}{A_2}$.

Now let us assume that c1 is not=0 and c2is not=0 then the general solution represents non-rectilinear paths. Let us we have y=c1 B1 e to the power lambda 1t+c2 B2 e to the power lambda 2t and x=c1 A1 e to the power lambda 1 t+c2 A2 e to the power lambda 2 t, again lambda 1, lambda<0 we have assumed that so now what will happen is that we should write this portion as c1 B1/c2 because c2 is not=0 we can divide /c2 and e to the power lambda 2 t.

Also we can divide in the numerical denominator so, c1b1/c2 e to the power lambda 1-lambda2*t+B2/c1+A1/c2 e to the power lambda1-lambda 2* t +A2. Now lambda 1 is <lambda 2 so lambda 1-lambda 2 is negative as t goes to +infinity, e to the power lambda 1-lambda 2*t goes to 0 and so this part goes to 0 and this part goes to 0 and therefore limit of y/x as t goes to infinity=B2/A2.

So, all these paths non-rectilinear paths they enter 0,0 with limiting slope okay limiting slope B2/A2, so we can see that in the figure here, see they are non-rectilinear paths, they are all non-rectilinear paths this one this one this one and this one, they are all non-rectilinear paths, these ones they all enter 0,0 okay with limiting slope B2/A2 you can see these paths enters okay next slope B2/A2 at the origin these two lines B2x=A2y.

It is tangent to all these paths this line is having slope B2/A2 so all these non-rectilinear paths enter 0,0 with limiting slope B2/A2. And thus we have the same situation now let us see let us assume that lambda 1 >lambda 2 >0 so we now assume that lambda 1 and lambda 2 are both positive then the general solution is still of the same form x=c1A1 e to the power 1t, c2 A2 e to the power lambda 2 t y=c1 B1 e to the power lambda 1 t +c2 B2 e to the power lambda 2 t.

Now we choose c2=0 so x=c1 A1 e to the power lambda 1 t y=c1 B1 e to the power lambda 1 t and x=c2 A2 e to the power lambda 2 t if we choose c1 0, if we chose c1=0 then x=c2 A2 e to the power lambda 2t y=c2 B2 e to the power lambda2t, so we get the solutions of the same form as in the case of lambda 1 and lambda 2 both being negative.

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Thus we have the same situation as before except that all the paths approach and enter (0,0) as $t \to -\infty$. The qualitative diagram remains unchanged except that all the arrows now point in the opposite direction. Hence the critical point (0,0) is still a node but it is unstable in this case.

Now what is the difference here we have the same situation here before except that all the paths approach and enter 0,0 as t goes to - infinity because lambda 1 and lambda 2 are positive so they enter 0,0 when t goes to -infinity. The qualitative diagram remains unchanged except that all the arrows now point in the opposite direction. Okay so hence the critical point 0,0 is still a node but it is unstable in this case.

Now see we had this situation earlier okay, now these directions will be opposite to this direction and therefore we will still have a node but it will be unstable node. Now we have come to the end of this lecture. Thank you very much for your attention.