Ordinary and Partial Differential Equations and Applications
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Lecture – 15
Power Series Solution of Second Order Homogeneous Equations

Hello friends, welcome to my lecture on power series solution of second order homogenous differential equations.

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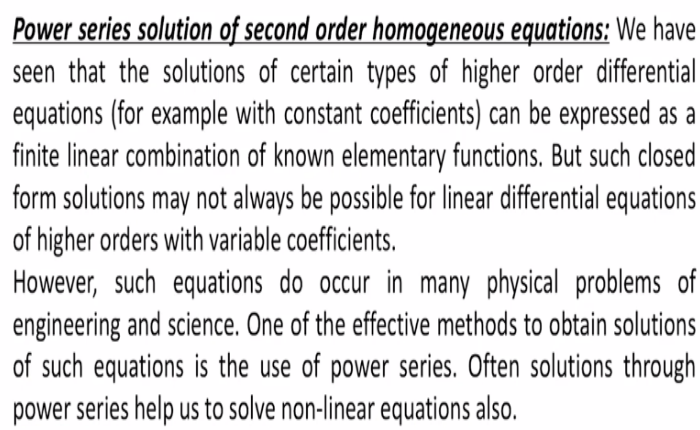

We have seen that the solution of certain types of higher order differential equations for example, higher order differential equation with constant coefficients can be expressed as a finite linear combination of known elementary functions. But such closed form solutions may not always be possible for linear differential equations of higher orders with variable coefficients. However, such differential equations do occur in many physical problems of engineering and science.

One of the effective methods to obtain solutions of such equations is the use of power series. Often solutions through power series help us to solve non-linear equations also. Now, let us consider a second order homogenous linear differential equation. a0x * y double dash + a1x * y dash + a2 x * y = 0 and suppose that this equation has no solution that is expressible as a finite linear combination of known elementary functions.

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Consider the second order homogeneous linear differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \quad (1)$$

and suppose that this equation has no solution that is expressible as a finite linear combination of known elementary functions.

Let us assume, however, that it does have a solution that can be expressed in the form of an infinite series. Specifically, let us assume that it has a solution expressible in the form

$$c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \ldots\ldots = \sum_{k=0}^{\infty} c_k(x-x_0)^k, \quad (2)$$

where, $c_0, c_1, c_2, \ldots\ldots$ are constants.

So let us assume that how about that it does have a solution that can be expressed in the form of an infinite series. It is specifically, let us assume that it has a solution expressible in the form c0 + c1 * x - x0 + c2 * x - x0 whole square and so on which in abbreviated form we can write as sigma k = 0 to infinity. Ck * x - x0 to the power k, where c0, c1, c2 and so on are constants. An expression of the form 2 is called power series solution of the differential equation 1.

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We can write the equation (1) in the equivalent normalized form as

$$y'' + P_1(x)y' + P_2(x)y = 0, \quad (3)$$

$$\text{where } P_1(x) = \frac{a_1(x)}{a_0(x)} \text{ and } P_2(x) = \frac{a_2(x)}{a_0(x)}.$$

So let us see now the question arises under what conditions this assumption of ours that the solution of the given equation 1 is expressible in the form of an infinite series under what conditions this solution is valid. So to answer this question, let us write the equation 1 in the normalized form by dividing the coefficient of y double dash throughout the equation and then we get y double dash + P1(x) * y dash + P2(x) * y = 0 where P1(x) = a1(x)/a0(x) and P2(x) = a2(x)/a0(x).

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Definition: The point x_0 is called an ***ordinary point*** of the differential equation (1) if both of the functions $P_1(x)$ *and* $P_2(x)$ occurring in (3) are analytic at x_0.

Now a function f is said to be analytic at a point x0 if it is Taylor series about the point x = x0 exists and converges to f in some interval containing the point x0. Here let us look at the normalized form.

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We can write the equation (1) in the equivalent normalized form as

$$y'' + P_1(x)y' + P_2(x)y = 0, \qquad (3)$$

$$\text{where } P_1(x) = \frac{a_1(x)}{a_0(x)} \text{ and } P_2(x) = \frac{a_2(x)}{a_0(x)}.$$

This is the normalized form of the equation 1. So P1(x) is a1(x)/a0(x) and P2(x) is a2(x)/a0(x). A point x0 is called an ordinary point of the differential equation (1), if both the functions P1(x) and P2(x) which occur in the normalized form are analytic at x0. That means the functions P1(x) and P2(x) are the Taylor series of the functions P1(x) and P2(x) about the point x = x0 exist and converges in some open interval containing the point x = x0.

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Example: Consider $y'' + xy' + (x^2 + 2)y = 0$.

Here $P_1(x) = x$ *and* $P_2(x) = x^2 + 2$.

Now let us consider y double dash + x * y dash + x square + 2 * y = 0. So here we can see that P1(x) = x and P2(x) = x square + 2. So P1(x) is a polynomial and P2 (x) is a polynomial. Both polynomials are analytic functions. We can write the Taylor series of P1(x) and P2(x) about any point x = x0 and it will be convergent for all values of x for all x. So these are analytic functions for all real values of x and therefore every point x of the real line is an ordinary point of the differential equation y double dash + x * y dash + x square + 2y = 0.

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Example: Consider $y'' + \frac{x}{x-1}y' + \frac{1}{x(x-1)}y = 0$

Here $P_1(x) = \frac{x}{x-1}$ *and* $P_2(x) = \frac{1}{x(x-1)}$.

For the given differential equation, all real numbers except $x=0$ and $x=1$ are ordinary points

Now here in this case y double dash + x/x - 1 * y dash + 1/x * (x - 1) * y = 0 we see that P1(x) is x/x - 1. P2 (x) is 1/x * x - 1. Now P1(x) is not analytic at x = 1 and P2(x) is not analytic at x = 0 and at x = 1. So we can write the Taylor series of the function P1(x) about any point other than x = 1.

And therefore this is every point x other than x = 1 is an ordinary point of P1(x) and in the case of P2(x) every point other than x = 0 and x = 1 is an. This function P2(x) is analytic about any x other than x = 0 and x = 1. So the quotient has ordinary point other than x = 0 and x = 1. So for the given differential equation all real numbers except x = 0 and x = 1 are ordinary points.

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Theorem: Let x_0 be an ordinary point of the differential equation (1). Then (1) has two non-trivial linearly independent power series solutions of the form $\sum_{n=0}^{\infty} c_n (x - x_0)^n$ and these power series converge in some interval $|x - x_0| < R$, (where R>0) about x_0.

Okay, now let us state a theorem which will be used to find the power series solution of a given differential equation. Let x0 be an ordinary point of the differential equation 1, then (1) has 2 non-trivial linearly independent power series solution of the form sigma n = 0 to infinity cn (x-x0) to the power n and these power series converge in some interval mod of x - x0 < R where R is positive about the point x = x0. So we will use this theorem to find the powerful solution of a given second order homogenous linear differential equation with variable coefficient.

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The method of solution:

$$y'' + P_1(x)y' + P_2(x)y = 0$$

$$y = \sum_{n=0}^{\infty} c_n(x-x_0)^n, \quad |x-x_0| < R$$

$$y' = \sum_{n=1}^{\infty} n c_n(x-x_0)^{n-1}, \quad |x-x_0| < R$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n(x-x_0)^{n-2}, \quad |x-x_0| < R$$

$$\sum_{n=2}^{\infty} n(n-1)c_n(x-x_0)^{n-2} + \left\{\sum_{n=1}^{\infty} n c_n(x-x_0)^{n-1}\right\}P_1(x) + P_2(x)\sum_{n=0}^{\infty} c_n(x-x_0)^n = 0, \quad |x-x_0| < R$$

$$k_0 + k_1(x-x_0) + k_2(x-x_0)^2 + \cdots = 0$$

Identity principle, $k_0 = 0, k_1 = 0, k_2 = 0 \ldots$

A relation between the coefficients c_n is obtained.

$$y = A_1 y_1 + A_2 y_2$$

Now what we do is suppose we are given a differential equation say y double dash + p1(x) * y dash + P2(x) * y = 0. Then what we will do? We will assume the solution to be y = sigma and x = x0 even the ordinary point of this differential equation. Then we will assume the solution to be y = sigma n = 0 to infinity cn(x - x0) rise to the power n and let us say that this series converges in the region mod of x - x0 < R that is it has a radius of convergence R.

So then we will find y dash sigma n = 1 to infinity n times cn(x - x0) rise to the power n - 1 and this series also has the same radius of convergence. We will also find y double dash because we need the value y double dash so sigma n = 2 to infinity n * n - 1 * cn(x - x0) raise to the power n - 2 and mod of x - x0 < R. Now, we shall then substitute the values of y, y dash, y double dash in the given differential equation and then what we will have sigma n = 2 to infinity.

n(n - 1) cn(x - x0) raise to the power n - 2 + sigma okay. n = 1 to infinity n * cn(x - x0) raise to the power n - 1 and this thing is multiplied by P1(x) okay and then we have P2(x) * y. So y is sigma n = 0 to infinity cn(x - x0) rise to the power n okay and = 0. So this is valid for mod of x - x0 < R okay. Then we will collect the coefficients of various powers of x - x0 okay and we will have say k0 + k1(x - x0) + k2(x - x0) whole square and so on = 0 okay.

Then what will happen this k0, k1, k2 will depend on the coefficients cn of the given n of the power series okay. Now the identity principles or the coefficients of various powers of x - x0 will be 0, so k0 = 0, k1 = 0, k2 = 0 and so on okay and once we have the values of k0, k1, k2. Once we put them 0 we will get relation between the coefficient c will be obtained from where we will be able to find the values of the coefficients c1, c2, c3 and so on.

C0, c1, c2, c3 and so on and then support y1 is 1 solution and y2 another solution. Then the general solution will be y = c1 * y1 + or you can say A1 * y1 + A2 * y2. So we will get 2 power series solution and the 2 independent solutions y1, y2 we will have by establishing the relation between the coefficient cn. A 2 power series solution will be obtained and A1, A2 are 2 arbitrary constants.

So y = A1Y1 + A2Y2 will then give the general solution of the given differential equation. Now we shall be solving polynomial the power series solution we shall find for homogenous linear differential equation with variable coefficients. P1(x) and P2(x) will be taken as polynomial coefficients here. So what we do is let us consider the case of the differential equation by

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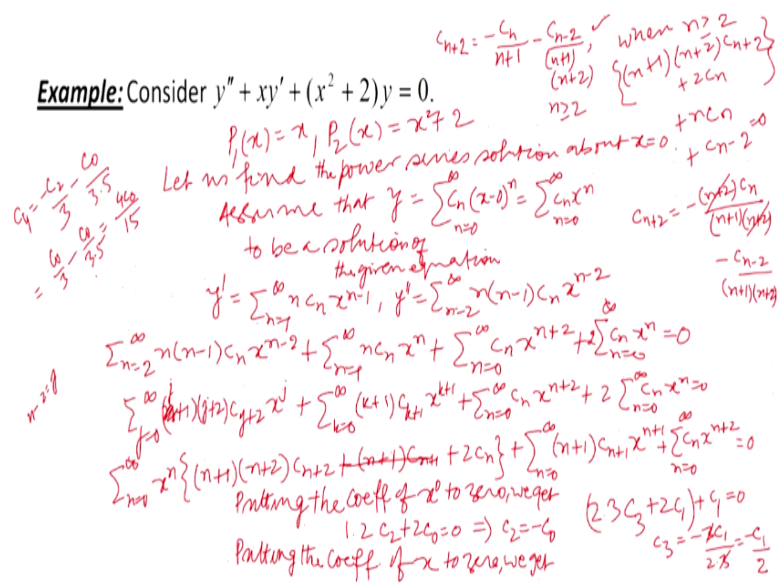

Y double dash + x y dash + x square + 2 y = 0. So what we see here is that P1(x) = x and P2(x) = x square + 2. So both P1 (x) is a polynomial of x and P2 (x) is a polynomial in x of degree 2 so both are analytic functions. The Taylor series converges for all values of x that is the radius of convergence is infinite. So they are analytic functions so every point every x belonging to R is an ordinary point of the given differential equation.

Let us find the solution of this differential equation about x = 0 okay. So let us find the power series solution throughout. We can find this power series solution about any x = x0. For convenience, let us take x0 to be = 0. So let us assume that y = sigma cn x - 0 to the power n. x0 = 0 here that is we have sigma cn * x to the power n, n = 0 to infinity. Let us assume this to be a solution assume that this y = to this to be a solution of the given differential equation.

Okay and let us say R means radius of convergence. So then the series convergences for mod of x < R and R is of course infinity here. So the series converges for all x belonging to R. So what we have. So y dash then = sigma n = 1 to infinity n * cn * x to the power n - 1, y double dash = sigma n = 2 to infinity. n * n - 1 cn * x to the power n - 2. So let us put it here. What do we get? Sigma n = 2 to infinity from the by putting the value y double dash n * n - 1 cn * x to the power n - 2 then x * y dash.

So we multiply x to y dash and we get sigma n = 1 to infinity n * cn x to the power n and then we will multiply x square first to y then we multiply x square to y we get sigma n = 0 to infinity cn * x to the power n + 2 and then we multiply 2 to this. So sigma n = 0 to infinity 2 times cn * x to the power n = 0. Now, this equation holds for all real values of x okay. So what we do? Let us replace n - 2 = y = 0 okay.

So this will give you j = 0 to infinity n + 2 * n + 1 so n + 1 sorry j + 1, j + 2, cj + 2 x to the power j and here we put n - 1 = k so we put k = n - 1 should k = 0 to infinity and k + 1 and then ck + 1 x to the power k + 1 okay and this is sigma n = 0 to infinity cn x to the power n + 2 + 2 times sigma n = 0 to infinity cn x to the power n = 0. You will have to put it like that and then let us collect the coefficient of x to the power n okay.

So summation n = 0 to infinity x to the power n is multiplied by n + 1 * n + 2 * cn + 2. Then from here n + 1 cn + 1 okay. This is wait we are not writing. We are writing the coefficient of x to the power n so not this. So this will be + 2 * cn okay. Sigma n = 0 to infinity x to the power n multiplied to n + 1, n + 2, cn + 2 + 2cn and then we have sigma n = 0 to infinity n + 1 cn + 1 x to the power n + 1 + sigma n = 0 to infinity cn x to the power n + 2 = 0.

Now let us notice that this series I mean this identity holds for all values of x. So the coefficients of powers of x will be = 0. Here 1 x = then we put n = 0, x to the power 0 we get okay. And here when we put n = 0 the least power of x that we get is x and here the least power of x is 2 okay. So first let us put n = 0 so that we get the coefficient of x to the power 0 okay which we put as 0. So when we put n = 0 putting the coefficient of x to the power 0 to 0 we get okay.

So let us put n = 0 so 1 * 2 okay 1 * 2 c2 + 2 c0 = 0 and what we get? c2 = - c0. So we get the value of c2. Now x to the power 0 term is not here, here also it is not there. Now let us put the coefficient of x = 0. We get okay you put n = 1 here to get the coefficient of x so 2 * 3 and then we get n = 1 we are writing so c3 and then we get 2 * c1 okay this or here we will get the coefficient of xy taking n = 0.

So we get 1 and then n = 0 means c1 these = 0 x term is not here. So we get 2 * 3 c3 = - 3 c1. So c3 = - 3 c1/2 * 3 so that means - c1/2. Now we go to the left higher power of x that is x square that is x to the power 2 we get. So we are taking n = 2 now what we get. So all terms now will give the contribution when n takes values > = 2. So when n is > = 2 let us see what we get okay. When n is > = 2 what will happen.

Here the coefficient of x to the power n we are having okay. So, x to the power n so n + 1 okay, n + 2 then cn + 2 then we get 2cn this coefficient of x to the power n. Here the coefficient of x to the power n will be we place n by n - 1. So we get n * cn. Here the coefficient of x power n we will get by replacing n by n - 2 + cn - 2. So this is = 0 when 1 is > = 2 okay so what do we get. cn + 2 okay. So cn + 2 = let us write n + 1, n + 2 okay.

n - n + 2 cn/(n + 1) (n + 2) and then - cn - 2/(n + 1)(n + 2). So this n + 2 cancels and we get cn + 2 = - cn/n + 1 and - cn - 2/(n + 1) (n + 2) and n is > = 2. So take n = 2 now. When we take n = 2 you get the value of c4. C4 will be coming out to be - c2/2 * 3 when we put n = 2 and then - c0/n = 2. So we are getting 3 * 5. Now we have the value of c2 with us. c2 = - c0. So we get c0/3 - c0/3 * 5. So this we can find and this is = 15 so 5c0 - c0 so we get 4 c0 okay.

So this is how we get the value of c4 we can find then c5, c6, and so on from this relation, which is differential relation okay. We know the value of so what will happen here you will have 2 arbitrary constants c0 and c1. C2 depends on c0, c3 depends on c1 okay. So c2, c4, c6 they will come in terms of c0 by c3, c5, c7

They will come in terms of c1 okay and then we write the power series solution y = sigma, n = 0 to infinity cn x to the power n we collect the coefficient of c0 that will give us 1 solution and we collect the coefficient of c1 that gives us the other solution okay and we will have the following situation okay.

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Hence the general solution is

$y = c_1 y_1 + c_2 y_2,$

$$\text{where} \quad y_1(x) = 1 - x^2 + \frac{x^4}{4} + \ldots\ldots.$$

$$\text{and} \quad y_2(x) = x - \frac{x^3}{2} + \frac{3x^5}{40} + \ldots\ldots..$$

$y = c_0 y_1 + c_1 y_2$

So we will have y = c1y1 + c2y2 and the coefficient of c1 will come out to be 1 - x square + x4/4. Now one part here is the coefficient of c1 is 1 - x couples x to the power 4/4. The other part is x - x cube/2 + 3 x to the power 5. Here the powers of x are odd okay. Powers of x are odd means that we have the coefficients. These are the coefficients of c1 and here this is the coefficient of c0 actually. This is c0/1 + c1/2 okay.

So, that we have written as c1y1 + c2y2, but we will have y = c0/1 + c1y2 okay. The coefficient of c0 will be y1(x) 1 - x square + x the power of 4/4 and the coefficient of c1 will be 1 - x2/2 + 3 x to the power 5/40 and so on. And so you can see that this solution y1 (x) is a power series in x and y2(x) is also power series in x and they are not known functions okay. They do not represent known functions.

So solution of this differential equation is not in the and cannot be obtained in the (()) (25:29) form. It is given by the power series which do not represent known elementary functions. The general solutions will be y = c0y1 + c1y2. The coefficient of c0 is y1 that is 1 - x square + x to the power 4/4 and the coefficient of c2 is y2 x which is x - x cube/2 + 3 * x to the power 5/4 t and so on and the functions y1(x) and y2(x) do not represent known elementary functions.

And therefore we can say that the power series solution of the given differential equation cannot be obtained in the form of the finite linear combination of known elementary functions. It is given by power series which does not represent known elementary functions. With that I will come to the end of this lecture thank you very much for your attention.