Ordinary and Partial Differential Equations and Applications
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Lecture – 13
Power Series

Hello Friends, welcome to my lecture on power series.

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A power series about x=a (or centered at x=a) is an expression of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$, where a and c_n are real numbers. The c_n's are called the coefficients of the series.
In the discussion of power series, the convergence of the series is a major question that we have to deal with.
A power series may converge for some values of x and diverge for other values of x.
We shall show that there is a number R so that the power series will converge for |x-a|<R and will diverge for |x-a|>R.
This number R is called the radius of convergence of the series. Let us note that the power series may or may not converge if |x-a|=R.

A power series about a point x = a, centered at x = a, is an expression of the form sigma n = 0 to infinity, Cn (x - a) to the power n, where a and Cn are real numbers. The Cn's are called the co-efficient of the power series. Now in the discussion of power series, the convergence of the series is a major issue. So we will have to deal with this convergence question. A power series may converge for some values of x and diverge for some other values of x.

We shall show that there is a number R so that the power series converges for mod of (x - a) < R and diverges for mod of (x - a) > R. The number R is called the radius of convergence of the series. The power series we shall see may or may not converge for mod of (x - a) = R. That means (a - R) at x = (a - R) and at x = (a + R). The convergence of the power series we will have to checked separately and it may turn out that at these 2 points the power series may or may not converge.

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What happened at these points will not change the radius of convergence. The interval of all x's, including the end points if need be, for which the power series converges, is called the interval of convergence of the series.

The interval of convergence must contain the interval $a-R<x<a+R$, since the power series converges for these values. Thus, to completely identify the interval of convergence, we have to determine if the power series will converge for $x=a-R$ or $x=a+R$. If the power series converges for one or both of these values then we shall include those in the interval of convergence.

Note that the power series converges for $x=a$ because

$$\sum_{n=0}^{\infty} c_n (a-a)^n = c_0.$$

So the but this will not change the radius of convergence. The interval of all x is, including the end points if need be for which the power series converges, is called the interval of convergence of the series. The interval of convergence therefore must contain the interval (a - R) < x < (a + R) that is the open interval a - R to a + R. Since the power series converges for these values thus, to completely identify the interval of convergence we have to determine if the power series convergence for the end points x = a - R or x = a + R of the interval a - R to a + R.

So if the power series converges for 1 or both of these values then we shall include them in the interval of convergence. Now we can see 1 thing that the power series always converges at its center, that is x = a. Because if you put x = a in the series expression sigma n = 0 to infinity Cn (x - a) to the power n, you see that except the term C0 all the other term varies. So sigma n = 0 to infinity Cn (x - a) to the power n at x = a gives C0 okay and therefore the series always converges for x = a.

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Thus, even if the series may not converge for any other value of x, it is guaranteed that it will converge for x=a.

Thus, even if the series may not converge for any of any other value of x, it is guaranteed that it always converges at its center that is x = a. Now how to determine the radius of convergence of the power series, we can apply the ratio test or we can also apply the root test.

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Determining the radius of convergence of a power series:

Theorem: If $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L,$ $\lim_{n\to\infty}|c_n|^{1/n} = L$

where L>0, L=0 or L=∞, then the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence $R = \frac{1}{L}$, where if L=0 then R=∞ and if L=∞ then R=0.

If you want to apply the ratio test, then C is the coefficients of the series are Cn's. We will have to find out the limit of mod of Cn + 1/Cn, if the limit is = L, where L is of course positive because we are taking the limit of positive real numbers, okay. So where L is > 0 or L is = 0 or L could even be infinity and then the power series will have the radius of convergence R = 1/L. So if it gives L = 0, then R will be infinity and if L is = infinity then R will be = 0 okay.

So in some cases we shall see that we need we get better to use instead of this ratio test we can use the root test. That is we can find the limit n tends to infinity mod of Cn raise to the power 1/n. So if this limit is = L, then the conclusion of this theorem will also hold for this root test. So if limit n tends to infinity mod of Cn to the power of 1/n is = L, where L is > 0.

Or L is = 0 or L is = infinity, then the power series sigma Cn (x - a) to the power n, where n varies from 0 to infinity, will have a radius of convergence R = 1/L and so where L is 0 or R is infinity and if L is = infinity then R is = 0. So we can either apply the root test or apply the ratios test this we will decide from the form of the power series.

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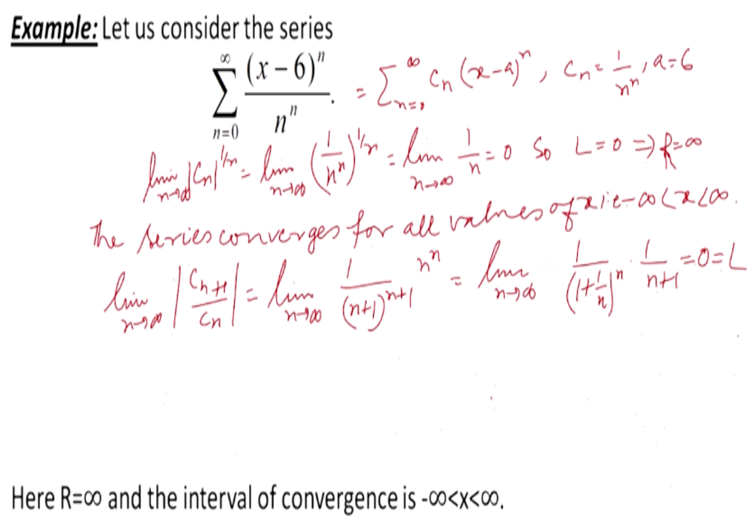

Okay for example let us consider the series sigma n is = 0 to infinity (x - 6) to the power n/n to the power n. So this series is of the form sigma n is = 0 to infinity Cn (x - a) to the power n, here Cn is = 1/n to the power n and a is = 6. So we have the power series with center at x = 6 and the coefficients of series are given by 1/n to the power n. Okay now let us find so if you can see here when you have Cn is = 1/n to the power n it is better to apply the root test.

So limit n tends to infinity mod of Cn to the power 1/n let us find this, so this is limit n tends to infinity 1/n to the power n raise to the power 1/n. So this will be = limit n tends to infinity 1/n which is = 0. And so L is = 0 here. Which implies that R is = infinity okay R is = infinity and as we have seen that these power series always converges at its center. So the given power series converges at x is = 6.

And therefore the series converges for all values of x from - infinity to infinity okay. So radius of convergence in infinity what we find is the conclusion is that the given series converges for all values of x, that is – infinity < x < infinity. Now here you can also to determine the value of L you can also use instead of root test the ratio test. Suppose you want to apply the ratio test, then you will have to find limit n tends to infinity mod of Cn + 1/Cn okay.

So this is limit n tends to infinity now Cn + 1 will be 1/(n+1) raise to the power n + 1. Okay and then Cn is 1/n to the power n so n to the power n/n+1 to the power n. n to the power n. What I will get 1/n + 1 1 + 1/n to the power n * 1/n + 1 right. So when we apply the ratio test, what we have to find limit n tends to infinity mod of Cn + 1/Cn and what we get is 1/n + 1 to the power of n + 1 into the n to the power n as n goes to infinity.

And you can write it as n to the power n/1 + 1/n + 1 to the power n which is 1/1 + 1/n to the power n * 1/n + 1. So as n goes to infinity 1/1 + 1/n to the power n tends to E and the 1/n + 1 goes to 0, so this is = 0 okay. So L is = 0 and therefore R is = infinity. The series converges at x is = 6. So, the series converges for all real values of x. So, we can apply both the tests, the root test as well as ratio test here and we can obtain the value of R and the interval of convergence.

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Example: Let us consider the series

$$\sum_{n=0}^{\infty} \frac{n!(5x+3)^n}{(n+1)^2+4n}. = \sum_{n=0}^{\infty} \frac{n!\,5^n\left(x+\frac{3}{5}\right)^n}{(n+1)^2+4n}$$

Here $C_n = \frac{n!\,5^n}{n^2+6n+1}$ Hence $\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right| = \lim_{n\to\infty} \frac{(n+1)!\,5^{n+1}(n^2+6n+1)}{\{(n+1)^2+6(n+1)+1\}\,n!\,5^n}$

$$= \lim_{n\to\infty} \frac{(n+1)5(n^2+6n+1)}{n^2+8n+8} = \infty = L$$

$$\Rightarrow R = \frac{1}{L} = 0$$

Here R=0 and the interval of convergence is $x = -\frac{3}{5}$.

Now let us consider another example sigma n is = 0 to infinity n factorial (5x + 3) to the power n (n + 1) whole square + 4 to the power n. The given series first will have to be brought in the form of the extended power series. So sigma n is = 0 to infinity and we have to take n factorial 5 to the power n and then (x + 3/5) raise to the power n/(n + 1) factorial sorry (n + 1) square + 4n. Okay so here Cn is = n factorial 5 to the power n then over (n + 1) square so (n + 1) square means n square + 2n + 1 so n square + 6n + 1.

Okay now hence limit of mod of Cn + 1/Cn. Okay, so n + 1 factorial 5 to the power n + 1/n + 1 whole square + 6 * (n + 1) + 1 * 1/Cn so n square + 6n + 1 and then we have n factorial and 5 to the power n okay. So this is how much? Now let us see (n + 1) factorial/n factorial is n + 1, 5 to the power n + 1/5 to the power n is 5 and we have then n square + 6n + 1/n + 1 whole square means n square + 2n + 1 + 6n + 6 + 1 so means we have n square + 8n and then we have 1 + 6 + 6, 1 + 6 + 1 so we have 8.

Okay now let us see there is a polynomial an of degree 2 and the denominator is also polynomial an of degree of 2. So n square + 6n + 1/n square + 8n + 8 will tend to 1 as n goes to infinity and n + 1 will go to infinity so the limit is infinity okay. And therefore L is = infinity here, which implies that R is = 1/L, R is = 0. So R is = 0 and since we know that this series always converges at its center. So the series converges for x is = – 3/5. So the given series converges only for x is = – 3/4 – 3/5 and nowhere else okay.

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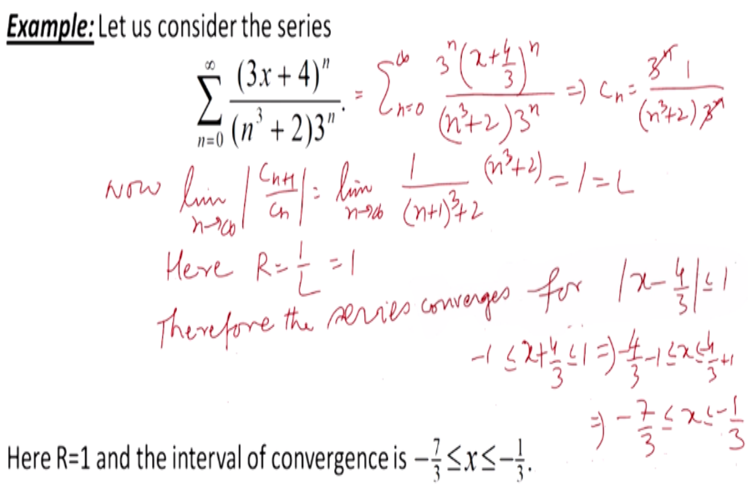

Let us now consider the series sigma n is = 0 to infinity (3x + 4) to the power n/n cube + 2 * 3 to the power n. So here again we let us write in the standard form n is = 0 to infinity and 3 to the power n and then we have x + 4/3 raise to the power n/n cube + 2 * 3 to the power n okay. So this gives Cn is = 3 to the power n/n cube + 2 * 3 to the power n. So we get 1/n cube + 2. Now let us find limit n tends to infinity mod of Cn + 1/Cn, so this is limit n tends to infinity 1/n + 1 whole cube + 2 and here we have n cube + 2.

Okay, so numerator is a polynomial an of degree 3, denominator is also polynomial an of degree 3. So we see that limit is = 1. Okay so here radius of convergence is R = 1/L, this is L. So L = 1 so R = 1. So radius of convergence is = 1 and therefore the series converges for mod of x – 4/3 < = 1 okay.

Or I can say x – 4/3. - 1 is < = x – 4/3 < = 1 which gives you but 4/3 – 1 < = x < = 4/3 + 1. So what we get is this is x + 4/3 yah because the center is – 4/3. So this is + 4/3. So I get here – 4/3 here also – 4/3. So this is – 7/3 < = x < = – 1/3. So this is the interval of convergence.

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For any x in the interval of convergence, the sum of the series defines a function

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n .$$

Then the question arises whether f is continuous, differentiable etc.?

Can the series be differentiated or integrated term by term etc.?

The answers to these questions depend on whether the series converges uniformly?

Okay now for any in the interval of convergence, the sum of the series defines a function of x which, let us denote by f(x). f(x) is = sigma n is = 0 to infinity Cn x - a to the power n. Now the question arises whether some function f(x) is continuous or it is differentiable? Whether the series can be differentiated term by term or integrated term by term? So the answered to these questions lie in the uniform convergence of the series. Whether the series converges uniformly and where? Okay. So let see we have to answer this question.

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Theorem: A power series converges uniformly to its limit in the interval of convergence.

Proof: Let us assume the power series to be $\sum_{n=0}^{\infty} c_n(x-a)^n$ and let R be its positive radius of convergence. Let us choose a no. say ρ such that $0<\rho<R$.

For any x in the interval $|x-a|<\rho$, let us see

$|c_n||x-a|^n < |c_n|\rho^n = M_n$

Then the series $\sum_{n=0}^{\infty} M_n$ is a convergent series by Root test $\lim_{n\to\infty} M_n^{1/n} = \lim_{n\to\infty} |c_n|^{1/n}(\rho^n)^{1/n} = \lim_{n\to\infty} |c_n|^{1/n}\rho = L\rho = \frac{\rho}{R}$

Since $R>0$, $\frac{1}{R}$ is a finite quantity

By Weierstrass M-test, the series $\sum c_n(x-a)^n$ converges uniformly in $|x-a|<\rho$ or $a-\rho<x<a+\rho$ since ρ is arbitrary

$\Rightarrow$ the given series converges uniformly in $(a-R, a+R)$

The theorem, which answer this question is, a power series converges uniformly to its limit in the interval of convergence. So this means that if f(x) will be some function of the series and the radius of convergence of the power series is R then the center of the series is a then the series always converges uniformly in the open interval a – R < x < a + R okay. So let see how we proved this. So let us assume that the power series given by and let R be its radius of convergence.

Okay now let us choose a number, let R be we are taking the radius of convergence to be positive okay. Because if R = 0 then the series converges only at the center, so the question of uniform convergence does not arise. So let us take R to be a positive number. So let R be its positive radius of convergence that is R is > 0 okay. So let us choose a number say rho such that 0 is < rho < R. So then what we do is let us take in x okay, so let say this is your interval 0 to R.

Okay these are interval 0 to R. We take a positive number rho which lies in between 0 and R okay so rho is > 0 but < R. Now let us choose a point x okay such that okay lets then okay we can say for any x in the interval in the interval x – sorry we have mod of x – a < rho. Okay let us see what happens, let us see what happens to the power series okay. So convergence of power series so mod of Cn x mod of x – a raise to the power n is < mod of Cn rho to the power n.

Okay let us take the nth term of the power series which is Cn x – a to the power n. Then we noticed that the nth term of the power series modulus of that mod of Cn mod of x – a to the power n for any x in the interval mod of x – a < R satisfy this equality that is < mod of Cn * rho to the power n. Let me take it as Mn okay. Then the series sigma n = 0 to infinity Mn is a convergent series. Let us see why it is a convergent series.

We can apply ratio test okay Mn is mod of Cn raise to power 1/n I am sorry we can apply the root test. So by root test limit n tends to infinity Mn to the power 1/n is = limit n tends to infinity mod of Cn raise to the power 1/n * rho to the power n whole to the power 1/n. So this is limit n tends to infinity mod of Cn raise to the power 1/n into rho. Now we know that the radius of convergence of the given power series R.

So limit n tends to infinity mod of Cn raise to power 1/n is = L. So this L * rho, but L is = 1/R so this rho/R. Okay rho/R means a positive. Because we are taking R to be > 0, so since R is > 0. Okay rho/R is a finite quantity. It is a finite quantity. It is a finite quantity and therefore the series sigma n = 0 to infinity Mn converges by root test. So this means that by Weierstrass M test okay by Weierstrass M test.

The series sigma n = 0 to infinity Cn x – a raise to the power n okay convergent uniformly in the interval mod of x – a < rho okay that means a – rho < x < a + rho. Now since rho is arbitrary. Rho is any number which is > than 0 but < R okay we can take the rho as close as R as possible okay this implies that the given series converges uniformly in the interval a – R to a + R. So the power series uniformly to its some function in the interval of convergence that is a – R to a + R, if a is the center of the series.

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Hence we can conclude that the limit function f(x) is continuous in (a-R, a+R).

Corollary: Suppose $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges uniformly for $|x-a| < R$. Then, if $-R < \alpha < \beta < R$ we have

$$\int_{\alpha}^{\beta} f(x)\,dx = \sum_{n=0}^{\infty} c_n \int_{\alpha}^{\beta} (x-a)^n\,dx.$$

Okay now so we conclude that the limit function f(x). Now we know from the theory of functions of real variable that if infinite series are functions of x converges uniformly over an interval I, then some functions of the series is continuous so the limit function f(x) will be continuous in a – R to a + R. Now let us suppose we can now state this corollary, suppose f(x) is = sigma n = 0 to infinity Cn x – a to the power n converges uniformly in the range mod of x – a < R.

Then, if mod R if alpha beta is a close interval, which is lying in mod of x – a < R. okay, then if – R < alpha < beta < R we have integral alpha to beta f(x) dx = sigma n = 0 to infinity. So we can integrate the function f(x) over the interval alpha beta okay. This we know again from the theory of functions of real variable that if we have infinite series which converges uniformly over a close interval I then the n the sum function is f(x).

Then integral over I f(x) dx is = sigma n = 0 to infinity integral over I of the nth term of the series. So here integral over alpha beta f(x) dx will be sigma n = 0 to infinity Cn integral over alpha to beta x – a to the power n dx. So the term series can be integrated term by term over any close interval alpha beta contended mod of x – a < R. So that is what I have to do in this lecture, thank you very much.