Ordinary and Partial Differential Equations and Applications
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Lecture - 11
Solution of Homogeneous Linear System with Constant Coefficients-III

Hello friends, welcome to my lecture on solution of homogeneous linear system with constant coefficients. We shall discuss the case of equal roots of the characteristic equation in this lecture. So if the characteristic polynomial of a matrix A does not have n distinct roots then A may not have n linearly independent eigen vectors, for example let us consider the matrix A = 110, 010, 002.

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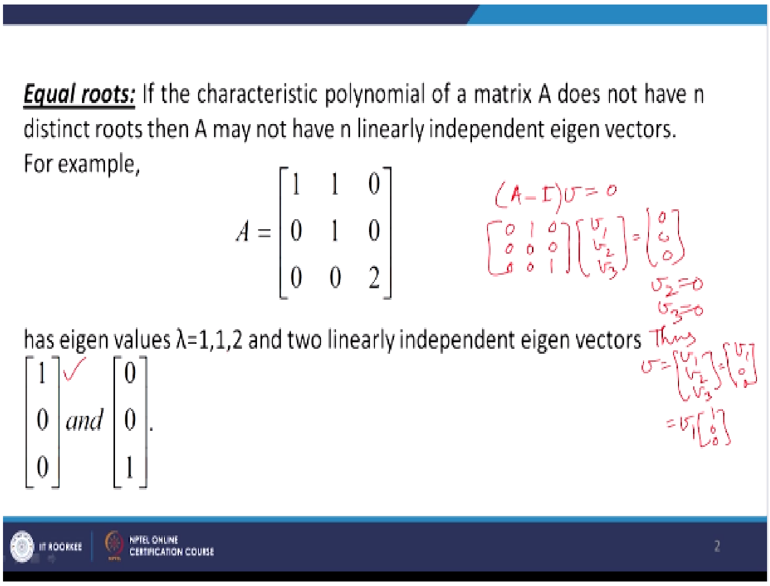

Then you can see that it is an upper triangular matrix, it is diagonal elements are therefore it is eigen values so the diagonal elements are 112 and therefore it has eigen values 112 okay. Now corresponding to eigen value lambda = 1 if you find the eigen vector then what you get. A-I * v = 0 so you subtract identity matrix from A and what you get is 010, 000 and then 001. So this is v1, v2, v3 and you get 000 here and what you get is okay.

So v1*0 v2*1 + v3*0 = 0 so you get v2 = 0 and then second equation is 0 equation, third equation gives you v3 = 0 and thus v = v1 v2 v3 reduces to v100 okay, or we can write it as v1 times 100, so taking v1 = 1 you have an eigen vector 100 corresponding to eigen value lambda = 1 and for lambda = 2 similarly you can find an eigen vector 001. So you see here that we do not have 3 linearly independent eigen vectors corresponding to the eigen value.

Eigen values 112 only 2 linearly independent eigen vectors we have so how to find n linearly independent solutions of the vector differential equation x dot = Ax that is the question, okay. So thus if you consider the vector differential equation x dot = Ax it will have only 2 linearly independent solutions of the form e to the power lambda t corresponding to lambda 1 = 1 we got one vector which is 100.

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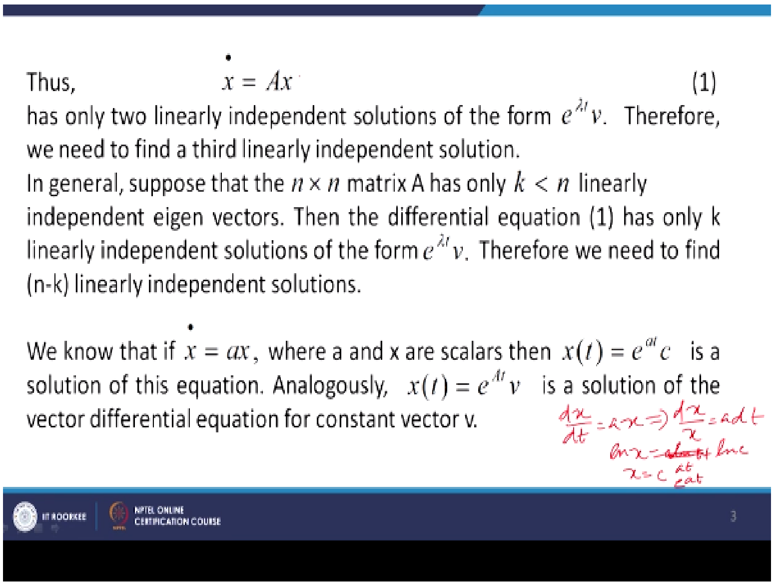

So one solution you get and then corresponding to lambda = 2 you got one eigen vector 001, so you get another linearly independent solution of the form e to the power lambda t * v. Now therefore we need to find a third linearly independent solution because in order to write the general solution of the vector differential equation x dot = Ax we need to have n linearly independent solutions of this equation.

And here A is the 3 * 3 matrix so we need 3 linearly independent solutions. Now in general suppose that the n * n matrix A has only k < n linearly independent eigen vectors then the differential equation x dot = Ax has only k linearly independent solutions of the form e to the power lambda t * v. Therefore, we need to find n – k linearly independent solutions in order to write the general solutions of the vector differential equation x dot = Ax.

We know that if you take these scalar first order differential equation x dot = Ax where A and x are scalars then xt = c times e to the power At is the solution of this equation that you can see it easily, you have dx/dt = Ax, so that gives you dx/x = Adt and so when you integrate we get lnx = lnt A times At + constant let us take lmc then you get x = c times you have, so lnx = At + lmc so x = c times e to the power At.

So x = c times e to the power At is the solution of this equation. Analogously x t = e to the power At * v is the solution of the vector differential equation x dot = Ax for a constant vector v. Let us see how it is the solution of this vector differential equation. First of all let us see the definition of e to the power At.

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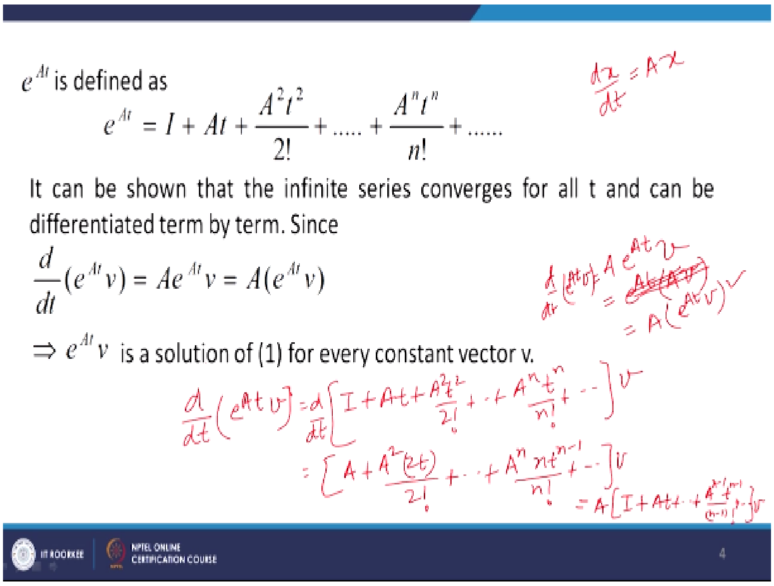

E to the power At where A is n * n matrix, is defined as I + At + A square t square/2 factorial and so on and A to the power n t to the power n * n factorial and so on where I is an identity matrix of the same order as A that is I is the identity matrix of order n. Now it can be shown that the infinite series which is giving e to the power At it converges for all t and can be differentiated term by term.

Now if you find d/dt of e to the power At * v what you get you see d/dt of e to the power At * v so this is = d/dt of e to the power At means I + At + A square t square/2 factorial and so on. A to the power n, t to the power n/n factorial and so on * v where v is the constant vector. So when we differentiate with respect to t what we get? This is independent of t so this will be 0.

So we get At when we differentiate this with respect to t we get A + A square * 2t/2 factorial * 2t/2 factorial and then so on A to the power n * n, t to the power n-1/n factorial and so on v, okay, or I can write it as A * I + here we have At then we have A to the power n-1, t to the power n-1/n-1 factorial and so on v, okay, so this is e to the power At. So we get A * e to the power At * v that is the derivative of e to the power At * v.

So d/dt of e to the power At * v is A times e to the power At * v and since we can write it as e to the power At okay * Av so what we get, we have to write it as this is A e to the power At * v. So what we have d/dt of e to the power At * v is A * e to the power A2 * v and therefore our differential equation is dx/dt = Ax. So in this if we replace x by e to the power At * v what we get is d/dt of e to the power At * v = A times e to the power At * v that is what we get here.

So e to the power At * v is the solution of the vector differential equation 1 for every constant vector v.

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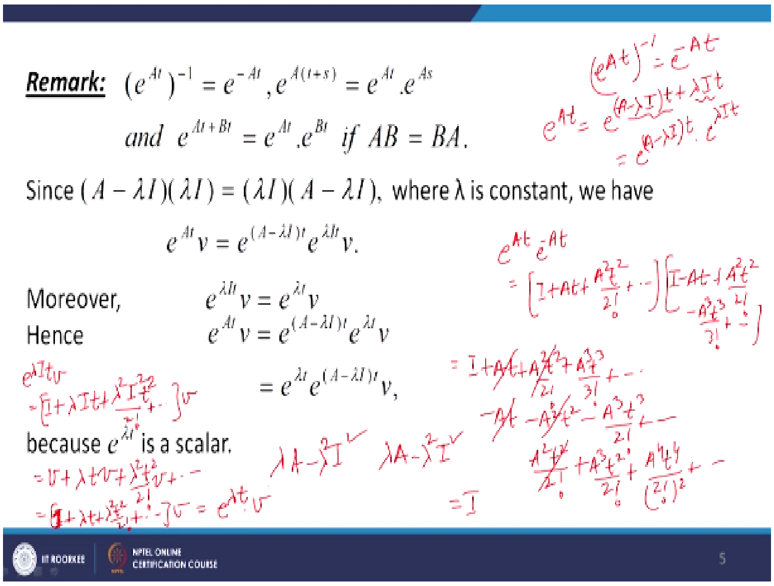

Now we can easily show that e to the power At inverse = e to the power –At and e to the power At + s = e to the power At * e to the power As, e to the power At + Bt = e to the power At * e to the power Bt if AB = BA. This we can see very easily. See e to the power At * e to the power –At if you write what we get I + At+ A square t square/2 factorial and so on and then we multiply by e to the power – At so on I – At + A square, t square/2 factorial – A cube t cube /3 factorial and so on, okay.

So we multiply by I first so I + I At + A square, t square/2 factorial and so on then –At. So we get –At then we multiply –At here so –A square t square then we multiply –At here so –A cube t cube/2 factorial and so on then you multiply by A square t square/2 factorial so you get

A square t square/2 factorial okay, then A cube t square/2 factorial and then you multiply by A square t square so A4 t4/2 factorial square and so on and you get like this.

So At will cancel with At A square t square/2 factorial + A square t square/2 factorial is A square t square, so it will cancel with A square t square. Similarly, A cube t tube/3 factorial okay –A cube t cube/2 factorial and then –A cube t cube/3 factorial so what we will get all the terms will cancel except I and what we get is e to the power At * e to the power –At = I. So we can say that e to the power At inverse is = e to the power –At.

And similarly if you multiply the series for e to the power At and then the series for e to the power At you will get the series for e to the power At + s, but her when you multiply e to the power At * e to the power Bt then you need AB = BA in order to get e to the power At + Bt. So e to the power At + Bt is true only if AB = BA, so e to the power At + Bt is e to the power At * e to the power Bt provided AB = BA.

Now let us look at these 2 matrixes A-lambda I * lambda I same as lambda I * A-lambda Iy because A lambda I is lambda A okay and then – lambda * lambda is lambda square I square so we get lambda square I so lefthand side is this and righthand side is what, lambda * IA that is lambda * A and then lambda I * lambda I is –lambda I square I square so lambda I square I. so lefthand side and righthand side are same.

So A- lambda I * lambda I = lambda I * A-lambda I, so A-lambda I end, lambda I commute okay where lambda I is constant. Now using this property e to the power At * B is e to the power A - lambda I * t + lambda It, okay, e to the power At = e to the power A - lambda I t + lambda It. Now use this property, A - lambda I and lambda I since they commute okay, we can write it as e to the power A - lambda I * t * e to the power lambda It.

So e to the power At can be written as e to the power A - lambda * t * e to the power lambda It because A - lambda I and lambda I commute and we are using this property e to the power At + vt = e to the power At* e to the power vt. Now e to the power lambda It = e to the power Lambda t * v this we can see very easily e to the power lambda It.

Let us write e to the power lambda It * v this is equal to we have I + lambda I t + lambda square I square t upon 2 factorials and so on * v. So what you get is so this is v + okay I v = v and then lambda t * v I v = v again then lambda square I square t square here okay, so lambda square t square I square * v is v so lambda square t square/2 factorial * v and so on or I can write it as scalar 1 + lambda t + lambda square t square/2 factorial and so on * v.

Which is = e to the power lambda t * v, okay. So e to the power lambda I t * v is e to the power lambda t * v and therefore e to the power Atv = e to the power A - lambda It * e to the power lambda t * v. Now e to the power lambda t is a scalar quantity so we can write it outside this matrix, e to the power lambda t * e to the power A - lambda I t v okay. So ultimately what we have e to the power Atv = e to the power lambda t * e to the power A -lambda It * v, okay.

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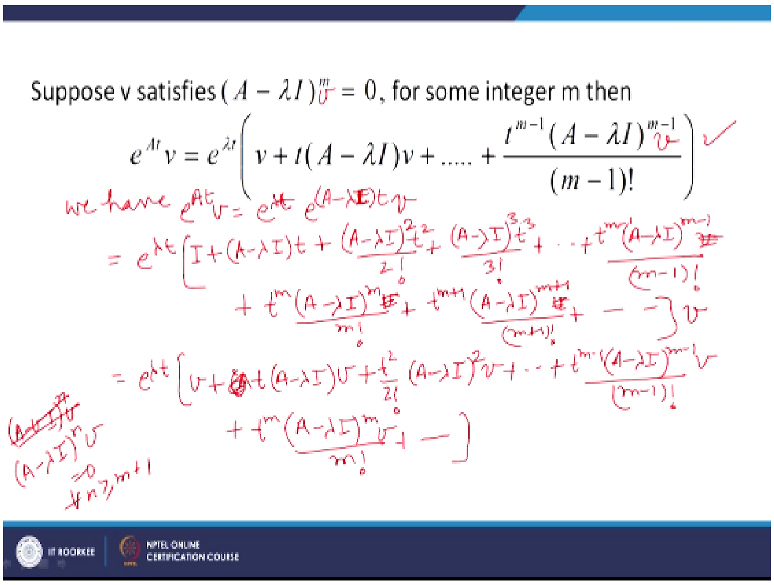

Now suppose A - lambda I mv = 0 for some integer m then e to the power Atv = e to the power lambda t * v + t times A - I v + and so on t to the power m - 1 A - lambda t raise to the power m – 1 * v and then m-1 factorial, this we can see from the previous equation we have e to the power At * v = this okay, so we have e to the power A - lambda It * v. This is what we have, e to the power At * v = e to the power lambda t * e to the power A - lambda I * t operating on v.

So this equal to e to the power lambda t and let us write the expansion series for this, so I + A - lambda I t + A - lambda I whole square t square/2 factorial A - lambda I whole cube t cube/3 factorial and so on, okay, t to the power m A - lambda I raise to the power m-1 v upon m-1 factorial and then we have t to the power m A - lambda I to the power m v upon m factorial and then so on, t to the power m+1 A - lambda I to the power m+1 v upon m+1 factorial and so on.

So here I should not have written v, v we can write outside here, okay. So then what happens is when this series operates on v, you get v + t times A - lambda I * v you get e to the power lambda t v + t times A - lambda I v + t square/2 factorial A - lambda I whole square v and so on. We get t to the power m-1 A-lambda I to the power m-1 v upon m-1 factorial and then we have t to the power m A-lambda I to the power m v upon m factorial and so on.

Now if v satisfies A - lambda I m v = 0 for some m for some positive integer m then what will happen this term will be 0, t to the power m * A - lambda I to the power mv/m factorial = 0 and after this we will have t to the power m+1 A-lambda I to the power m+1v upon m+1 factorial that term will also be 0 because you can operate on this equation by A-lambda I so that you get A-lambda I to the power m+1 v =0.

So if A-lambda I to the power n * v = 0 for all n >= m+1, so this A-lambda I to the power m * v = 0 implies this = 0 for all and >= m+1 so this series will then reduce to this series okay. All terms starting with the power of t as m and more will vanish and we will have e to the power At*v = e to the power lambda t v + t times A-lambda I * v and so on t to the power m-1 A-lambda I to the power n-1 v/m-1 factorial.

So there will be only m terms inside the bracket. Now let us see how we can find n linearly independent solutions of the vector differential equation x dot = Ax.

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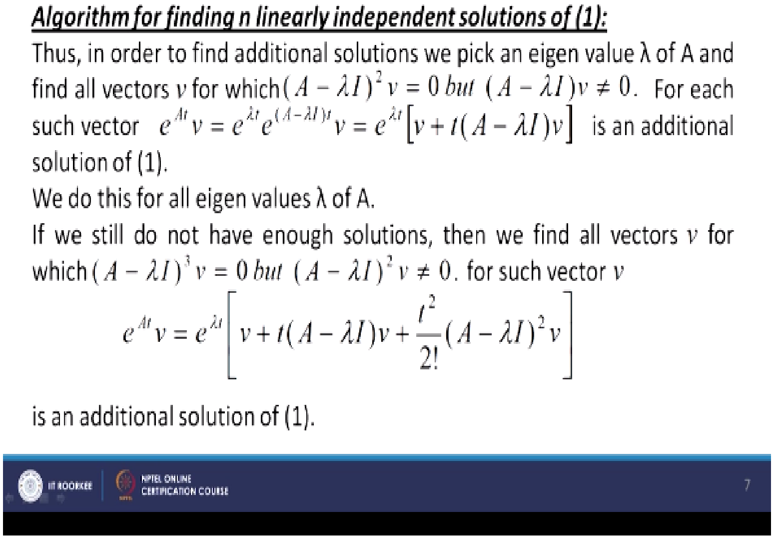

So in order to find additional solutions suppose we have k solutions in linearly independent solutions in order to find n–k solutions we pick an eigen value lambda of A and find all vectors v for which A-lambda I whole square v = 0, but A-lambda I v is != 0 okay. So for each such vector e to the power At * v = we have seen just now e to the power At * v = e to the power lambda t * e to the power A-lambda * t * v will reduce to e to the power lambda t * v + t times A-lambda I * v.

Because we are assuming that A-lambda I whole square v = 0 so here m = 2 and this means that only v + t times A-lambda I * v will remain inside the bracket all other terms will vanish, So we will have e to the power At*v = e to the power lambda t * v + t times A-lambda I * v this will give us an additional solution of x dot = Ax. Now we do this for all eigen values lambda of A.

If we still do not have enough solutions that is we do not get all n solution then we find all vectors v for which A-lambda I raise to the power 3 v = 0, but A-lambda I whole square v is ! = 0. For such vector v, e to the power At * v will then give you e to the power lambda t * e to the power A-lambda I * t * v which will reduce to this okay, so this e to the power A-lambda I * tv will give you this expression inside the bracket.

All other terms will vanish so e to the power At * v will be e to the power lambda t v+t times A-lambda * v + t square/2 factorial A-lambda whole square v which will give us an additional solution of x dot = Ax.

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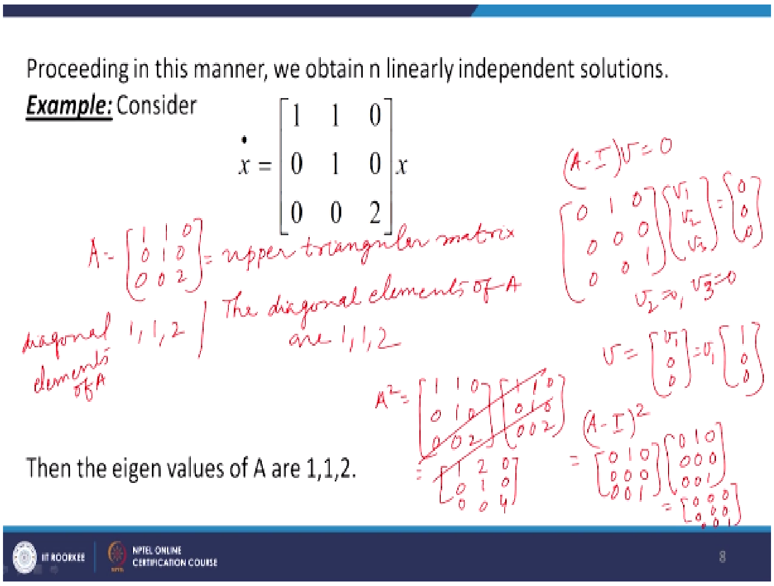

Now proceeding in this manner we obtain n linearly independent solutions of the equation x dot = Ax. So let us now take an example where x dot = Ax is given, A is a 3 x 3 matrix, so our aim will be to determine 3 linearly independent solutions of this vector differential equation, so here A = 110, 010 and 002, okay. Let us note that A is a upper triangular matrix so the Eigen values of A are the diagonal elements of A.

The diagonal elements of A are 112 these are the diagonal elements and this is an upper triangular matrix. So the diagonal elements are the eigen values of A okay. The diagonal elements of an upper triangular or lower triangular matrix are it is eigen values. The diagonal elements of A are 112 so we have the eigen values of A 112. Now let us find the eigen vector for lambda = 1 so we have to consider A-I v =0.

We have this 110, let us find A-I v = 0 so what we will get this will give you 010, 000 and then 001. So here you can see v2 = 0 and v3 = 0 so we get the eigen vector v as v100 and you can take it as 100 so we get corresponding to eigen value lambda = 1 we get an eigen vector 100.

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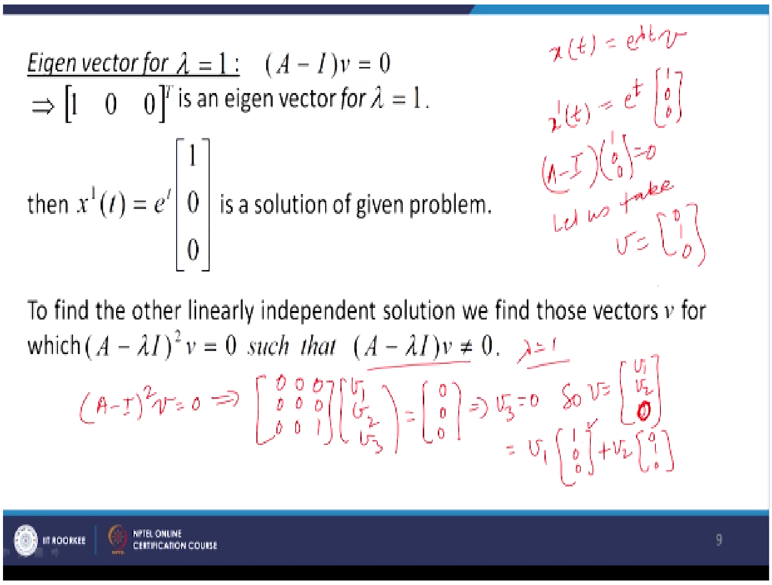

And therefore we can have one solution of the equation x dot = Ax we know that xt = e to the power lambda t * v, okay, if lambda is an eigen value and v is the corresponding eigen vector of the matrix A then xt = e to the power lambda t * v is the solution of x dot = Ax. So one solution x1t will be = lambda is 1 here so e to the power t and the eigen vector we have taken as 100 so we have 1 solution e to the power t * 100.

Now to find the other linearly independent solution, you know one solution we shall get corresponding to lambda = 2 okay, but corresponding to lambda = 1 we have got only one linearly independent solution so let us find one more solution corresponding to lambda =1. So what we do is to find the other linearly independent solution. We find those vectors v for which A-lambda whole square v = 0 so that A-lambda I v is != 0 where lambda = 1.

This we are doing for lambda = 1. So what we will do is let us find A-lambda I whole square. Lambda I is 1 here so A-I whole square v = 0 gives you if you write A-I matrix so A square will be = 110, 010, 002 and you multiply 110, 010, 002 so what you get when you multiple 100 here what you get is 100. When you multiply 110, when you multiply you get 1 + 1 that is 2 and 110 here we get 1*00, 1*11 okay.

And then 110 you multiply 002 so we get 0 and then 002 when we multiply we get 0 here and 002 we multiply here we get 0 and here we multiply we get 4. So A square, this is A- oh I do not need this I need A-I whole square, sorry I need A-I whole square, so A-I whole square is 010, 000, 001 and 010, 000, 001 so let us see what it is. First column is 000, so when you multiply to the all rows here you will get the first column as 000 and then 100 you multiply so 100 will give 0 here, 100 will give 0 here, 100 will give here again 0.

And 001 you multiply you get 0 here you multiply you get 0 and here you multiply you get 1, okay, so we get this matrix. So A-I lambda A-I whole square v =0 gives you 000, 000, 001 v1, v2, v3 = so what we get v3 = 0 and so v = v1, v2, v3. So v1, v2 are arbitrary and we get v3 as 0 here, okay, now I can write it as v1 times 100 + v2 times 010, so one solution for A-lambda I whole square v can be taken as 100.

The other solution can be taken as 010, but if you take 100 then we have seen A-I 100 = 0 because A-Iv = 0 gives a solution 100, so A-I 100 = 0 so this condition will be not valid if we choose 100, so let us choose 010, okay, so let us take solution of A-lambda I whole square v = 0 such that A-lambda I v != 0 as 010, okay.

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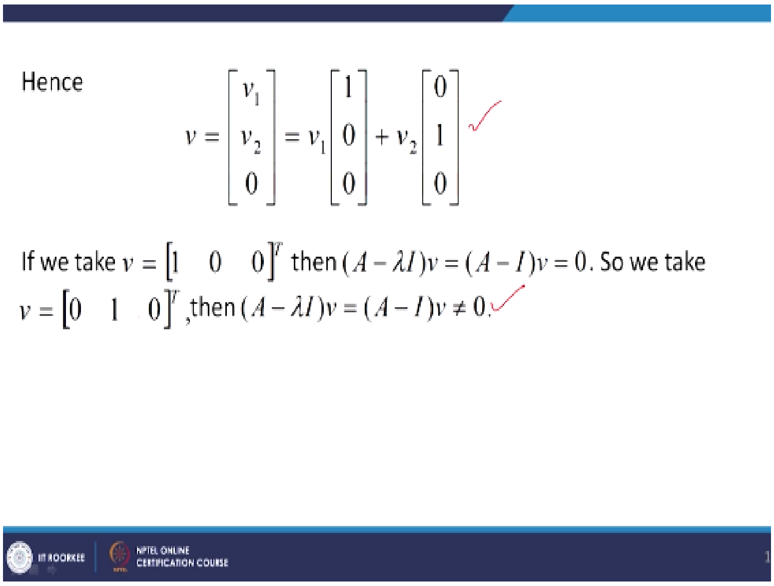

So we take the solution and yeah this is what I have written v = linear combination of these 2 vectors and if we take this 100 transpose then A-lambda I v = 0 so we take 001 transpose and then A-lambda I v which is A-lambda I v is != 0 this you can check A-lambda I v is != 0 if you take v to be 001 vector, okay.

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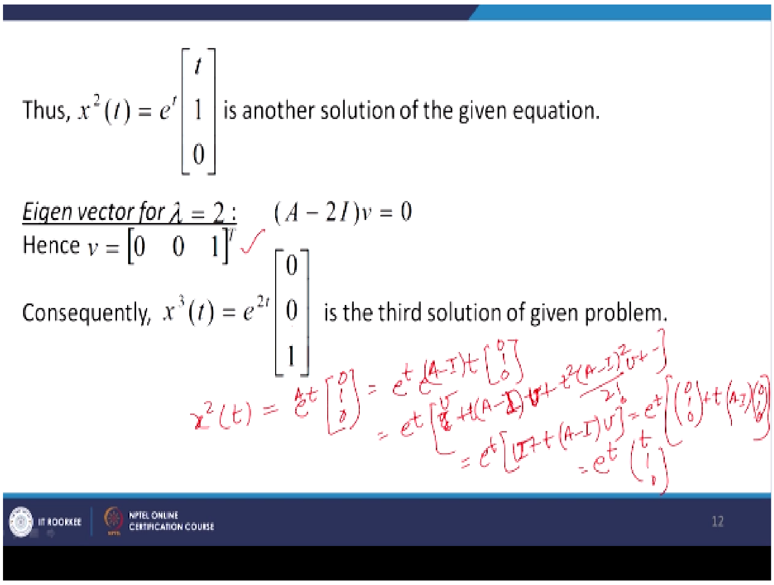

Thus x2 t = e to the power lambda t lambda is 1 here and then what you have okay, x2t let us find, so x2t we have to find, okay, so in order to find x2t what we have e to the power lambda t that is e to the power t * we have taken 010 okay, so 010 now e to the power t we can write as e to the power t, e to the power A-I * t, we have x2t = e to the power t * e to the power A – okay, so this is e to the power At.

Because x2t is e to the power At * 010 this is a solution of this, so x to the power A-I * t this is e to the power t so I + A-lambda I * 010 also we have to write so A-lambda It + and then A-I whole square, this is v here and this is t here, so t square A-I whole square/2 factorial v and so on this reduces to e to the power t I + t times A-I * v and here also v, here also v, okay, so what we get, e to the power t then we have v is 010 so 010 + t times.

So you put the value of A-I here t times A-I * v, v = 010, what we get is e to the power t times t10. So e to the power t * t10 is another solution of the equation and for lambda = 2 we have the equation A-2I v = 0 if we solve this we will get v = 001 transpose, so third solution is x3t = e to the power 2t 001 so thus we have 3 linearly independent solutions.

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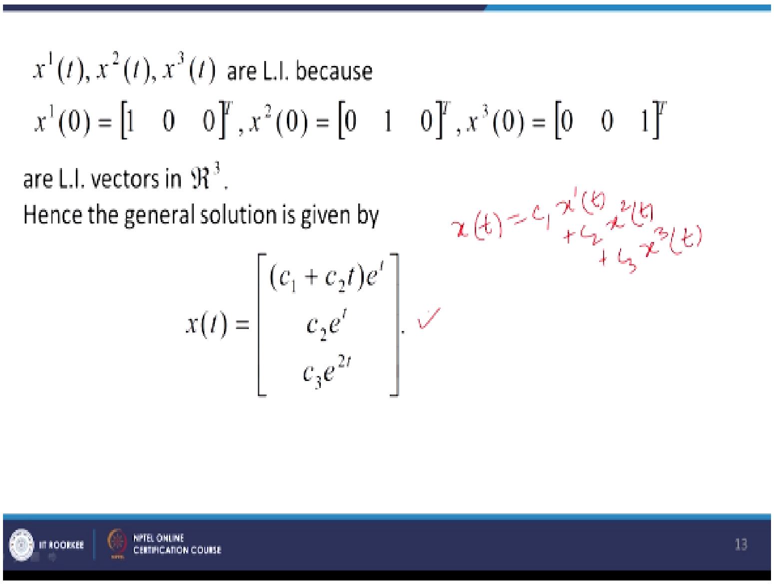

How they are linearly independent see x1t, x2t, x3t are linearly independent because x10 is 100, x20 is 010 and x30 is 001, you can see, x30 is 001, x20 is 010 and x10 is 100. So these 3 vectors are linearly independent vectors in R cube and therefore x1t, x2t, x3t are linearly independent. Now so the general solution we can write xt = c1 times x1t + c2 times x2t + c3 times x3t and when you add the 3 column vectors what you get is this column vector, okay.

So this is the general solution of the given vector differential equation. Now in the case of this initial value problem x dot = 213, 02-1, 002 x x0 = 121 you can see that lambda = 2 is an eigen value of A with multiplicity 3.

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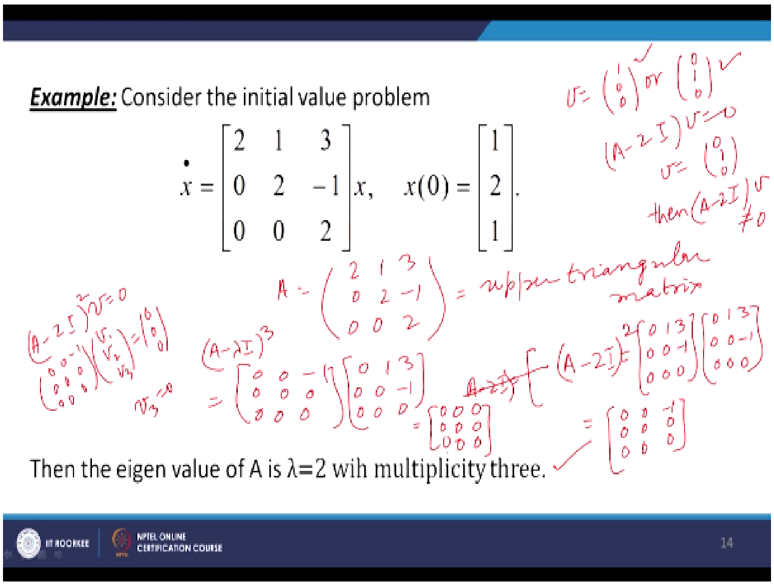

A = 213, 02-1, 002 it is an upper triangular matrix so it is eigen values are it is diagonal elements and which are 222 so eigen values of A are 222, okay that is lambda = 2 occurs with multiplicity 3.

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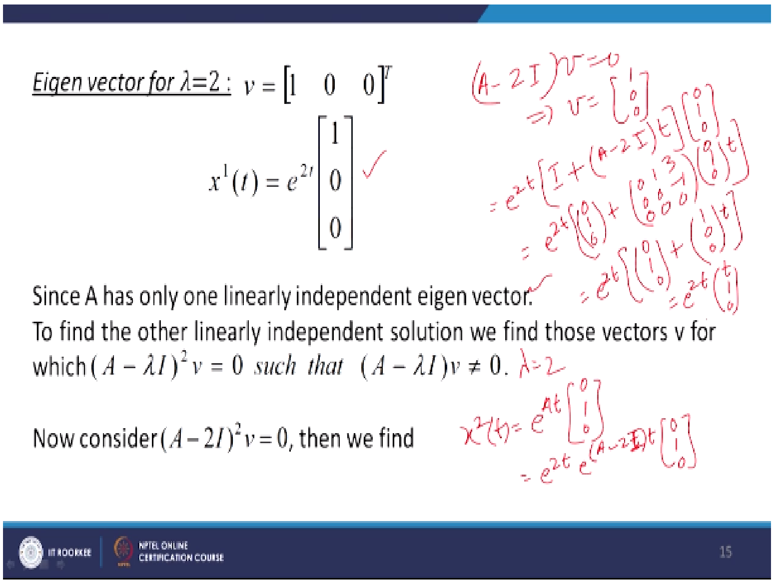

Now eigen vector for lambda = 2 if you find then you will see that A-2I v = 0 gives you solution by v = 100, so e to the power 2t 100 is 1 linearly independent solution of x dot = Ax. Since A has only one linearly independent eigen vector we get one solution, to find other linearly independent solution we find those vectors v for which A-lambda I whole square v = 0 such that A-lambda I v is != 0, lambda = 2 here.

So now consider A-2I whole square v = 0 then we find, so A-2I is this 213 okay, so A-2I will give you what, A-2I will give you 013, 00-1, 000 and when we multiple 013, 00-1, 000 what we get. So this will give you 000 first column is 000 when it is multiplied 2 rows will give you first column as 0 then 100, 100 will also give 000 and then 3-10 will give what, 3*00-1*1 is -1 and then 3-10 will multiply to this.

This is 0, this is 0, this is 0 and this is also 0 so what we get A-2I whole square this is, okay. So A-2I whole square v = 0 if you do then what you will get? So we get v3 = 0 that means v1 and v2 are arbitrary okay. So v1 and v2 are arbitrary means we can take v as 100 or we can take v as 010 okay. Now corresponding to lambda = 2 the one solution that we have got is what 100. So if you choose 100 here then A-2I v will be = 0.

Okay, because we have already seen that so let us choose 010 okay, corresponding to lambda A-2I square v = 0 and A-2I v != 0 so if you choose v as 010 then A-2I v will not be = 0 so we can consider this vector okay, and when you do this then what will be x2t, so x2t will be = e to the power At * v, v is 010 we are taking okay, so e to the power 2t and then e to the power A-2t * A-2I t 010 so this gives you what?

E to the power 2t times I + A-2I t and then 010, okay, so we get e to the power 2t 010 okay + A-2I, A-2I is the matrix 013 and then we have 00-1 and then we have 000 * 010 * t, okay so this is what we get. So e to the power 2t 010 and then 010 when you multiply here what you get 010 you multiply so you get 01 here and 010 when you multiply here you get 0, 010 you multiply here you get 0 and you get t here.

So what you get e to the power 2t t10, so one solutio is this we have got okay.

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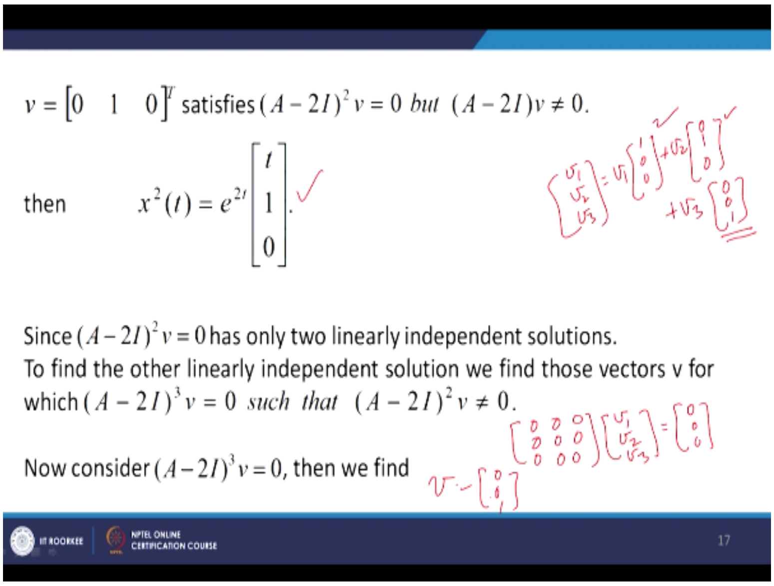

Now so we have x2t = e to the power 2t t10 this is one another solution. Now we have to still get one more solution because A-2I whole square v has given only 2 linearly independent solutions so let us consider A-2I whole cube v = 0. So when you consider A-2I whole cube v = 0 such that A-2I square v is != 0 then A-2I whole cube v = 0 gives you what? A-2I whole square is this.

If you multiply again A-2I what you get 00-1, 000 and 000 and we multiply to this 013 and we multiply 00-1, 000 okay so this is A-lambda I whole cube. So what do you get first column vector is 0 so vector, so we get 0. Second 100 gives you 0 and third 3-10, 3-10 when you multiply here you get this okay, so A-lambda I whole cube v = 0 okay, A-2I whole cube v = 0 when we do what we get 000, 000, 000 and then v1, v2, v3 = 0.

Okay, so v1, v2, v3 are arbitrary vectors here okay, so now we can consider then v1, v2, v3 is the linear combination of 100, 010 and 001, okay, so 100 we already have got 100 vs solution of A-2I * v = 0, 010 we got as a solution of A-2I whole square v = 0 so here we want that vector for which A-2I whole square v is != 0 okay, but A-2I v cube = 0 so this we cannot take because for this vector A-2I v = 0.

So A-2I whole cube v will be = 0, A and A-2I whole square v will also be 0 and this vector also we cannot take so let us take 001, so we take here v = 001 and for this we then find the solution, third solution x3t = e to the power At * v.

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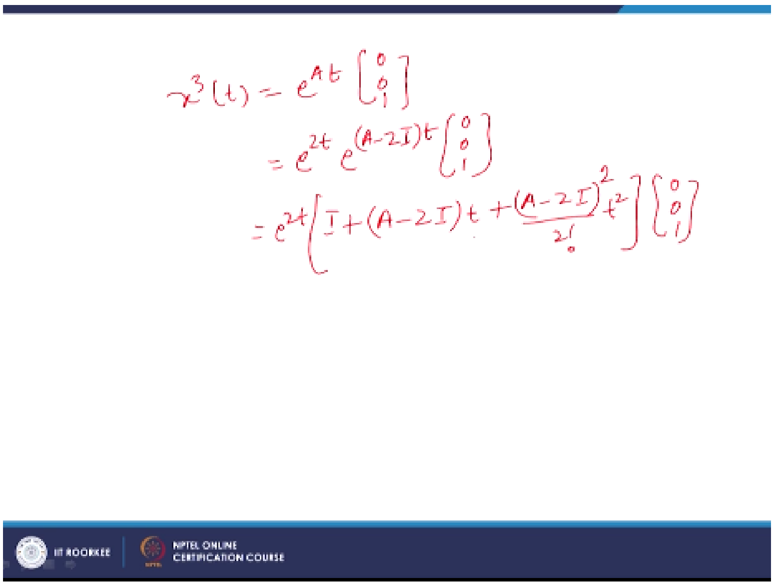

So we get 001 and we write it as e to the power 2t, e to the power A-2I * t 001 and then we write e to the power 2t I + A-2I * t + A-2I whole square upon 2 factorial * t square other terms will vanish operating on 001. So you know the value of the matrix A-2I, you also know A-2I whole square so substitute here this I is identity matrix when you take this product of these 2 matrixes this matrix *001 what we get is this x3t as e to the power 2t 3t-1/2 t square – t1.

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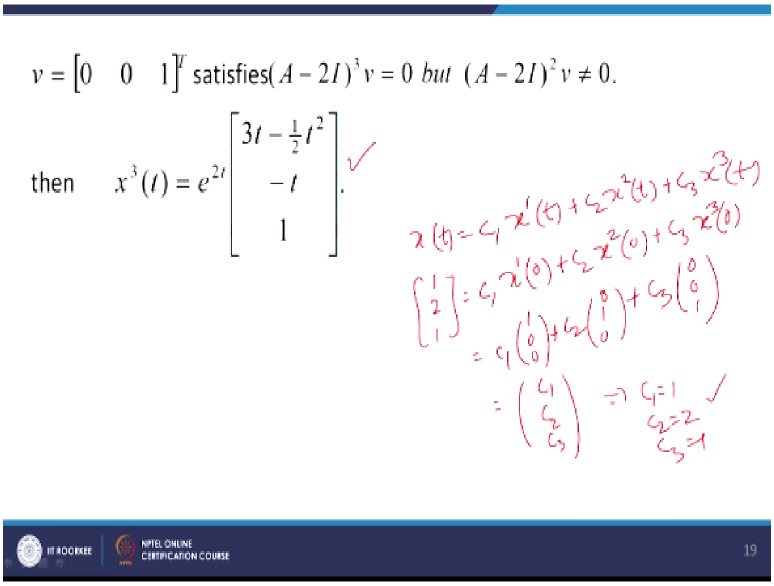

So what we get xt now is the linear combination of these 3 linearly independent solutions x1t, x2t, x3t. Now we use the initial condition, the initial condition is x0 = 121, so when we use x0 = 121 we will have c1 x10 + c2 x20 + c3 x30, okay, c1 times. Now x10 is what, you put t = 0 here, so we get 100 and here you put t = 0 then but you get 010 and here you put t = 0 then you get 001.

So we get 100, 010 and 001, so what we get is c1 c2 c3, so then c1 = 1, c2 = 2, c3 = 1, so put these values of c1, c2, c3 here okay, you get the solution of the initial value problem.

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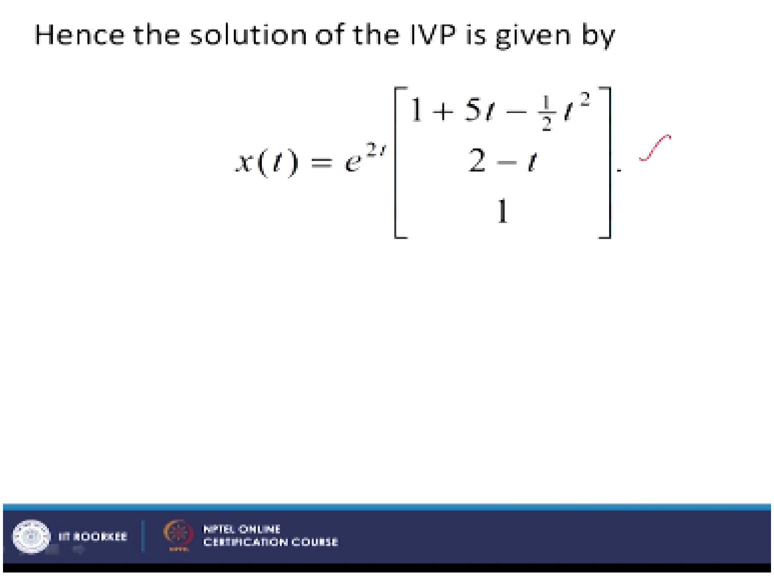

So the solution of the initial value problem is xt = e to the power 2t times this column vector 1+5t - 1/2t square 2-t and 1, so this is how we solve the homogeneous vector differential equation with constant coefficients in the case where the characteristic equation has equal roots, so in the next lecture we shall discuss nonhomogeneous vector differential equation with constant coefficient. Thank you very much for your attention.