Ordinary and Partial Differential Equations and Applications
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Lecture - 02
Introduction to Differential Equations - II

Hello friends. Welcome to the lecture. If you recall in previous lecture we discussed some basic concepts of ordinary differential equation and in this lecture we will continue our discussion of ordinary differential equation. So in previous lecture we have discussed the classification based on the dependent function. In fact, we have discussed what is linear differential equation and what is nonlinear differential equation.

Now in this lecture we discuss the classification based on the conditions provided along with the differential equation. So consider the following differential equation y dash t+alpha y t=0.

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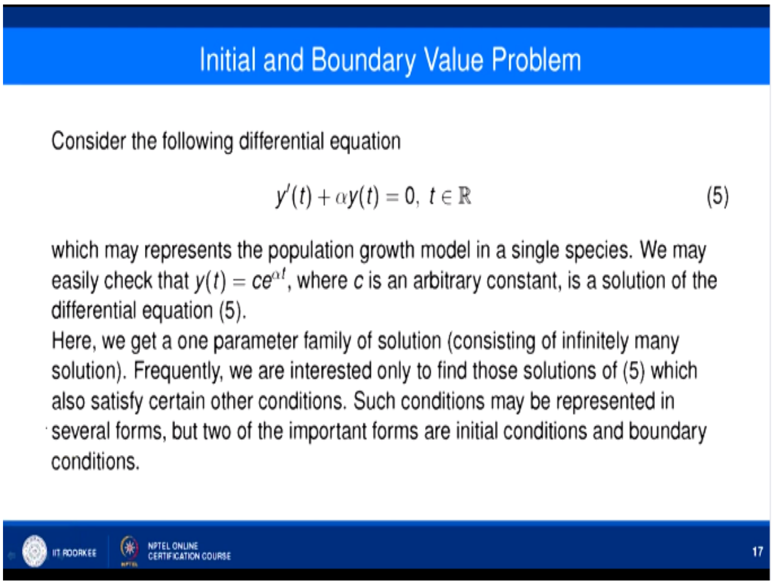

So here this is a linear differential equation given to us, here t is an independent variable and t is a belonging to R which may represent the population growth model in a single species. Here we have already discussed that this represent a population growth model in a single species and we may easily check that y t=ce to power alpha where c is an arbitrary constant is a solution of differential equation 5.

Here we get a one parameter family of solution, if you look at here the solution is one parameter. So you change your c, here c is a constant and if you look at this alpha, alpha is a parameter present here. So we can say that this c if you keep on changing the c we have a different, different solutions. So here we can say we can get a one parameter family of solution and this one parameter family of solution consist of infinitely many solutions.

And frequently we are interested only to find out one solution because whenever we have a real world problem and we are formulating it as an ordinary differential equation we only interested in finding a unique solution rather than having infinitely many solutions. So we are interested only to find out those solutions of 5 which also satisfy certain other condition such conditions may be represented in terms of in several forms but 2 of the important forms are initial conditions and boundary conditions.

So here we can say that these are initial condition and boundary conditions. There are several other forms are also available but here we consider only these 2. There are initial conditions, there are nonlinear non local conditions, there are impulsive conditions, there are several other conditions available but here we will consider restrict ourself to initial condition or boundary conditions.

So what are these initial condition or boundary condition let us discuss this.

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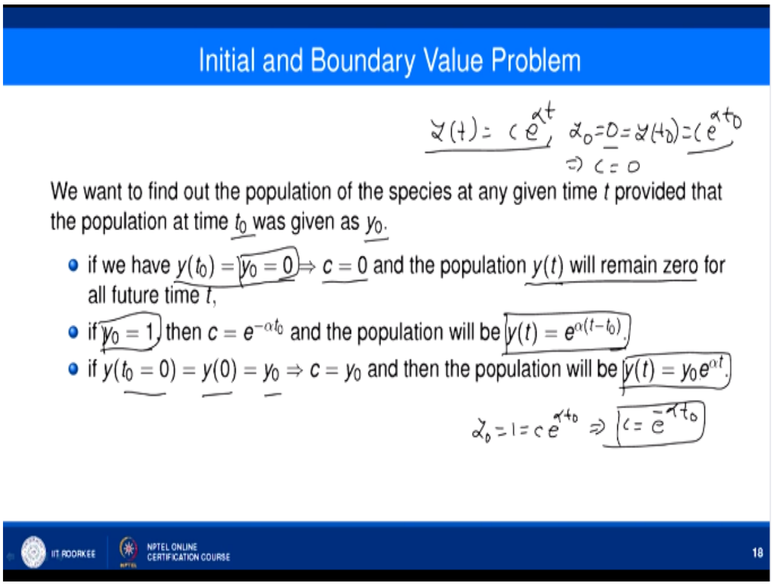

So we want to find out the population of the species at any given time t provided that the population at time t0 was given as y0. So along with the differential equation it is also given that at time t=t0 the population is y0. So with the help of this we want to find out the solution.

So if we look at this possibility that initial population is given as 0 so it means that at time t0 y of t0=y0 and it is given that y0=0.

Then, this implies that we have only c=0 and this gives that we have population y t at any time it is coming out to be 0. How it is, if you look at we have what we have solution as y t=c to power alpha t. Now it is already given that at time t=t0 the initial condition is given as 0, so here we can say that y0=0=y t0=ce to power alpha t0. Now this 0=ce to power alpha t0. Now e to power alpha t0 is never 0 so c has to be 0.

So c has to be 0, if c=0 then solution is nothing but a 0 solution. So in this case when initial condition when the condition given at t0 is coming out to be 0 then the population will remain 0 for all future time t. Now if we consider the initial population as 1, we have y0=1 then we can check that c is coming out to be e to power –alpha t0 and the population will be given as e to power alpha t-t0 that also you can check, here in place of 0 it is 1.

So if it is 1=ce to power alpha t0 then c is coming out to be – so here if we consider y0=1=ce to power alpha t0. So we can check that here c is coming out to be e to power –alpha t0. So using this value we can write our solution as y t=e to power alpha t-t0. So here if you look at if your initial condition is changed from 0 to 1 then your solution is also changed. So here solution is 0, here solution is y t=e to power alpha t-t0.

Now if initial point t0 is given as 0 and y0 is given as y0 some value then the c is coming out to be y0 and we can say that the population is given as y t as y0 e to power alpha t. So if initial population when initial time is given as 0 then solution is given as y t=y0 e to power alpha t. So here every time solution is depending on the initial condition. So if the condition given at the initial point t0 is 0 then we have 0 solution.

If it is some non-zero quantity that is 1, then we have a solution like this and if initial point is 0 then solution is coming out to be this.

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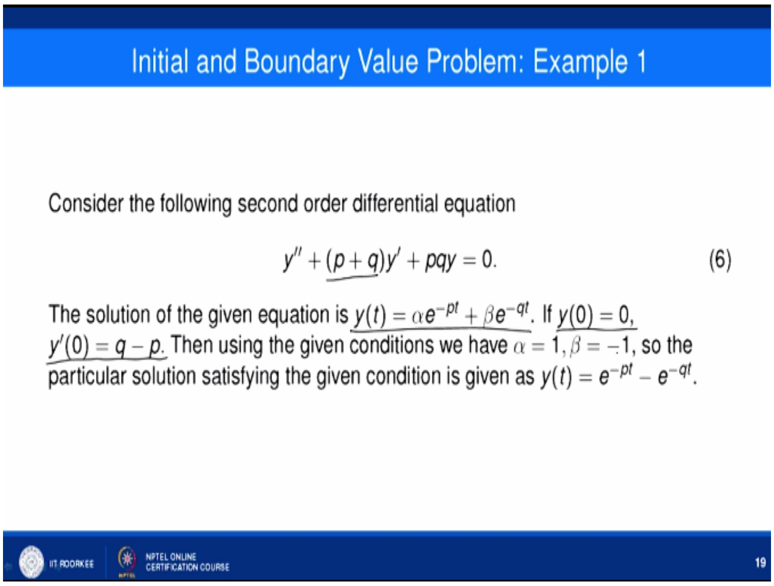

Now look at the next example, consider the following second order differential equation, y double dash+p+q y dash+pqy=0. So this is a second order differential equation with constant coefficient, here p and q are constant, so we can find out the solution of the given equation as y t=alpha e to power –pt+beta e to power –qt. Now if the condition along with this differential equation is given as this y0=0 and y dash 0=q-p since it is a second order differential equation we need 2 conditions.

One condition given at 0 is 0 and one condition is given at 0 in terms of derivative. So y dash 0=q-p, we can simplify this and we can say that we have alpha=1 and beta=-1.

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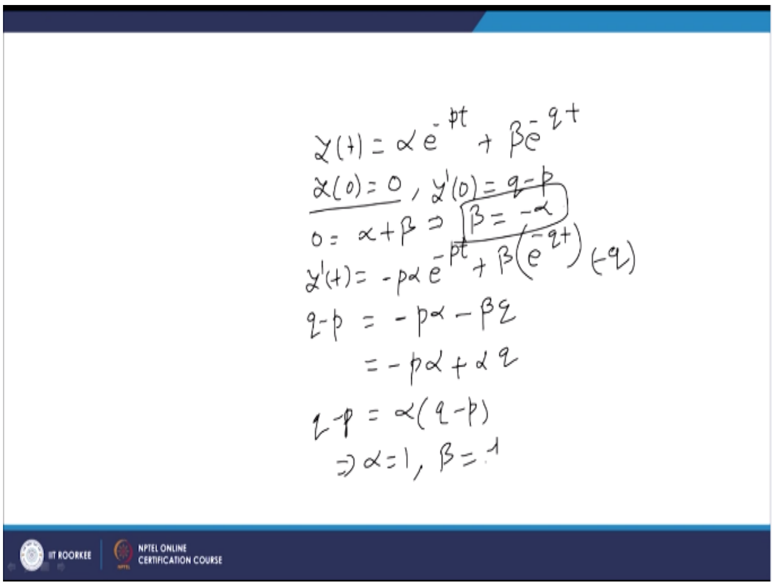

So how we can look at this, so here we have y t=alpha e to power –pt+beta e to power –qt and condition given as y of 0=here we have taken y0 is 0 and y dash 0 is q-p, it is 0 and y dash 0=q-p and we want to find out the alpha and beta. So here use this condition y0=0 then it is 0=alpha and e to power –p*0 it is coming out to be 1 only, so 0=alpha+beta and if you look at the y dash t, y dash t=--p alpha e to power –pt+beta e to power –qt.

And if you differentiate this e to power –qt it will give you e to power –qt*-q and now put y dash t as 0 so this will give you q-p=-p alpha and this will come out to be 1 and –beta q. So –p alpha-beta q is coming out to be this. So we can write it this as alpha+beta so here I can say that this implies that you can write beta as –alpha. So if you can write beta as –alpha-p alpha-beta is coming out to be alpha so alpha q.

So here we can take out alpha and we have what q-p that is q-p here. So this implies that alpha=1 and if you look at this equation beta= -alpha then this implies that beta= -1 and that is what is written here that alpha=1 and beta= -1. So the particle solution satisfying the given condition is given as y t=e to power –pt-e to power-qt. So here condition is given at only one point that is 0 here.

And if you look at the previous example here also condition is given at only one point that is t0 here.

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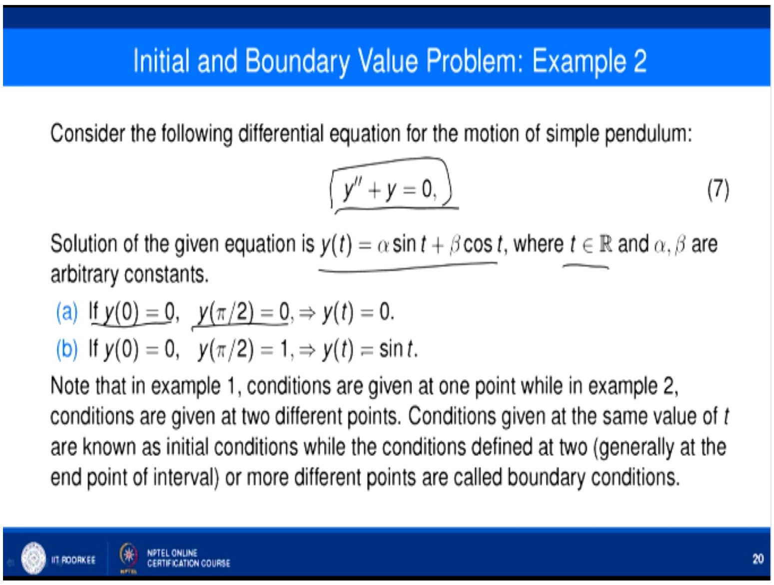

Now look at the second example. Second example is this, consider the following differential equation for the motion of simple pendulum and we already know that this is a very famous example of simple pendulum and your equation is coming out to be y double dash+y=0 and we know that the solution is given as y t=alpha sin t+beta cos t. Now here alpha, beta are arbitrary constant and t is from R.

Now since it is a second order differential equation, we need 2 conditions so if we consider the following 2 cases first condition if y0 is 0 and y pi/2=0 we can check that y t is coming out to be 0 and if y0=0 and y pi/2=1 then solution is coming out to be y t=sin t.

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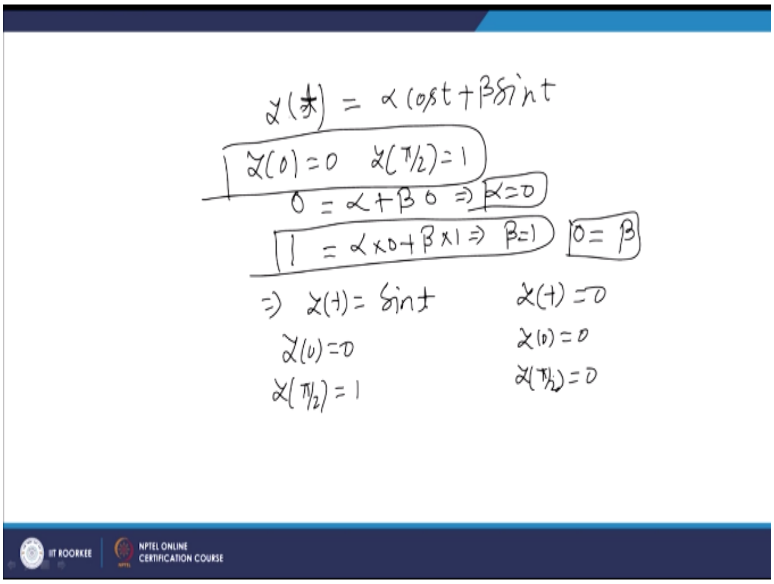

This you can easily check when you put y0=0 what is given here, here it is y of t=alpha cos of t+beta sin of t that is what we have written here, alpha sin t+beta cost t. Anyway here we have written in a different manner but we need to find out the solution of this equation provided that y0=0 and y pi/2=1 here. You want to find out the solution of this, so when you put y0=0 it will give you 0=alpha cos of 0 is 1+beta sin of 0 is 0, so you will get alpha=0.

Now consider the second equation, it is y pi/2=1=cos of t is basically cos of pi/2 is 0 so it is alpha times 0+beta sin of pi/2=1, so this implies that beta is coming out to be 1 so we can say that the solution is given is what solution is given as y t=now alpha is 0 and beta is 1 so solution is coming out to be sin t. In case when y0=0 and y pi/2=1 which is given here that if y0=0 and y pi/2=1 then y t=sin t.

If you look at the first case if y pi/2 =1 in place of 1 if we write 0 then the second equation this equation is replaced by the following equation, it is 0=beta. So here this will give you beta=0, alpha is already 0 so in this particular case your solution is coming out to be 0. So this is the solution when y0=0 and y of pi/2 is also coming out to be 0 and this is the solution when y of 0=0 and y of pi/2 is coming out to be 1.

So if you change your condition, your solution will also change and if you look at in example 1 the previous example conditions are given at one point while in this example conditions are given at 2 different points. For example, it is 0 here and pi/2 here. So in previous example your condition is given only at 0 and earlier one it is given only at t0 right. So it means that in first example conditions are given at one point while in example 2 conditions are given at 2 different points.

So conditions given at the same value of t are known as initial conditions right. So it means that if all the conditions are defined only at one point we call that problem conditions are initial condition and the corresponding problem is known as initial value problem while the conditions define at 2 or more than 2 points we call the conditions are boundary conditions. So here these 2 points are generally the end points of the interval.

So in the case when conditions define at 2 which are generally at the end point of the interval or more different points are called boundary conditions. So if along with ordinary differential equation if the conditions are prescribed at only one point, it is an initial value problem, if along with ordinary differential equation if the conditions are defined more than one point or day 2 point or more than 2 points then it is known as boundary value problem.

And conditions are known as initial conditions and boundary conditions respectively okay. Now let us consider the following remark.

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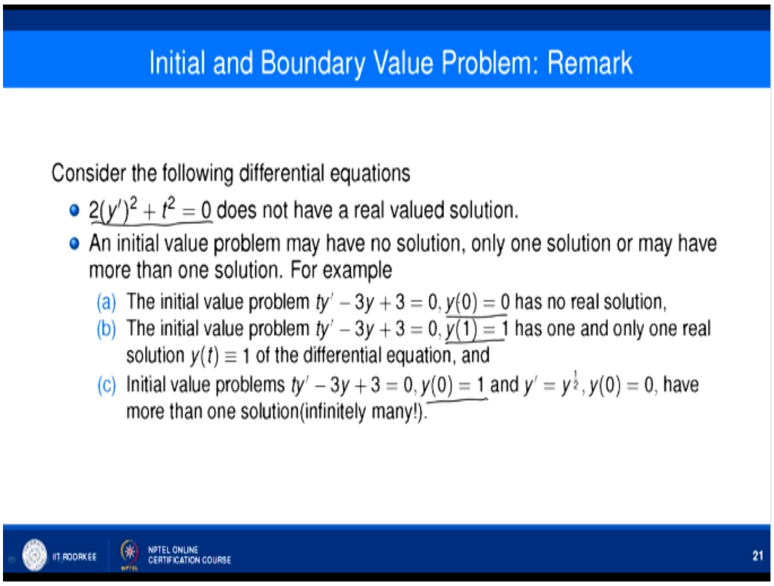

So consider the following differential equation 2 y dash whole square+t square=0, so here if you look at whatever condition you define this will not have any solution. Here if you look at this is a nonlinear problem because here dependent variable y dash is coming in terms of square so this is a nonlinear problem so it is a nonlinear ordinary differential equation but here whatever initial or boundary condition you prescribe this will not have any real value solution.

Now if you look at the second remark it says that an initial value problem may have no solution, only one solution or may have more than one solution. For example, if you consider the initial value problem ty dash-3y+3=0. If we look at in a, b, c we have the same differential equation ty dash-3y+3=0 and in a, b, c we are changing only the initial condition. For example, in a part it is y0=0.

In b, it is y1=1 and in c it is y0=1, so here in all the 3 cases, we have condition given only at one-point right and the only thing is that values are changed. So if you look at the first thing the initial value problem ty dash-3y+3=0 along with y0=0 has no real solution. The initial value problem ty dash-3y+3=0 given with y1=1 has one and only one real solution y t ideally=one of the differential equation.

And if you look at this initial value problem ty dash-3y+3=0 along with this condition y0=1 then it will have infinitely many solution or more than one solution and if we consider one more problem y dash=under root y where y0=0 this will also have a more than one solution and here I am not working it out right now and this you may consider as an exercise to find out that indeed in these cases we have the corresponding results.

So here we have seen that an initial value problem here if you look at differential equation is same, the only difference is the conditions given at one point. So only conditions are changing and we have seen that we have no real solution, one solution or more than one solutions. So the conditions are playing very, very important role. The similar thing is also happened in case of boundary conditions.

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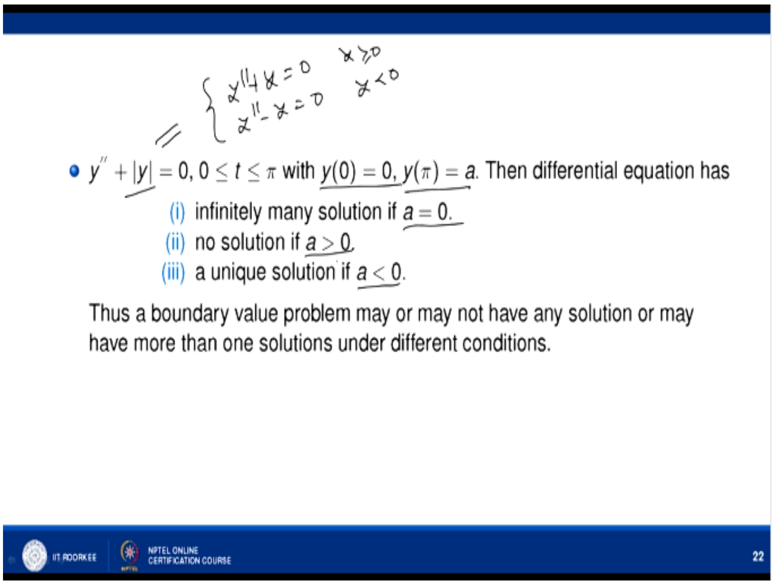

So if you look at this y double dash+mod of y=0. If you look at this is a nonlinear ordinary differential equation. Here t is lying in this interval 0 to pi and conditions are given as y0=0 and y pi=some a where a is a constant. So here if you look at conditions are given at 2 points and pi, so it is the case of boundary value problem, so differential equation along with boundary condition. So it is known as boundary value problem or we write BVP.

Then differential equation has infinitely many solutions if a=0, no solution if a is>0 and an unique solution if a is<0. So again this I give as an say exercise to verify whether this will have the result stated here that it has infinitely many solution if a=0 and correspondingly all other stated result. So here we can say boundary value problem may or may not have any solution or may have more than one solution under different boundary conditions.

I may give you one hint for this particular problem. If you look at the solution is what, here we need to consider the case depending on this mod of y right. So here we have to consider the case when y is positive and y is negative. So this is not only one differential equation but it is a set of 2 differential equations. So depending on this you try to consider these things. This is just a hint. Hint is that this is what this is given as y dash+y=0 when y is positive.

And y double dash-y=0 when y is <0, so taking this and involving your boundary condition you may verify this okay. So here our purpose is to show that a boundary value problem will also may have infinitely many solutions, no solution or unique solution depending on the boundary conditions. Similar thing we have already seen for initial value problem.

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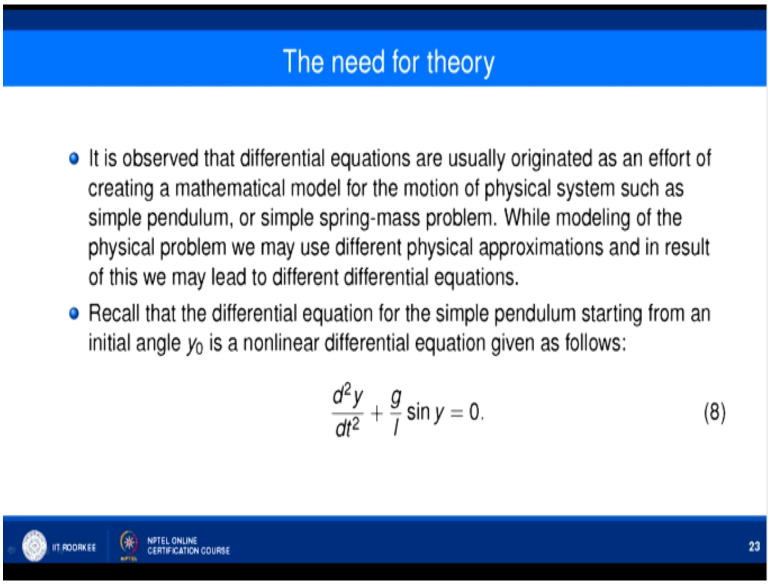

So here we say that why we want to discuss all these things, why we discuss these examples that we need some kind of theory to deal with these kinds of problems. So we can say that it is observed that differential equations are usually originated as an effort of creating a mathematical model for the motion of physical system. It means that given a physical system if you want to study we have various ways but the most common way is to propose a theory right.

And once we have a theory we try to write down the equations governing that theory in terms of mathematical terms and most of the times it is coming out to be differential equation. So we say that for example cases are simple pendulum. If we consider the case of simple pendulum, your differential equation is coming out to be y double dash+y=0 after simplification after assuming certain things.

Simple spring-mass problem is also give rise to a differential equation, while modeling of the physical problem we may use different physical approximation and in result of this we may lead to different differential equation. It means that if we assume certain set of assumptions we have a differential equation. For example, if you consider the simple pendulum, if we assume that sin of y can be approximated by y it is coming out to be y double dash+y=0.

It is a linear second order differential equation but if we do not assume this assumption that sin y is approximately=y then this is y double dash+sin y=0 some constant time sin y=0 which is a nonlinear ordinary differential equation and we can say we have already seen that linear and nonlinear differential may vary in a very different manner. So it depends on the approximation which gives rise to the different ordinary differential equation coming out from the same particular physical system.

So it is very, very important that a physical system may give rise to more than one ordinary differential equation if we change the set of assumptions. So recall that the differential equation for the simple pendulum starting from an initial angle y0 is a nonlinear differential equation given as follows this, this is what I am talking about that initially we write it like this. Then, if we assume that sin y is approximated by y by assuming that initial angle is very, very small.

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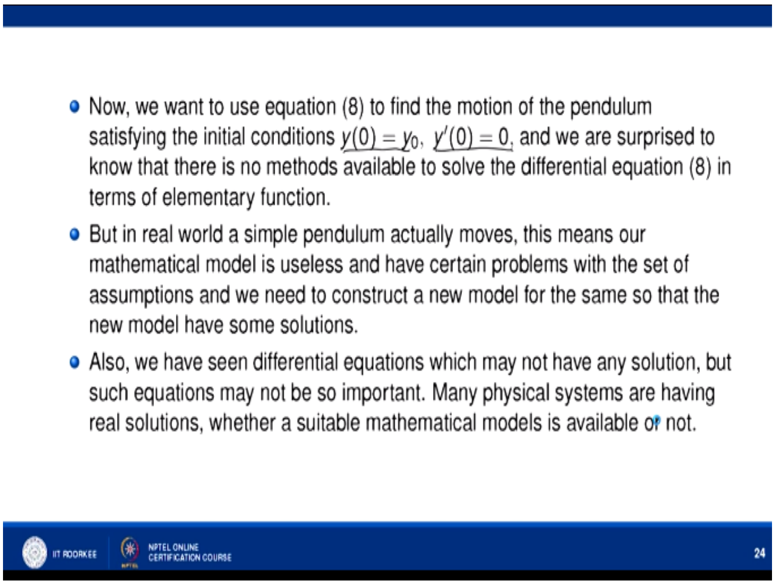

In that case we can say that this is a simple equation y double dash+g/l y=0 right. Look at here, we want to use equation 8 to find the motion of the pendulum satisfying the initial condition y0=y0 and initial velocity y dash 0=0 and it looks very strange but it is true that there is no methods available to solve the differential equation 8 in terms elementary functions.

But we already know that in real world a simple pendulum actually moves. It means that it does not stop somewhere it actually moves it means that our mathematical model is useless because we are not able to solve this mathematical model in terms of elementary function. So we can say that this particular mathematical model is useless and have certain problems with the set of assumptions and we need to construct a new model for the same physical system so that the new model have some solution.

So this is one instance we want to consider and also we have seen differential equation which may not have any solution but these are very rare formulas or we can say that this equation may not be so important and many physical systems are having real solution whether suitable math models is available or not, so it means that look at the simple pendulum we already know that it moves whether we have suitable mathematical formula or mathematical models available or not.

For example, if you look at this model then we may not have solution given in terms of elementary function but if you consider this then of course we have a beautiful theory to find out the solution in terms of elementary functions.

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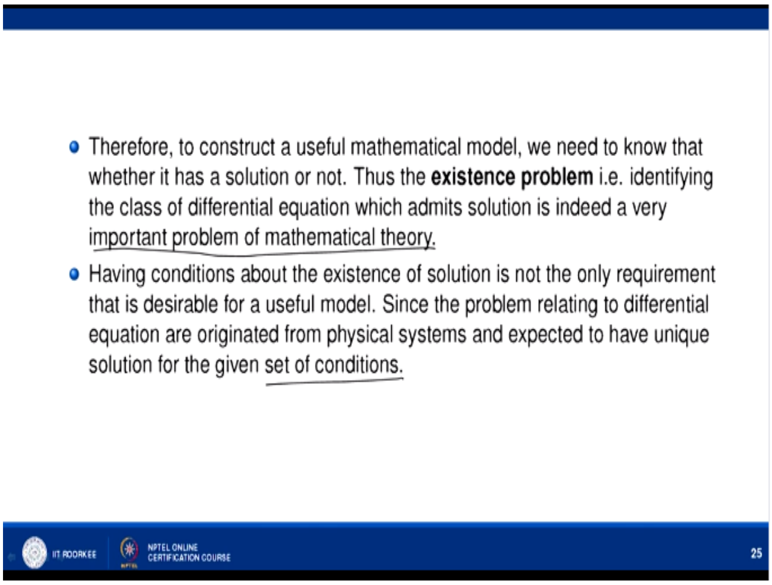

So therefore to construct a useful mathematical model we need to know that whether it has a solution or not right. Once we have mathematical model first thing we want to consider whether it has a solution or not and this is the existence problem. So it means that identifying the class of differential equation which admits solution is indeed a very, very important problem of mathematical theory.

Because once we have mathematical model if it does not have a solution then that model is useless kind of thing. So we want to first consider the existence problem and having condition about the existence of solution is not the only requirement that is desirable for a useful model. It means that once we have a solution that may not be sufficient at all because we want to find out a solution which satisfies certain conditions.

That since the problem relating to a differential equation are organized or originated from physical system and expected to have unique solution for the given set of condition. So given set of condition we must have a unique solution so that is also important.

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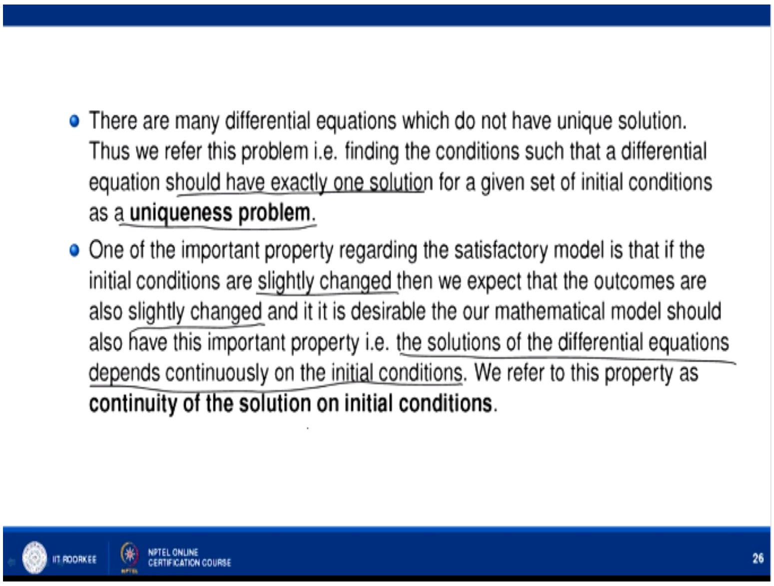

So there are many differential equations which do not have unique solution. We are not saying that those are not important but we want to find out unique solutions. So we refer this problem finding the conditions as that a differential equation should have exactly one solution and this kind of initial conditions are considered as this condition is sufficient for uniqueness problem.

So we can say that finding the condition such that a differential equation should have exactly one solution for a given set of initial conditions are considered as uniqueness problem and along with this we also want to see that one of the important property regarding the satisfactory model is that if the initial conditions are boundary conditions are slightly changed, perturbed then we expect that the outcomes are also slightly changed.

So it means that if there is slightly change in conditions then there is a slightly change in solutions also and it is desirable that our mathematical model should also have this important property. It means that the solution of the differential equation depends continuously on the initial condition and we refer to this property as continuity of the solution on initial condition.

Why this is important because whenever we consider a mathematical model it is coming from physical system along with some assumption and it may happen that while finding out that conditions or that differential equation we may have some kind of equipment problem. So it means that rather coming y0 it may have some different say it is y0+some delta kind of thing where delta is small enough.

But if y0 has some solution and y0+delta has some other solution which is drastically different from the original solution then we simply say that our mathematical model is not good enough. So for example we can consider the following differential equation.

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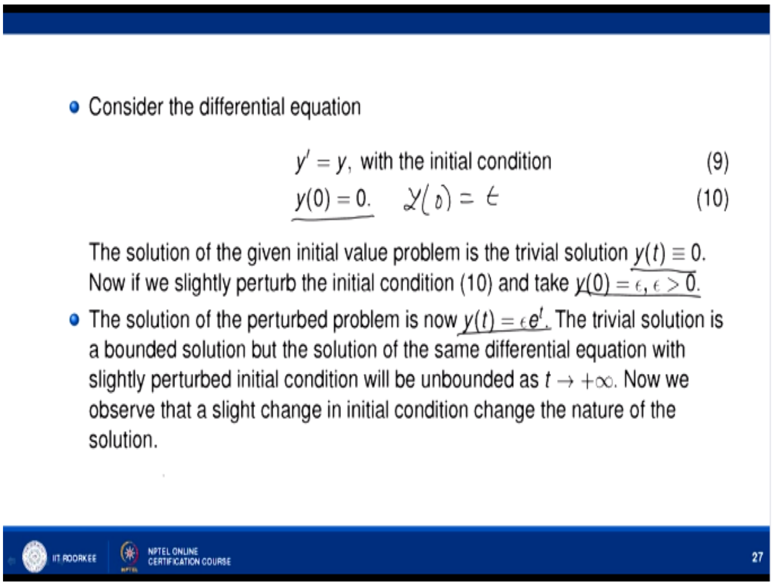

We have y dash=y with the initial condition y0=0 and we can simply find out the solution and solution is coming out to be y t identically = 0 but if we perturb this initial condition by a epsilon so we simply write in place of y0 we have y0=epsilon and in this case where epsilon is>0 and we can say that if the solution of the perturb solution problem is now y t=epsilon e to power t.

When y0=0 we have a identically 0 solution but when we have y0=epsilon we have y t=epsilon e to power t. You can say that why these 2 are so different but if you look at the trivial solution is a boundary solution but the solution of the same differential equation we slightly perturb initial condition will be unbounded means this is unbounded problem, unbounded solution while this is a bounded solution.

If you take t tending to infinity, then this is going to be infinity as well. So we can say that in this case we have a bounded solution but in this case we have a unbounded solution but we have only perturbed our initial condition slightly, epsilon is very, very small here. So here we can observe that a slight change in initial condition changes the nature of the solution and may be in drastic manner.

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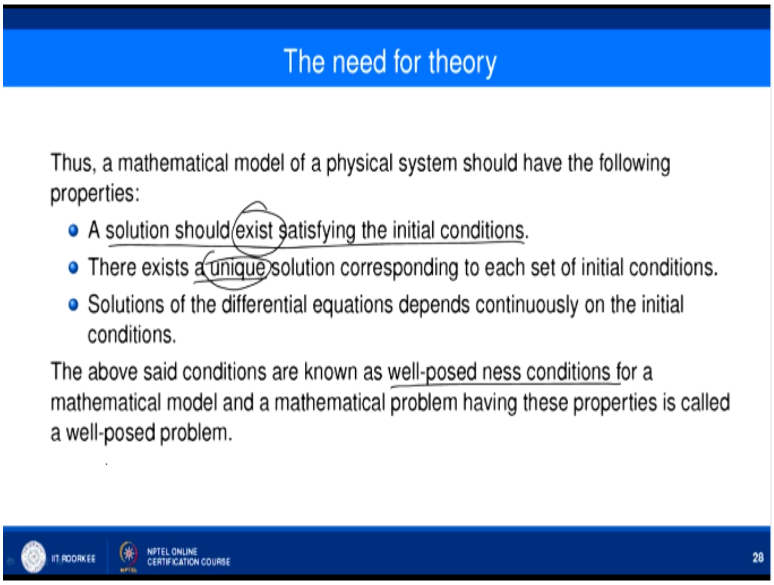

So here if we want to summarize here we say that thus a mathematical model of a physical system should have the following properties. First property is solution should exist satisfying initial condition. Second, there exist unique solutions. So first is existence and second is a uniqueness solution corresponding to each set of initial condition. If we have one set of initial condition, we have a unique solution.

If we have another set of initial condition or boundary condition, we have another solution exist and that is unique and solution of the differential equation depend continuously on the initial condition. It means that if we change our condition slightly then we have a slight change in solution. So it means that solution will depend on the data in a continuous manner.

The above said conditions are known as well-posed ness conditions for a mathematical model.

And this is known as Hadamard properties for well-defined, well-posed problem and we say that a mathematical model having these properties is called a well-posed mathematical model.

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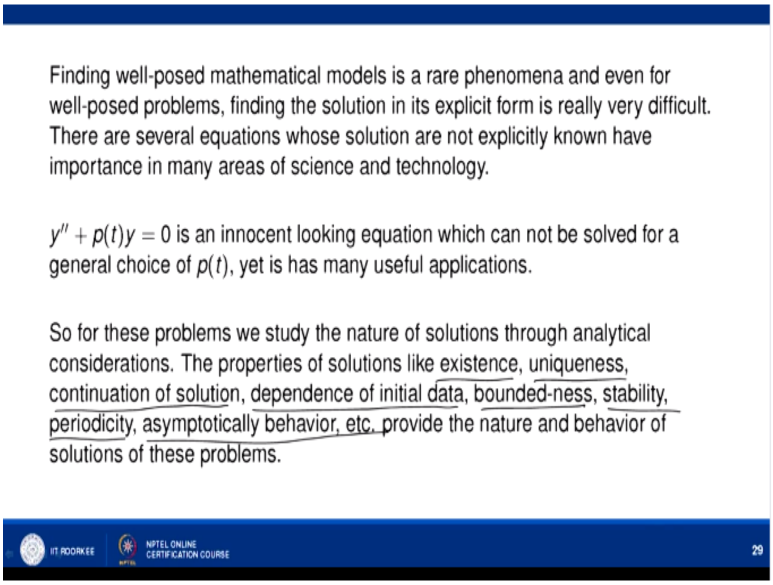

So finding well-posed mathematical model is a rare phenomena, it is not so common here and even for well-posed problems finding the solution in its explicit form is really very difficult. So there are several equations whose solutions are not explicitly known but they are very, very important in many areas of science and technology. For example, this particular y double dash+p t y=0.

This looks very, very simple one and we can show that this cannot be solved for a general choice of function p t. So it means that this equation may be solved for simple choice of p t. For example, if I take p t as 1/t square it is solvable but if we take p t as sin t, cos t or some other thing this may not be so easy to solve this and sometimes we cannot solve this problem in terms of elementary function.

So for these problems we study the nature of solution through an analytical consideration. So in these cases when we do not have explicit solution or solution is not given in terms of elementary function, we try to look at the other methods available which are known as analytical methods and we want to find out the properties like existence, uniqueness, continuation of solution, depending on initial data, bounded-ness, stability, periodicity, asymptotically behaviour.

And all that provided the nature and behaviour of solution of these problems. So here we stop and we have discussed the classification of ordinary differential equation and we have discussed certain examples where these initial boundary conditions are important to have a unique solution, one solution or infinitely many solution and also we have seen that why we need nice theory to deal these kinds of problems. So thank you for listening us. Thank you.