Ordinary and Partial Differential Equations and Applications

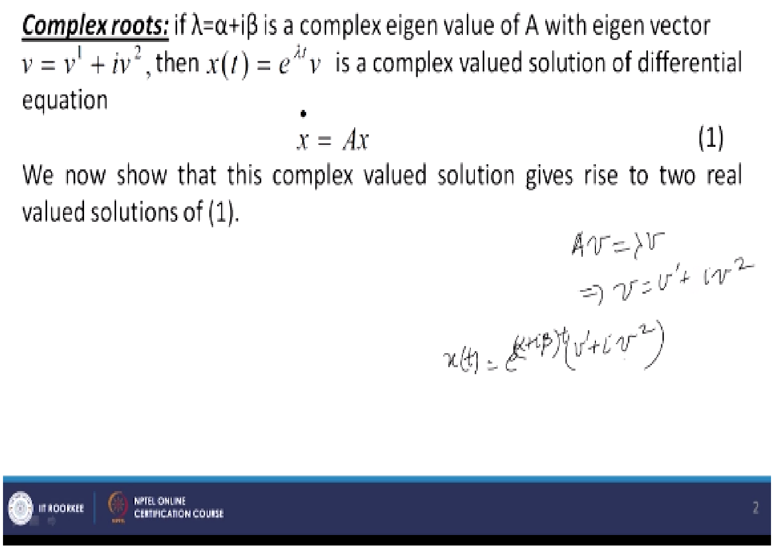
Dr. P. N. Agrawal
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture - 10
Solution of Homogeneous Linear System with Constant Coefficients-II

Hello friends, welcome to the lecture on solution of homogeneous linear system with constant coefficients. This is our second lecture on solution of homogeneous linear system with constant coefficients. In the previous lecture we consider the case where the characteristic equation had an distinct real roots, all the roots of the characteristics equation were real and equal.

Now we shall consider the case where the characteristic equation has complex roots, so then how we shall find the linearly independent solution of the vector differential equation x dot = x. So let us consider the case where suppose lambda = alpha + ibeta is a complex eigen value of A with eigen vector v = v1 + iv2, okay, here what we are assuming is that the characteristic equation determinant A – lambda = 0 has a complex root.

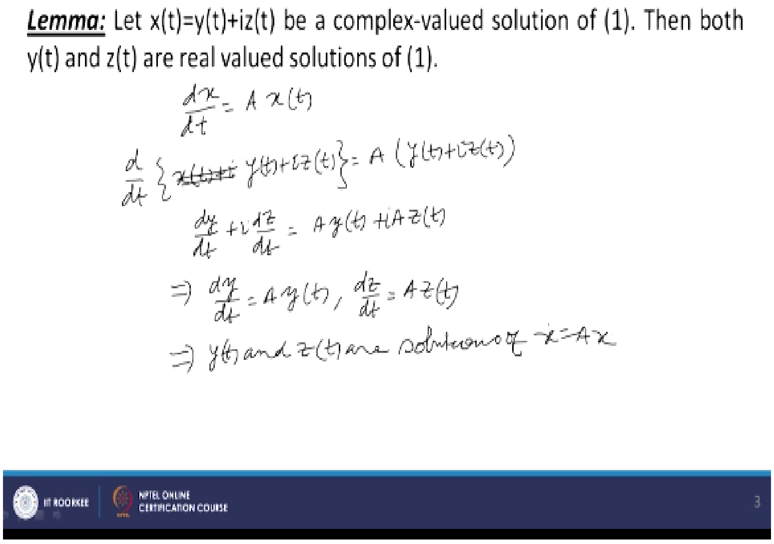
(Refer Slide Time: 01:30)



Lambda = alpha + ibeta, if lambda = alpha + ibeta is the root lambda = alpha – ibeta is also a root because we are assuming A to be a real matrix, so first we consider the case where lambda = alpha + ibeta and we assume that the associated eigen vector Av = lambda v for lambda = alpha + ibeta okay. It gives you v = v1 + iv2, okay, so v = v1 + iv2 is an eigen vector of A corresponding to the eigen value lambda = alpha + ibeta.

So then xt = e to the power lambda t * v will be xt = e to the power lambda t e to the power alpha + ibeta * v so v1 + iv2, okay, so it is a complex valued solution of the differential equation x dot = x. We now show that this complex valued solution gives rise to 2 real valued solutions of the equation x dot = Ax.

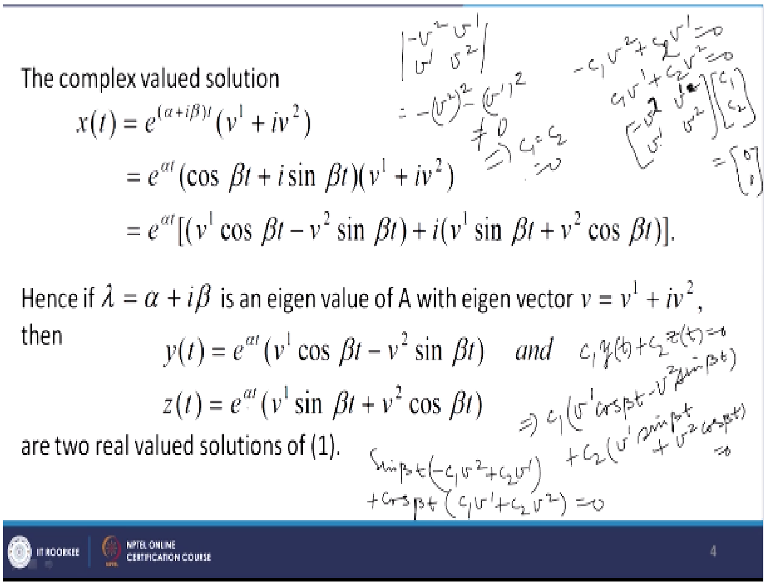
(Refer Slide Time: 02:48)



Suppose, first let us prove this lemma. Let xt = yt + izt be a complex valued solution of x dot = Ax. Then we shall see that yt and zt are real valued solutions of 1. So the equation 1 is dx/dt = Axt. Let us put xt = yt + zt in this so d/dt xt = yt + izt = A times yt + izt. So when you differentiate this vector valued function what you get is dy/dt + idz/dt = Ayt by matrix multiplication property to Ayt + iAzt.

Now equating real and imaginary parts both sides what we get dy/dt = Ayt and then dz/dt = Azt and these 2 equations then tell us that yt is the solution of dx/dt = Axt and zt is the solutions of dx/dt = Axt, so yt and zt are solutions of x dot = Ax, okay, so thus if xt = yt + izt is a complex valued solution of dx/dt = Axt then both yt and zt are real valued solutions of x dot = Ax.

(Refer Slide Time: 05:00)



So the complex valued solution xt = e to the power alpha + i beta * t * v which is v1 + iv2 let us split into real and imaginary parts then the real and imaginary parts of this complex valued solution will give us the 2 real valued solutions of x dot = Ax. So e to the power alpha + i beta t I can write as e to the power alpha t * e to the power i beta t which is cos beta t + i sin beta t and then here v1 + iv2.

Now let us multiply these 2 complex numbers so e to the power alpha t and this gives you v1 cos beta t, i square is -1 so v2 sin beta t iv1 cos beta t – v2 sin beta t + i times v1 sin beta t + v2 cos beta t, hence if lambda = alpha + i beta is an eigen value of A with eigen vector v = v1 + iv2 then yt is e to the power alpha t v1 cos beta t – v2 sin beta t and zt = e to the power alpha t * v1 sin beta t + v2 cos beta t are 2 real valued solutions of the homogenous system x dot = Ax.

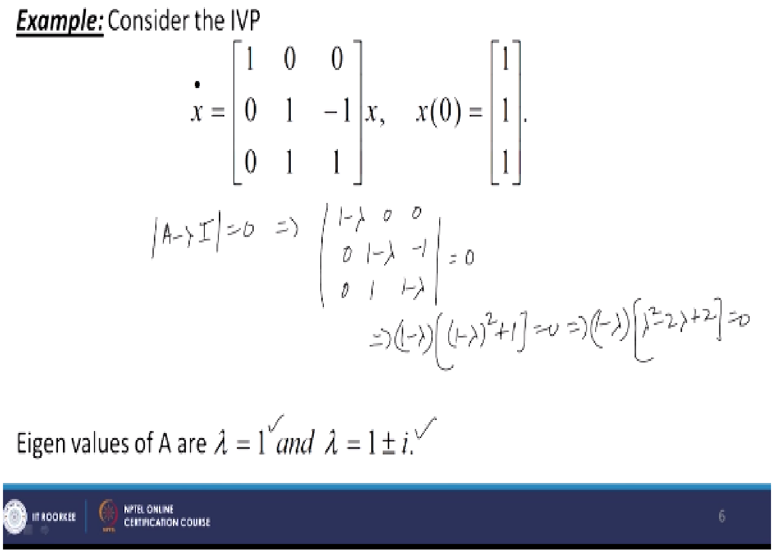
Now further we notice that these 2 solutions are linearly independent, we can see that, so let us put c1 yt + c2 zt = 0 then what we get we will get c1 times e to the power alpha t * v1 cos beta t – v2 sin beta t + c2 times e to the power alpha t v1 sin beta t + v2 cos beta t = 0, e to the power alpha t is never 0 so we can divide this equation by e to the power alpha t and this will give you c1 times v1 cos beta t – v2 sin beta t + c2 times v1 sin beta t + v2 cos beta t = 0 after dividing by e to the power alpha t.

Now let us collect the coefficients of cos sin beta t and cos beta t so we get sin beta t times what you get c2 v1 here we get –c1 v2 + c2 v1 and then + cos beta t times c1 v1 + c2 v2 = 0. Now we know that sin beta t and cos beta t are linearly independent therefore their coefficients must be 0 so – c1 v2 + c1 v1 = 0 and we have c1 v1 + c2 v2 = 0, okay, this gives us the matrix equation v1 v2 c1 v1 this will be, this is c1 b2, c2 b1, so we get okay.

So we get here –v2 here, - c1 v2 and c2 v1 so this is v1, okay, and then here c1 v1 + c2 v1 okay, so we get here –v2 v1 v1 v2, now so c1 c2 to prove that c1 c2 = 0 we need to prove that the determinant of this matrix is nonzero okay, and what is the determinant –v2 v1 and v1 v2 so determinant of this is –v2 whole square – v1 whole square, okay, now this is 0 if and only if v1 and v2 both are 0s.

But v1 and v2 both cannot be 0 because v is a nonzero vector okay, so this is !=0. So this is ! = 0 implies c1 = c2 = 0. So we get yt and zt are 2 linearly independent real valued solutions of equation 1, so these solutions are linearly independent.

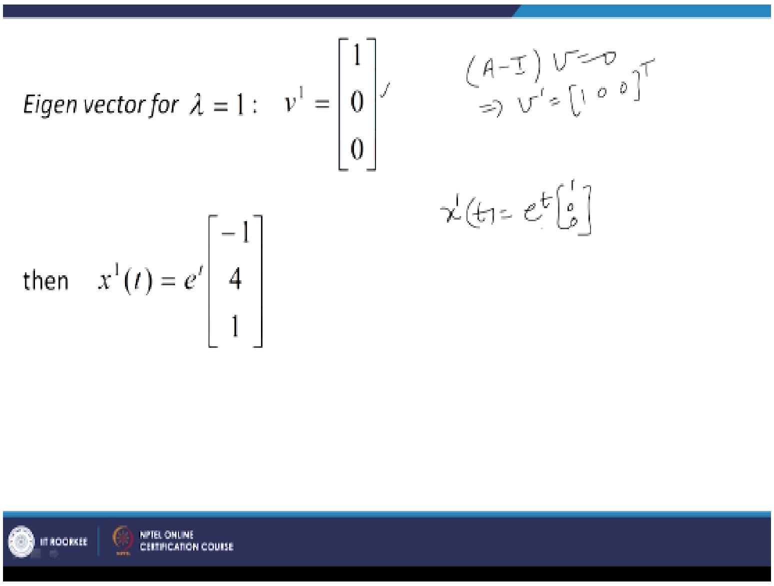
(Refer Slide Time: 10:20)



Now consider the initial value problem x dot = 1 0 0, 0 1 -1, 0 1 1 x x0 = 1 1 1. So eigen values of A are if you solve the determinant of A – lambda = 0 that is you have the equation 1–lambda 0 0, 0 1-lambda -1, 0 1 1-lambda if you solve this equation okay, then you can see here, you have this gives you 1-lambda times 1-lambda whole square and then here you have +1 = 0.

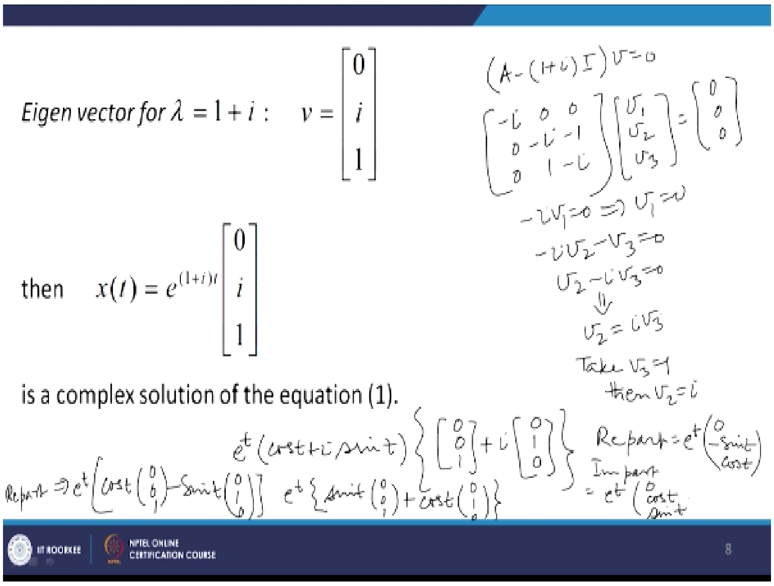
So what do you get 1-lambda okay and then here lambda square -2 lambda + 2 so you get lambda = 1 and then here lambda square – 2 lambda + 2 = 0 gives you lambda = 1+-i. So eigen vector for lambda = 1 we can find by solving A – iv = 0 then it turns out that v1 = 1 0 0 column vector 1 0 0 which we can write as 0 vector 1 0 0 transpose.

(Refer Slide Time: 11:35)



So here x1t = e to the power t then 1 0 0, okay so this is what we have, one solution we have got.

(Refer Slide Time: 11:49)



Now for lambda = 1 + i we have to solve A – 1 + i times i * v = 0 okay, so what do you get? A matrix is this 1 0 0, from this we subtract 1 + i throughout the diagonal elements, okay, so we get here 1 + i when we subtract from 1 0 0 we get –i 0 0 and then we get 0 –i -1 and lastly 1 + i we subtract, so 0 1 –i v1 v2 v3, now here you can see here, we have –iv1 = 0 this is first equation. So we get v1 = 0. Then we get –iv2–v3 = 0 this is second equation.

And third equation is v2 - iv3 = 0 so third equation gives you v2 = iv3, okay, so taking v3 = 1 v2 = -1 take v3 = 1 then v2 = i so we get v1 is 0 so 0 i 1 so that is the eigen vector corresponding to lambda = 1 + i. Now the eigen vector, okay then the solution is xt = e to the power 1 + i t 0 i 1, it is a complex solution of equation 1 from this we shall find the 2 real valued solutions of equation 1 okay.

So for that we need to write this as e to the power 1 + i t so e to the power t then e to the power i t can be written as cos t + i sin t and then 0 i 1 okay, 0 i 1 we shall write as real and imaginary parts in that form so we have 0 0 1 + i times 0 1 0, 0 i 1 can be written as real part, 0 i 0 real part as 0 0 1 + i times imaginary part is 0 1 0, okay, so what we do when we multiply this will give us e to the power t times cos t * 0 0 1 then i square is -1.

So –sin t * 0 1 0, this is the real part okay, and imaginary part will be e to the power t times sin t * 0 0 1 and then we shall have cos t times 0 1 0, so if you write it, simplify it, then real part is e to the power t times you see here we have cos t * 0 - sin t * 0 so we have 0 then cos t * 0 – sin t so we get – sin t and then here you have cos t, here you have 0 so this is real part and imaginary part = e to the power t times.

Here sin t * 0 cos t * 0 so 0 sin t * 0 cos t * 1 so cos t and then we have sin t * 1 so we have sin t. So we get 2 independent solutions of the equation 1 which are e to the power t 0 – sin t cos t and then i times 0 cos t sin t.

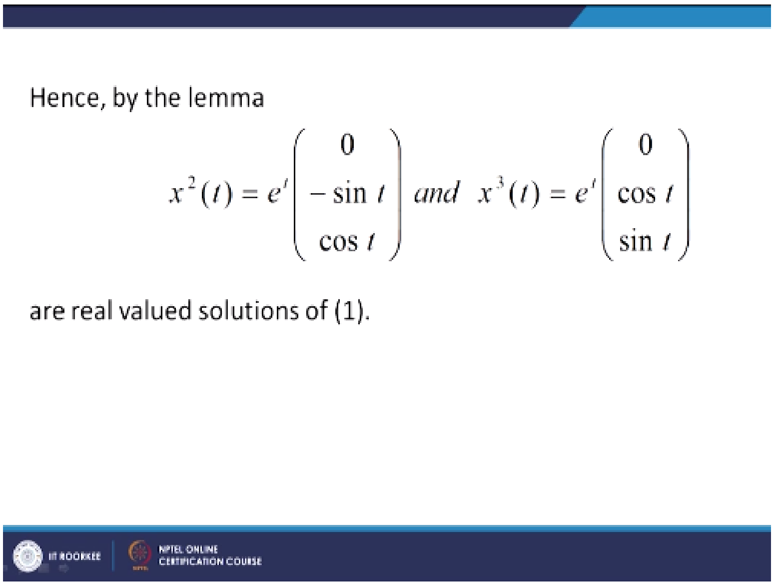
(Refer Slide Time: 16:40)

$$x(t) = e^{(1+i)t}\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

$$= e^{t}\left[\begin{pmatrix} 0 \\ -\sin t \\ \cos t \end{pmatrix} + \begin{pmatrix} 0 \\ \cos t \\ \sin t \end{pmatrix}\right]$$

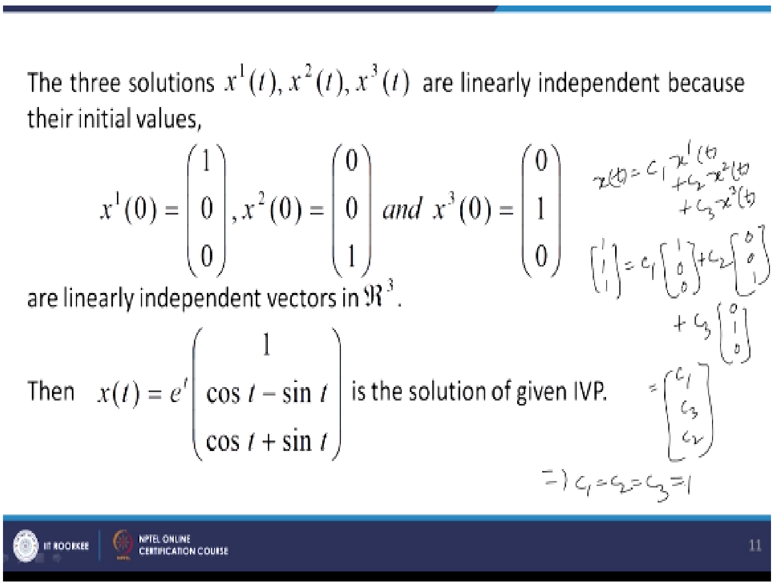
So xt can be written as e to the power t 0 – sin t cos t + i times 0 cos t sin t and we get therefore the 2 by using the lemma the 2 real independent solutions as x2t = e to the power t 0 – sin t cos t x3 t = e to the power t 0 cos t sin t.

(Refer Slide Time: 16:59)



So they are the 2 real valued solutions of x dot = Ax. The 3 solutions x1t, x2t, x3t are linearly independent because their initial values.

(Refer Slide Time: 17:11)

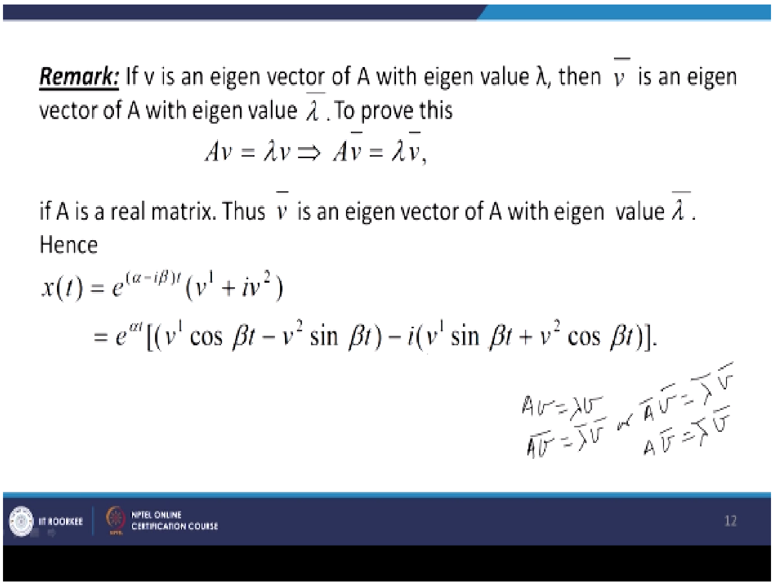


Let us calculate their values at t = 0, so you can see here x1t when you put t = 0 it gives you, t = 0 gives you 1 0 0 and here when you put t = 0 what you get 0 0 1 and here you get 0 1 0, okay, so the 3 vectors 1 0 0, 0 0 1, 0 1 0, we know they are from extended basis of R cube okay, so they are linearly independent vectors in R cube and therefore xt = e to the power t 1 cos t – sin t cos t + sin t, okay is the solution of the given initial value problem.

We need to use the fact here that x 0 = 1 1 1 okay, x0 = 1 1 1 we have to use so for that what you do is, you write the general solution, general solution is xt = c1 x1t + c2 x2t +c3 x3t okay, we have seen that they are linearly independent x1t x2t x3t so in that then you put t = 0, when you put t = 0 you get 1 1 1 = c1 times x10 is 1 0 0, then c2 times x2t 0 0 1 and then c3 times we have 0 1 0 okay.

So we get here c1 c3 c2, okay, so we get c1 c2 c3 = 1 and so the general solution is xt = e to the power t 1 cos t - and we can put here c1 c2 c3 = 1, so we get the general solution of the initial value problem.

(Refer Slide Time: 19:29)



Now let us note the following, if v is an eigen vector of A with eigen value lambda then v bar is an eigen vector of A with eigen value lambda bar, if A is the real matrix, okay, so Av = lambda v if you have then we can see Av bar = lambda bar v bar okay since (()) (19:53) A bar v bar = lambda bar v bar, okay, now A is real matrix so A bar = A, so Av bar = lambda bar v bar, so lambda bar is an eigen value of A and v bar is the corresponding eigen vector okay.

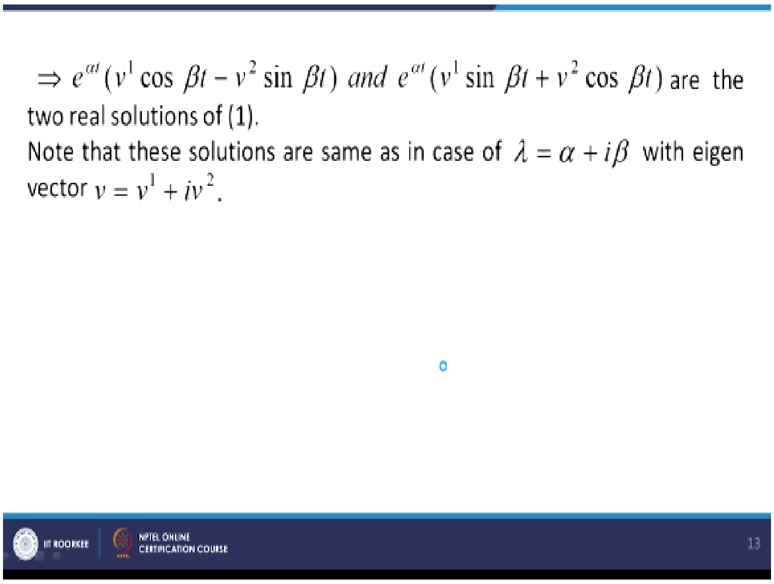
So if v is an eigen vector of A with eigen value lambda and A is the real matrix then v bar is an eigen vector of A with eigen value lambda bar. Now here we have seen that for alpha + i beta if v + iv2 is an eigen vector then for alpha – i beta v1 - iv2 will be an eigen vector from this result, okay, so xt = e to the power alpha – i beta t v1 + iv2.

Now if you simplify this multiply this you write as e to the power alpha t then cos beta t – i sin beta t and then v1 + iv2 and then you write it as real and imaginary parts you see that e to the power alpha t v1 cos beta t - v2 sin beta t and v1 sin beta t + v2 cos beta t we get the real part as this, the imaginary part is negative of this okay.

So we do not get any new solutions you see here corresponding to the eigen value lambda = alpha – i beta, we get the same 2 real valued solutions as in the case of alpha + i beta. The real valued solution in the case of alpha + i beta where e to the power alpha t v1 cos beta t – v2 sin beta t and here we had + v1 sin beta t + v2 cos beta t so if we know that if xt is a solution see xt also a solution.

So minus does not give us any new solution, so this is what we have, so corresponding to alpha + i beta we find 2 real valued solutions of the vector differential equation and that give us all the 3 solutions together with the solution for lambda = 1.

(Refer Slide Time: 22:00)



So these are the 2 real valued solutions of 1 and these 2 solutions are same as in the case of alpha + i beta with eigen vector v1 + iv2. So that is all in this lecture. Thank you very much for your attention.