Ordinary and Partial Differential Equations and Applications
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Lecture – 16
Regular Singular Points - 1

Hello friends, welcome to this lecture. In this lecture, we will discuss the Frobenius series solution method for linear second order ordinary differential equation and we have already discussed the power series solution method for certain early differential equation. Here we continue our discussion for the equation where this point is a singular point. So consider the differential equation L(y).

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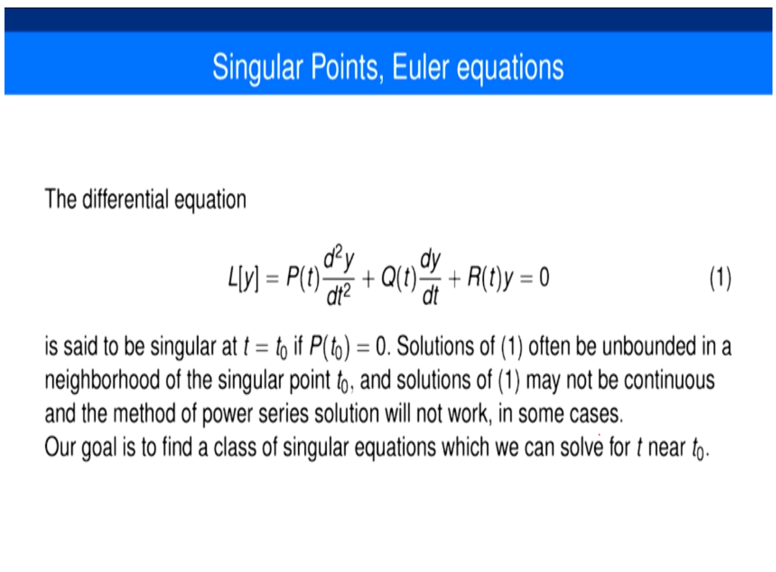

P(t) d2y/dt square + Q(t) dy/dt + R(t)y = 0. Here P(t) is the coefficient of y double dash. Q(t) is the coefficient of y dash and R(t) is the coefficient of y. So here we say that a point t = t0 is a singular point of this differential equation given as equation number 1, if P(t0) = 0. If P(t0) is != 0 for t = t0 we say that t0 is an ordinary point of this differential equation, but in place when t = t0 if P(t0) = 0 we say that the point t = t0 is a singular point of this differential equation and solution of 1 often be unbounded in a neighbourhood of the singular point t0.

So in general we cannot apply the power series solution method for the equation or near the point where solution is a singular differential equation. So here we say that solution of 1 may not be continuous and the method of power series solution will not work in some cases. So our goal is to find out a class of singular equations which we can solve for t in a neighbourhood of the singular point t0. So first let us say define this easiest singular differential equation which is known as Cauchy Euler equation so here the differential equation L(y).

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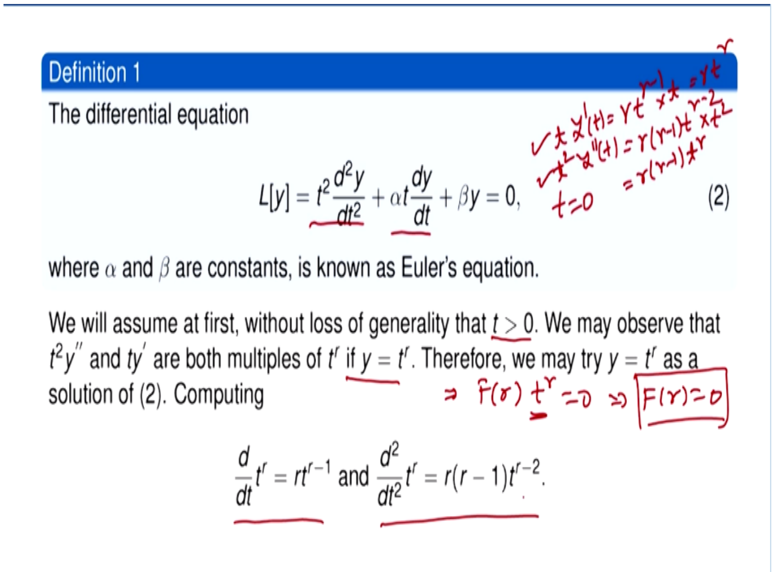

Given as t square d2y/dt square + alpha t *dy/dt plus beta y = 0. Her alpha and beta are constants and this equation is known as Euler's equation. If you look at here the coefficient of y double dash is t square and t square will vanish at t = 0. So it means that for this equation this t = 0 is a singular point. So here we say that here in general we cannot we should not apply power series solution.

So in place of power series solution what should be the method by which we can find out the solution? So here we first try to solve this Cauchy Euler's equation and with the help of working procedure here for this particular problem we try to extend this theory for a general ordinary differential equation where t = t0 is a singular point. So here we assume first of all that without loss of generality that t is > 0 here and we may observe that t square y double dash.

And t y dash are both multiple of t to the power r. So if y = tr, then you can easily calculate y dash. So y dash is y dash t = rt to the power of r - 1 and y double dash t = r(r - 1) t to the power r - 2. So we can say that if we multiply t in y dash then it will be what? So it is what r t to the power r and if you multiply t square here then it is what r (r - 1) t to the power r here. So we can say that if y(t) = t to the power r then t y dash t = rt to the power r and t square/double dash t is r(r - 1) * t to the power r.

So it means that this component and this component is multiple of t to the power r. So we can say that this can be written as when if y = t to the power of r then this equation can be written as some kind of (()) (04:41) in terms of r * t to the power r = 0 as a solution. So if t to the power r is nonzero we already know that this t to the power of r is nonzero. So it means that y = t to the power of r is a solution of this equation provided that F(r) = 0.

So this is the say summary of this method that we want to find out the values are for which t to the power r is a solution of this particular equation which is known as Euler's equation. So we may try y = t to the r as a solution of 2. So for that you just calculate first derivative of t to the power which is nothing but r t to the r - 1 and second derivative of t to the power r that is r (r - 1) * t to the power r - 2 here.

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We have that

$$\begin{aligned} L[t^r] &= r(r-1)t^r + \alpha r t^r + \beta t^r \\ &= [r(r-1) + \alpha r + \beta]t^r \\ &= F(r)t^r = 0 \quad \Rightarrow F(r) = 0 \end{aligned} \tag{3}$$

where

$$\begin{aligned} F(r) &= r(r-1) + \alpha r + \beta \\ &= r^2 + (\alpha - 1)r + \beta. \end{aligned} \tag{4}$$

So, $y = t^r$ is a solution of (2) if and only if, r is a solution of the quadratic equation

$$r^2 + (\alpha - 1)r + \beta = 0 \tag{5}$$

And simply we put y = t to the power r in our equation that is L(y) = t square d2y/dt square + alpha dy/dt + beta y = 0. So in place of y let us put t to the power r. So L[t to the power of r] is going to r (r - 1) t to the power r + alpha r t to the power of r + beta * t to the power r. So if you simplify we have this equation that is r(r - 1) + alpha r + beta * t to the power r. So let us denote this equation quadratic equation in terms of r as F(r) so F(r) * t to the power r.

So L[t to the power of r] is given as F(r) t to the power r. So we say that here F(r) is known as indicial equation and it is given as r (r - 1) + alpha r + beta. We can simply and we can have F(r) as this indicial equation that is r square + (alpha - 1) r + beta. So we can say that this L[tr] = 0 provided that F(r) = 0.

So it means that y = t to the power r is a solution of t if and only if r is a solution of the quadratic equation that is r square + (alpha - 1) r + beta = 0 that r should satisfy this quadratic equation that is F(r) = 0 here. So let us find out for what values this F(r) = 0. For that we can use the theory of quadratic equation.

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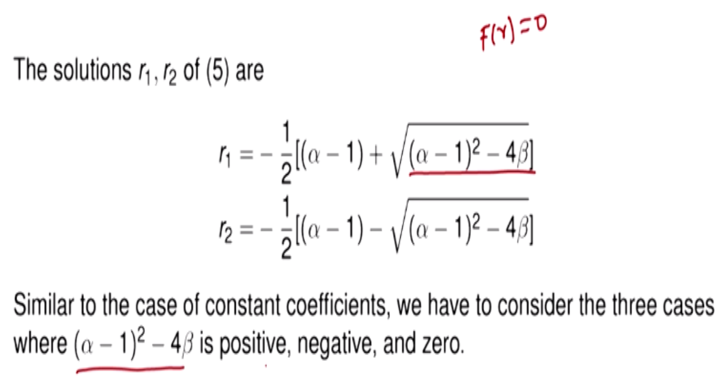

And we can find out the roots as r1 and r2 where r1 is given as - 1/2(alpha - 1) + under root (alpha - 1) whole square - 4 beta and r2 is the other root that is - 1/2 alpha - 1 - beta - under root (alpha - 1) whole square - 4 beta. So here we are able to find out this r1, r2 as a root of this equation that is F(r) = 0 here.

So here we say that as similar to case of constant coefficients we have to consider the 3 cases where it depend on this quantity that is alpha - 1 whole square - 4 beta whether it is positive, negative and = 0. So here let us consider all these cases and depending on these cases the solution will be t to the power r is the solution and r is taking these 2 values r1 and r2 and depending on this quality whether it is positive, negative, and zero we will find out the solutions here.

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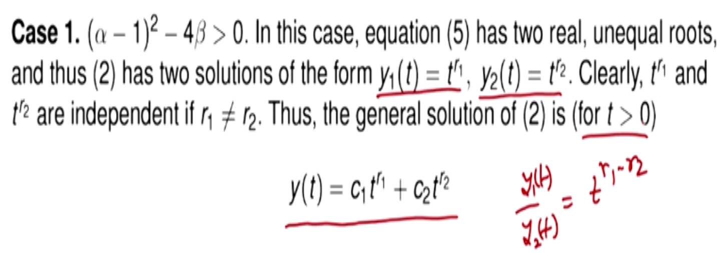

So let us see case 1 where we have this alpha - 1 whole square - 4 beta is > 0. So here if we look at this quantity in square root this quantity is positive. So it means that now we have 2 different value of r1 and r2 that is r1 = - 1/2 alpha - 1 + some quantity which you can is recalculate by taking the square root of this. So here we can say that in this case equation 5 has 2 real unequal roots and thus (2) has 2 solution of the form y1(t) = t to the power r1 and y2(t) = t to the power r2.

So here r1 and r2 is not equal, real, and it satisfies the equation F(r) = 0 and we can say that 1 solution is given as t to the r1 and another solution is given as t to the power r2 and we can easily check that these 2 obtain solution are linearly independent solution because we already know that we can easily check y1 t upon y2t = t to the power r1 - r2.

So the ratio of these 2 is not constant so we can say that y1 and y2t is or y1 and y2 are linearly independent solution. So if r1 ! = r2 we can say that y1t = t to the power r1 and y2t = t to the power r are linearly independent solution. So we can write our general solution like this y(t) = c1 t to the power r1 + c2 t to the power r2. Here please note down that here we are assuming that t is > 0.

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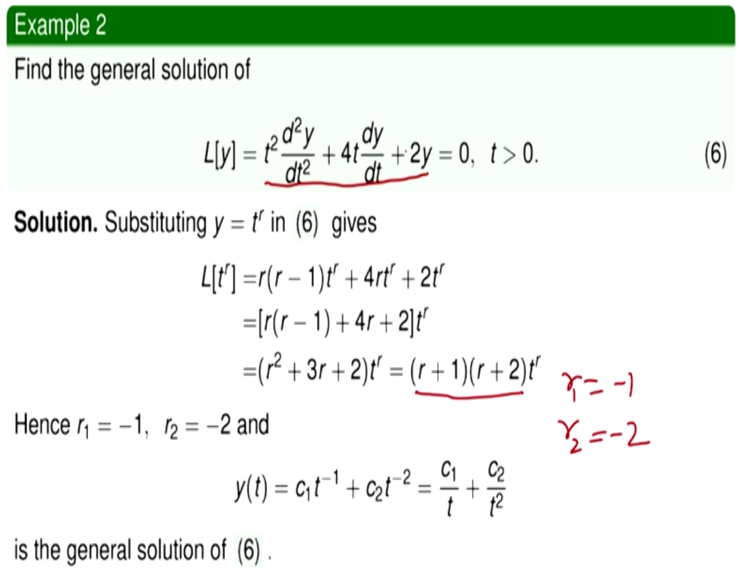

Now consider the second case, but before that let us take 1 example based on this theory. So find the general solution of this equation t square d2y/dt square + 4t dy/dt + 2y = 0. We denote this part as some equation and y = 0. So, here as we have discussed then let us put y = t to the power r because it is satisfying the Cauchy Euler equation. So it is a Cauchy Euler equation. Here alpha = 4 and beta = 2. So when you put y = t to the power r we can simplify this r(r - 1) t to the power r + 4 rt to the power r + 2t to the power r.

So taking the coefficient of t to the power r is r(r - 1) + 4r + 2 and if you simplify this it is r square + 3r + 2 and you can factor it out and we have (r +1) (r + 2) t to the power r. So it means that t to the power r will be a solution of this if r = - 1 and you can call it r1 and r 2 = - 2. So we here we have r1 = - 1 and r2 = - 2 and we can write down our general solution as y(t) as c1t to the power r1 that is - 1 + c2t to the power - 2 that is r2 because of this thing. So here we have general solutions c1/t + c2/t2 is the general solution of this equation number 6 here.

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Case 2. $(\alpha - 1)^2 - 4\beta = 0$. In this case

$$r_1 = r_2 = \frac{1-\alpha}{2}$$

$y_1(t) = t^{r_1} = t^{r_2}$

$y_2(t) = c(t)\, y_1(t)$

and we have only one solution $y = t^{r_1}$ of (2). A second solution can be found by the method of reduction of order. However, we would like to produce an alternative method of obtaining y_2. Observe that $F(r) = (r - r_1)^2$ in the case of equal roots. Hence

$$L[t^r] = (r - r_1)^2 t^r. \tag{7}$$

Taking partial derivative of both sides of (7) with respect to r, we have

$$\frac{\partial}{\partial r} L[t^r] = L\left[\frac{\partial}{\partial r} t^r\right] = \frac{\partial}{\partial r}[(r - r_1)^2 t^r].$$

Now moving on case 2, here we have this alpha - 1 whole square - 4 beta = 0. So in this particular case, we have equal real roots that is r1 = r2 = 1 - alpha/2. So here when r1 is = r2 we have only 1 solution available that is y1(t) you can call it as t to the power r1 whether you use r1 or r2 because r1 and r2 are equal so we have only 1 linearly independent solution. So how do we find out the second solution? We already know that this method of reduction is used to find out the second solution, but here method of reduction is what?

You assume that y2(t) = some function cty1t and we try to find out the function ct in a way that this y2t satisfy the given ordinary differential equation. So this process is based on method of variation of parameter and by using this we actually get linear differential equation of 1 or less that is why this method is known as method of reduction also. So here we will not use this method of reduction here. Here we will give us an alternative method to find out this y2. Why we are giving this alternative method, because this method we are going to generate for a general similar ordinary differential equation.

So let us observe that this F(r) which is given as r - r1 whole square, because r1 is a double root. So in this case, we can write L[t to the power of r] = (r - r1) whole square * t to the power r. So we already know that r1 satisfy this equation and we have 1 solution that is t to the power r. So here we simply say that if we take the partial derivative of this equation number 7 with respect to r what we will get dou/dou r of L[t to the power r] = L[dou/dou r * t to the power r].

If you look at L is what L represent an operative equation if you look at L and represent this kind of equation. L[y] = t square d2y/dt square + alpha t * dy/dt + beta y. So here we can say that this is an operator which input is y and output is this quantity t square t square d2y/dt square + alpha t dy/dt + beta y. So here if we look at this is a differential operator given in terms of d/dt. So here we are differentiating with respect to t.

So here in this procedure we are applying the partially derivative with respect to r. So since this L is independent of r, so we can write we can take this partial derivative inside. So we can write this is as dou/dou r * L[tr] can be written as L[dou/dou r * t to the power r]. So and if you look at the right side it is dou/dou rof this quantity r(r - 1) whole square t to the power r. So let us see what it will give us.

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$y = t^r$

Since $\partial(t^r)/\partial r = t^r \ln t$, we observe that

$$L[t^r \ln t] = (r - r_1)^2 t^r \ln t + 2(r - r_1)t^r. \qquad (8)$$

The right hand side of (8) vanishes when $r = r_1$. Hence,

$L(y) = 0$, $y = y_1$, $L(y_1) = 0$

$$L[t^{r_1} \ln t] = 0$$

which implies that $y_2(t) = t^{r_1} \ln t$ is a second solution of (2). Since t^{r_1} and $t^{r_1} \ln t$ are linearly independent, the general solution of (2) in the case of equal roots is

$$y(t) = (c_1 + c_2 \ln t)t^r, \quad t > 0.$$

So here we already know that dou/dou of r of t to the power of r is nothing but t to the power of r ln t. That you can easily verify. So here if you take y = t to the power r then apply the logarithmic and you can easily differentiate with respect to r here. So here using this special dou/dou r of t to the power t to the power r ln t we can write this left hand side that is.

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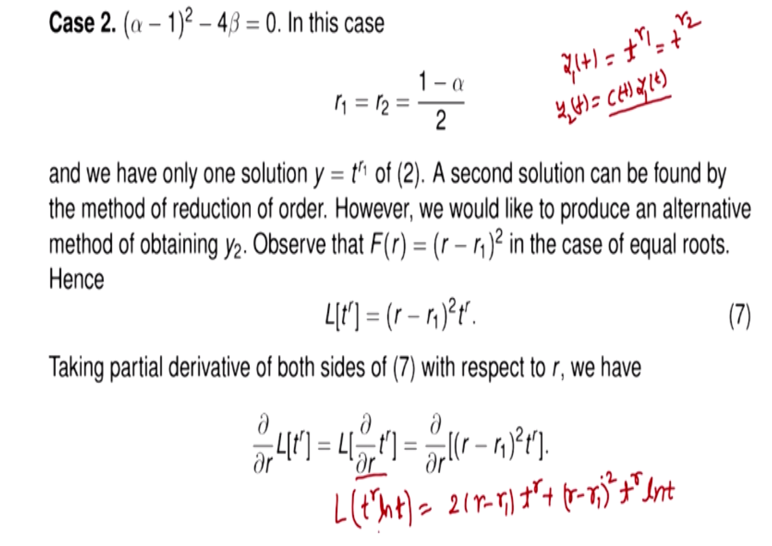

Case 2. $(\alpha - 1)^2 - 4\beta = 0$. In this case

$$r_1 = r_2 = \frac{1-\alpha}{2}$$

and we have only one solution $y = t^{r_1}$ of (2). A second solution can be found by the method of reduction of order. However, we would like to produce an alternative method of obtaining y_2. Observe that $F(r) = (r - r_1)^2$ in the case of equal roots. Hence

$$L[t^r] = (r - r_1)^2 t^r. \quad (7)$$

Taking partial derivative of both sides of (7) with respect to r, we have

$$\frac{\partial}{\partial r} L[t^r] = L[\frac{\partial}{\partial r} t^r] = \frac{\partial}{\partial r}[(r - r_1)^2 t^r].$$

Here you simply write this is what t to the power r ln t = so L of this quantity = if you differentiate this what you will get 2(r- r1) t to the power r + (r - r1) whole square and again t to the power r ln t here. So we will get this special for t to the power r ln t. So L[t to the power ln t] = (r - r1) square t to the power r ln t + 2(r - r1) t to the power r. We know that if we look at the right hand side if you put r = r1 then this right hand side will be 0.

So it means that this t to the power r1 ln t is a function if we put in place of y as t to the power of r1 ln t then we have 0 for this operative equation. So these are when we consider this L(y) = 0 we say that y = y1 is a solution of this operator equation if L(y1) is coming out to be 0 here. So we can say that y1 is a solution of this equation. So here we say that L of t to the power r1 ln t = 0 so please add this to the power r1 ln t is another solution of the differential equation L[y] = 0.

So here we say that since t to the power r1 and t to the power r1 ln t are linearly independent so we can say that in this case the general solution of 2 we have y(t) = (c1 + c2 ln t) t to the power r. So in case of equal roots solution is given by this that y(t) - (c1 + c2 ln t) t to the power r and we can easily check that this I am leaving it you that t to the power r1 and t to the power r1 ln t are linearly independent solutions. So this I am leaving it to you. Now let us take 1 example based on this.

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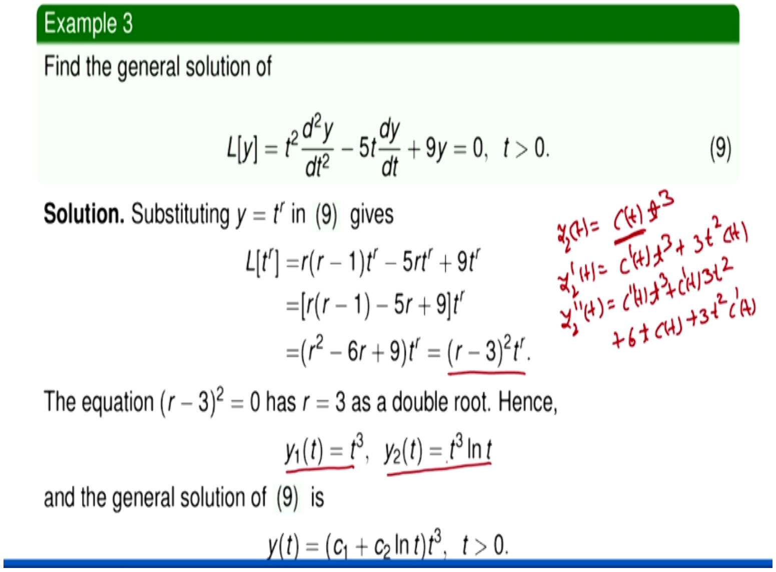

So here we can consider this linear second order ordinary differential equation L[y] = t square d2y/dt square - 5t dy/dt + 9y = 0. So this is Cauchy Euler equation. So we apply y = to the power r and try to find out the values of r for which it has a solution as t to the power r. So when we apply L as t to the power r we will have this r square - 6r + 9 * t to the power r which we can simplify as r - 3 whole square t to the power of r.

So this t to the power r will give you a 0 of this provided r = 3. So these are corresponding to r = 3 we have a 1 solution. So we say y1(t) = t to the power q as 1 solution and as we pointed out that if t cube is a solution then t cube ln t will also be a solution because here r = 3 is a double root of this indicial equation this equation. So in case of double root we have 1 solution as t to the power cube as it is and second solution is given as t cube ln t.

So here our general solution is y(t) = c1 + c2 ln t * t to the power cube where t is > 0 here. So again you may take this as an exercise that if you assume y2(t) as = c(t) t to the power cube and if you put it here then you can easily find out y2 dash (t) = c dash t * t cube + 3 t square c(t) as similarly you can find out y2 double dash t as c double dash t * t cube + c dash t * 3 t square + 6t * ct + 3t square c dash t. If you put it back and solve you can easily find out the differential equation in terms of c(t) and then also you can get a solution given as t cube * ln t. So you try it.

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Case 3. $(\alpha - 1)^2 - 4\beta < 0$. In this case

$$r_1 = \lambda + i\mu \text{ and } r_2 = \lambda - i\mu$$

with

$$\lambda = \frac{1-\alpha}{2}, \quad \mu = \frac{[4\beta - (\alpha - 1)^2]^{1/2}}{2} \tag{10}$$

are complex roots. Hence,

$$e^{ix} = \cos x + i \sin x$$

$$\begin{aligned}\phi(t) &= t^{\lambda + i\mu} = t^{\lambda} t^{i\mu} \\ &= t^{\lambda}(e^{\ln t})^{i\mu} = t^{\lambda} e^{i\mu \ln t} \\ &= t^{\lambda}[\cos(\mu \ln t) + i \sin(\mu \ln t)]\end{aligned}$$

is a complex- valued solution of (2).

So, now consider the last case that is case 3. Here we have alpha - 1 whole square - 4 beta < 0. So in this case, r1 and r2 are coming out to be complex root of this quadratic equation. So we have r1 as lambda + I mu and r2 = lambda - I mu where lambda is given as 1 - alpha/2 and mu is given as square root of 4 beta - alpha - 1 whole square/2. So here we are getting a complex roots, but this differential equation represent a real part phenomenon so it means that we must have real solution or in fact we are interested in the real part solution.

So out of this complex solution let us find out the real solution. So here solution will be what phi t = t to the power r should be a solution. What is r here? Here let me write it r as lambda + I mu and we can write this as t to the power lambda * t to the power I mu. The problem is that this t to the power I mu may not be defined. So here to define this t to the power I mu let us write t as e to the power ln t. So if you calculate e to the power ln t it will give you t.

So we write t to the power I mu as e to the power ln t to the power I mu. Now this has a meaning. So here we can write t to the power lambda * e to the power I mu ln t. So we already have this expression e to the power ix as cos of x + I sinx. So using this expression, we can write down our solution as t to the power lambda cos of mu ln t + I sin mu ln t. We can say that this is 1 complex-valued solution of the equation which is known as Cauchy Euler's equation. So since we are not interested in this complex solution so here we take out the real part and complex part out.

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But then

$$y_1(t) = Re\{\phi(t)\} = t^\lambda \cos(\mu \ln t)$$

and

$$y_2(t) = Im\{\phi(t)\} = t^\lambda \sin(\mu \ln t)$$

are two real- valued independent solutions of (2). Hence, the general solution of (2), in the case of complex roots, is

$$y(t) = t^\lambda[c_1 \cos(\mu \ln t) + c_2 \sin(\mu \ln t)]$$

with λ and μ given by (10).

Here we can say that this y1(t) that is real part of phi t that is t to the power lambda cos mu ln t and y2(t) as imaginary of phi t that is t to the power lambda sine mu ln t are 2 real valued independent solutions of 2. So it means that if we have a complex root of a differential equation then it is real.

And imaginary part will also satisfy that real ordinary differential equation. So using that result you can say that real part of phi t and imaginary part of phi t are 2 real valued independent solutions of 2 and we can write down our general solution as t to the power lambda c1 cos mu ln t + c2 sine mu ln t and in this way we can write down the solution.

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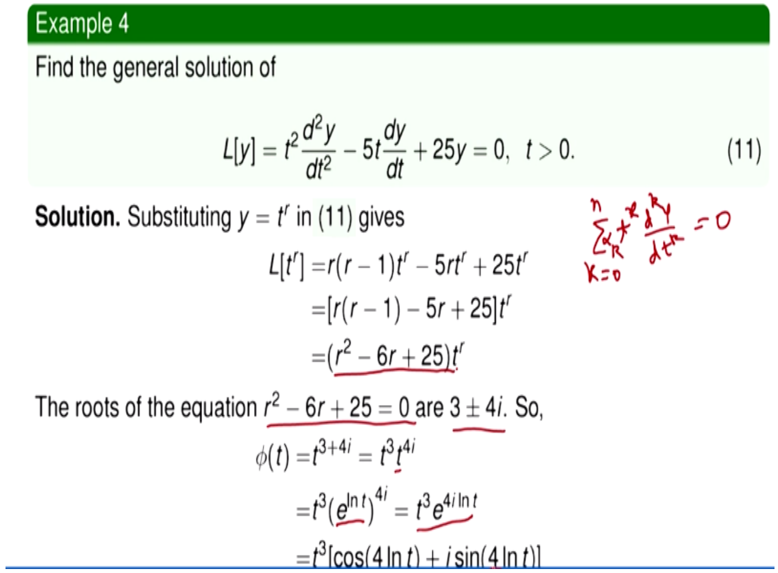

Now let us take 1 example based on this. So here we take L[y] as t square d2y/dt square - 5 t dy/dt + 25 y = 0. Please remember in case of Cauchy Euler equation the order of derivative is 2 here and here power of t is 2. So we can say that if we can write down this equation, so here we write that alpha k t to the power k dky/dtk = 0 here k is = say 0 to any number n is known as Cauchy Euler equation of r and n.

So here we say that the r represent in derivative is multiplied by t to the power (()) (23:18). So here we have d2y/dt square then it is multiplied by t square. Here we have dy/dt then it is multiplied by t * some constant and here we do not have any derivative. So here t to the power 0 is there. So we can write that this is a simple Cauchy Euler equation. So let us take y = t to the power r as a solution here and then you put L = t to the power r it is clear as r square - 6 r + 25 * t to the power r.

When we take the 0 of this then we have 0 are 3 + - 4 i. So it means that phi(t) is given as t to the power of 3 + 4i and I can write this as t cube * t to the power 4i, but t to the power 4iI is not defined in this current situation. So here we write t as e to the power ln t to the power 4i and you can write t cube * e to the power 4i ln t and this here using this formula e to the power ix we can write t cube cos of 4 ln t + I sin (4 ln t) to give out the real part and the imaginary part.

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Consequently,

$$y_1(t) = Re\{\phi(t)\} = t^3 \cos(4 \ln t)$$

and

$$y_2(t) = Im\{\phi(t)\} = t^3 \sin(4 \ln t)$$

are two independent solutions of (11). Hence, the general solution of (11), in the case of complex roots, is

$$y(t) = t^3[c_1 \cos(4 \ln t) + c_2 \sin(4 \ln t)], \quad t > 0.$$

$$L(y) = t^2 \frac{d^2 y}{dt^2} + \alpha t \frac{dy}{dt} + \beta y = 0$$

$$y(t) = t^r \quad F(r) = 0 \Rightarrow r_1 = , \ r_2 =$$

$$L(t^r) = F(r) t^r$$

We can write y1(t) as t cube cos 4 ln t and y2(t) is t cube sin (4 ln t) are 2 independent solutions of 11. So here general solution is given as y(t) = t to the power cube [c1 cos(4 ln t) + c2 sin(4 ln t)] and t > 0. So we have discussed the solution of this equation. L(y) = d2y/dt square + alpha t sorry here t square dy/dt + beta here we have only y = 0 and here this is Cauchy Euler equation and here we say that y(t) = t to the power r is a solution provided that this r satisfies this equation F(r) = 0.

Here because when we put y(t) = t to the power r then L(t to the power r) is given as F(r) * t to the power r. So this will be a 0 when F(r) = 0. So here we are trying to find out the real roots of r1 and r2 and once we have roots r1 and r2 then accordingly we can find out the solution of this equation.

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Since we have assumed in the beginning that $t \geq 0$. Let us see how the above analysis is sufficient for all t. In case when $t < 0$, one main difficulty is that t^r may not be defined. Other difficulty is that $\ln t$ is not defined for negative t. We may handle both of these difficulties with the following change of variable. Set

$$t = -x, \quad x > 0,$$

and let $y = u(x), \quad x > 0$. Observe, from the chain rule, that

$$\frac{dy}{dt} = \frac{du}{dx}\frac{dx}{dt} = -\frac{du}{dx}$$

and

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(-\frac{du}{dx}\right) = \frac{d^2u}{dx^2}.$$

$y(t) = u(x)$
$t = -x, x > 0$

$$= \frac{d}{dx}\left(-\frac{dy}{dx}\right) \times \frac{dx}{dt}$$

Now so far we have utilized only the case when t is positive. Now, let us consider the case when t is not positive, it may be the value t < 0. So here in case when t < 0 one main difficulty is that t to the power r may not be defined in the neighbourhood of t = 0 and another difficulty is that this ln t may not be defined because ln t is defined only for positive values. So it may not be defined for negative t.

So it means that whatever we have already discussed may not be applicable in the case when t is < 0. So here we will do 1 simple thing. We simply handle these difficulties by taking the change of variable and here we write t as - of x since t is negative so this x is going to be a positive. Now using this change of variable let us calculate dy/dt. dy/dt = du/dx what is u here? Here we write t is = u of x, here x is positive.

So here we are assuming y(t) = u(x) here. So t = - x here x is positive, because t is negative. So dy/dt = du/dx and dx/dt. On dx/dt you can easily calculate from this that is coming out to be - 1. So dy/dt = - du/dx. Similarly, we can calculate d2y/dt square that is d/dt of - du/dx and which is nothing but d/dx of - du/dx* dx/dt and if you simplify we have d2u/dx square. So we have dy/dt as - du/dx and d2y/dt square as d2u/dx square.

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Thus, we can write (2) in the form

$$(-x)^2\frac{d^2u}{dx^2}+\alpha(-x)\left(-\frac{du}{dx}\right)+\beta u=0$$

$t^2y''(t)+\alpha t\frac{dy}{dt}+\beta y=0$, $t>0$

or

$$x^2\frac{d^2u}{dx^2}+\alpha x\frac{du}{dx}+\beta u=0,\quad x>0 \tag{12}$$

But Equation (12) is exactly the same as (2) with t replaced by x and y replaced by u. Hence, Equation (12) has solutions of the form

$$u(x)=\begin{cases} c_1x^{r_1}+c_2x^{r_2} \\ (c_1+c_2\ln x)x^{r_1} \\ (c_1\cos(\mu\ln x)+c_2\sin(\mu\ln x))x^{\lambda} \end{cases} \tag{13}$$

depending on whether $(\alpha-1)^2-4\beta$ is positive, zero, or negative.

So we can write this equation - x square d2y/dx square + alpha * t, t is - x. - du/dx + beta u = 0 and if you simplify you will have this equation x square d2 u/dx square + alpha x du/dx + beta u = 0. Here x is positive and if you look at carefully this equation number 12 then it is exactly the same as 2. Here 2 is what? Here t square * y double dash t + alpha t sorry. So here we have alpha t dy/dt + beta y = 0. So if you look at and here we have assumed that t > 0. So if you look at these 2 equations then it is exactly the same.

The only thing is that y is replaced by u and t is replaced by x and we already know how to find out the solution. So in this case, solution is given by u(x) depending on the discriminant which is positive, negative, and zero. So, here when t is positive we have solution c1 x to the power r1 + c2x to the power r2 and in case of equal root we have c1 + c2 ln x * x to the power r1 and in case of when discriminant is negative.

We have general solution as c1 cos mu ln x + c2 sin mu ln x * x to the power lambda. So depending on whether this alpha - 1 whole square - 4 beta is positive, zero or negative.

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Now, observe that

$$x = -t = |t|$$

for negative t. Thus for negative t, the solutions of (2) have one of the forms

$$\begin{cases} c_1|t|^{r_1} + c_2|t|^{r_2} \\ (c_1 + c_2 \ln|t|)|t|^{r_1} \\ (c_1\cos(\mu\ln|t|) + c_2\sin(\mu\ln|t|))|t|^{\lambda} \end{cases}$$

$\alpha y'' + \beta y' + \gamma y = 0$ $\quad e^{rt}$ $\quad t^r$

$F(r)e^{rt} = 0$ $\quad F(r)t^r = 0$

$F(r) = 0$ $\quad F(r) = 0$

So if we summarize the solution given this case when x is > 0 or you can say that t is < 0. So in this case, we can say that x is written as - t. This I can write it modulus of t. So it means that if we combine the general solution given in case t positive and t negative we can write down the solution is given as c1 modulus of t to the power r1 + c2 modulus of t to the power r2 when discriminant is positive and discriminant 0 we have c1 + c2 ln modulus of t * modulus of t to the power r1.

And third case c1 cos mu ln modulus t + c2 sinu mu ln modulus t * modulus t to the power lambda. Here in this case, your discriminant is negative. So in this case, it means that we have solved this Cauchy Euler equation assuming that y = t to the power r is a solution. In fact, how to get this t to the power r here we can say that there are 2 ways by which we can say that t to the power r is a solution 1 is a resemblance of constant coefficient case.

Here if you look at the constant coefficient case is what here y double dash + you can simply write some alpha y double dash + beta y dash + gamma y = 0. So this is a simple been a second order differential equation and we know how to solve this here alpha, beta, gamma are constant. Here we have assumed that e to the power say r is a solution of this because while putting this e to the power r here as a quantity we can say that this is nothing but coming out to be F(r) e to the power r = 0.

So now this e to the power r is non zero so we can say that this e to the power r(t) is a solution provided that r satisfy this equation F(r) = F(r) = 0. So if you look at this Cauchy Euler equation has exactly the similar situation the only thing is that in place of e to the power of rt we have here solution as t to the power r and here when you put t to the power r here and then also it is coming out to be F(r) * t to the power r = 0.

So this t to the power r will be a solution provided that r satisfies this equation that is F(r) = 0. So this is say analogous to your constant coefficient case. So here I am not going to difference this, but I am saying that the same t to the power r you can get if you use the power series solution method. So here if we consider this as a case of power series solution and here we apply the power series solution then see that we will get this t to the power of r kind of a solution.

So I may give you this assignment that find out that y(t) = t to the power r is a solution if you apply the power series solution for this case also. Here this t = 0 is not an ordinary point is a singular point, but it is still your power series solution will be occurred and here we will get the t to the power r is a solution here. Now let us generalize this finding to a more general differential equation.

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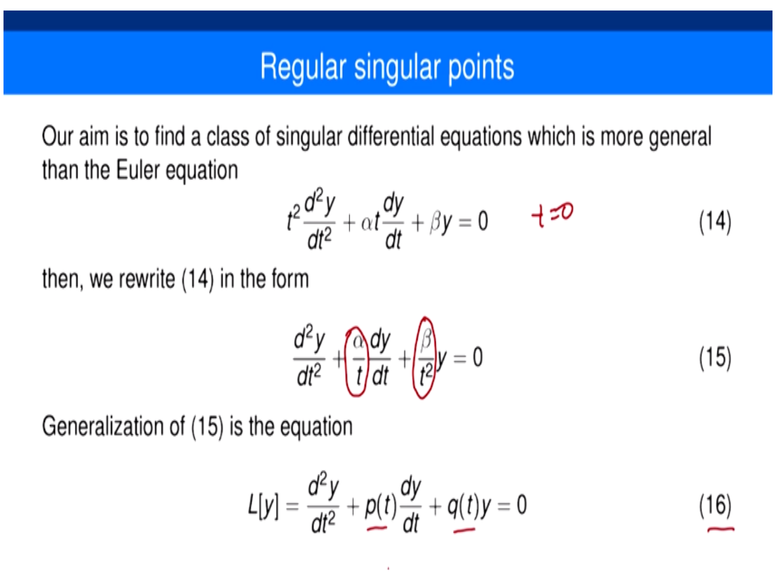

So now our aim is to find a class of singular differential equation which is more general than Euler equation. So Euler equation is if you remember it is t square d2y/dt square + alpha t dy/dt

+ beta y = 0. So here we can easily change that t = 0 is a singular point of this differential equation because this coefficient of d2y/dt square is coming out to be 0. So at t = 0 is a singular point of this differential equation.

But we know how to solve this and so let us see this can be written as this form d2y/dt square + alpha/t dy/dt + beta/t square y = 0. So in place of alpha/t and beta/t square I replace this by function of t and function of so this alpha/t I am writing p(t) and beta/t square I am writing as q(t). So we can say that this 14 is a particular case of 16 provided that p(t) = alpha/t and q(t) is beta/t square. Now let us say that let us consider more general case.

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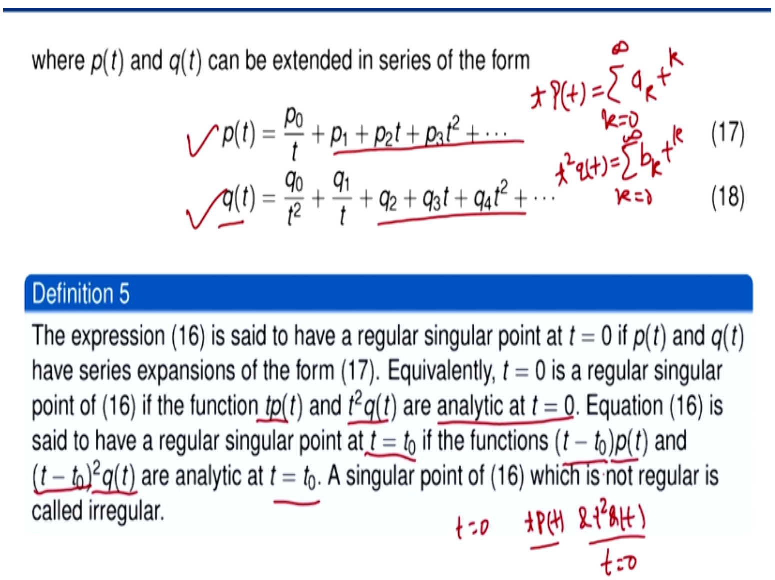

When this P(t) is not only p0/t, but it is p0/t + p1 + p2t + p3 t square and so on. So here you can say that this is nothing, but we have a power series solution power series we are adding here. Similarly, in q(t) and not only q0/t square term is there, but we have q1/t + q2 +q3(t) + q4(t) square is there. So here we are adding some more terms which are coming from some kind of power series.

So here we can say that this 17 or 18 can be written a way such that if you simply this then it is what t of p(t) = summation some ak t to the power k. k is from 0 to infinity. Similarly, your t square q(t) is again written as bk t to the power k, k is from 0 to infinity. So here we claim that the method which we have already discussed can be generalized before this kind of equation

L[y] = d2y/dt square + p(t) dy/dt + q(t) y = 0 provided that p(t) is more general than this alpha/t and beta/t square.

Or we can say that when tp(t) is given as this kind of power series or we can say that when tp(t) has Taylor series expansions at t = 0. Similarly, t square q(t) has a Taylor series expansion at t = 0. So we can say that the expression 16 is said to have a regular singular point at t = 0 if p(t) and q(t) have series suspension of the form 17. In equivalent sense t = 0 is a regular similar point of 16 if the function tp(t) and t square q(t) are analytic at t = 0.

So we can say that this is the situation at t =0. So at t = 0, first thing it should be a singular point. Second that this tp(t) and t square q(t) must have Taylor series expansion at near t = 0. So in this case we call t = 0 as a regular singular point and in general for t = t0 we say that equation 16 is said to have a regular singular point at t = t1 if the function t - t0 * p(t) and t - t0 whole square q(t) are analytic at t = t0 or analytic means t - t0 * p(t) has a Taylor series expansion.

And t - t0 whole square q(t) has a Taylor series suspension at point t = t0 and a singular point of 16 which is not regular is called irregular singular point. So here I will stop here and in next lecture we will continue the discussion of how to find out a solution for regular singular point case. So here I will stop. We will meet in next lecture. Thank you for listening us. Thank you.