Multivariable Calculus
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Lecture - 09
Chain Rule-I

Hello friends. So, welcome to lecture series on multivariable calculus. In the last lecture, we have seen some properties of partial derivatives. Now, we will see chain rule what chain rule is and how we can apply it.

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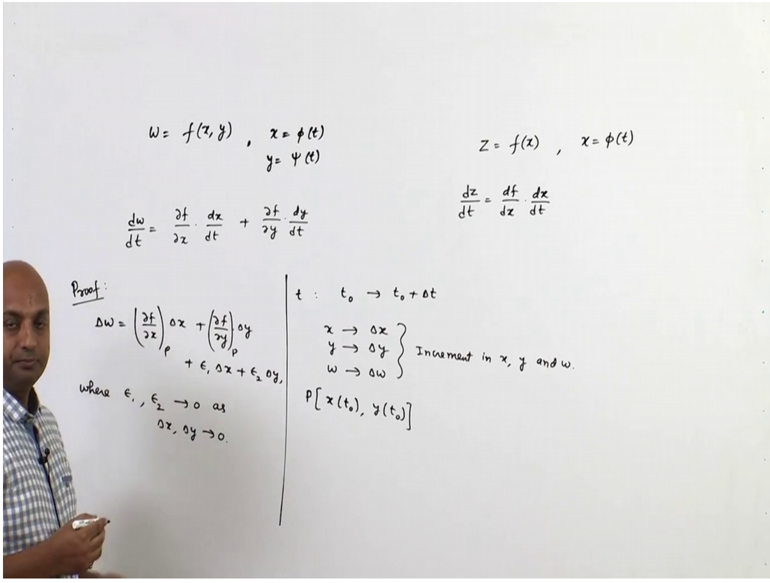

Now suppose w is equals to f x y function of 2 variables ok. Now first come to function single variable. Suppose, we have z equal to f x function of single variable and say x is a function of t some function of t ok. So, if you find d z by d t. So, it is nothing, but d f by d x into d x by d t. This is in single variable function, if we have a function of only one variable x, where x is a function of t then d z by d t or d f by d t is nothing, but d f by d x into d x by d t this is a chain rule for single variable function.

Now, we extend the same rule for several variable functions. Now here instead of one variable we are having 2 variables and say x is a function of t and y is also some function of t and we are assuming w x and y all are differentiable functions ok. We are assuming that w x and y all are differentiable functions. So, if we want to find out say d w by d t because when you substitute x is a function of phi t here and y is the function of psi t here. So, it will be a function of single variable t ok.

So, instead of partial derivative we will be having complete derivative d w by d t because we are having because now w is a function of only one variable t ok. So, this d w by d t can be written as now w is a function of x and y del f by del x into d x by d t because again x is a function of only one variable t here plus del f by del y into d y by d t because f is a function of x and y and x and y both of the functions of t. So, by chain rule we are having this expression to calculate d d w by d t.

Now, what the proof of this chain rule how we are obtaining this? So, proof is quite simple suppose, suppose, t is changing from t naught to t naught plus delta t and the corresponding change in x is delta x y in y it is delta y and in w it is delta w this is an increment in x y and w because t is changing from t naught to t naught plus delta t there is a change in delta t in t in the variable t the corresponding change in x y and w are delta x delta y and delta w respectively ok. Now, suppose p is a point x t naught and y t naught because x and y both are the functions of t at t naught we are assuming that this point is p which is given by x at t naught and y at t naught ok.

Now, since function w is a differentiable function of x and y; so, we can always take w as delta w which is equals to which is equals to it is del f by del x into delta x plus del f by del y into delta y plus epsilon 1 into delta x plus epsilon 2 into delta y where epsilon 1 in epsilon 2 both tends to 0 as delta x delta y tending to 0. This is by the definition of differentiability because we are taking w as a differentiable function of x and y. So, by that we can easily write delta w as del f by del x into delta x plus del f by del y into delta y into this plus this and we are taking this at say point p. So, we are taking this as point p now you divide the entire expression this entire expression by delta t.

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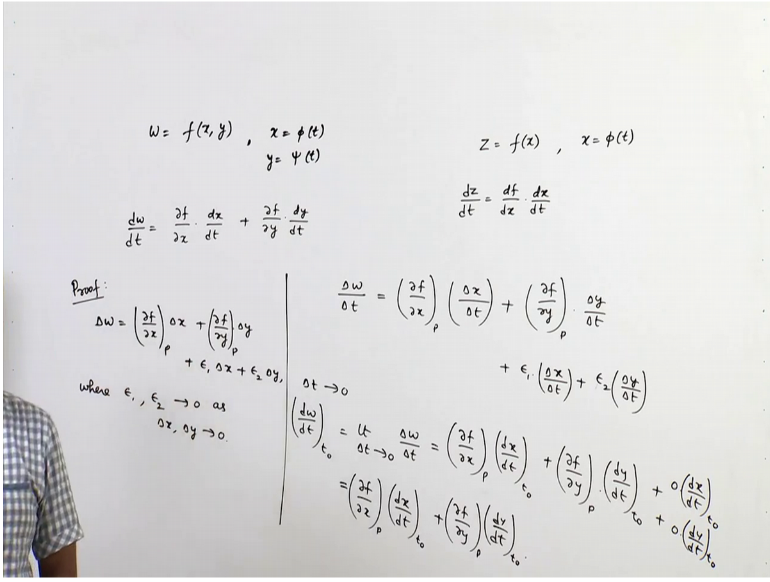

So, what we will obtain delta w by delta t is equals to del f by del x of p into delta x by delta t plus del f by del y of p into delta y by delta t plus epsilon 1 into delta x by delta t plus epsilon 2 delta y by delta t.

So, we have divided the entire expression by delta t ok. So, what we will obtain we obtain this expression. Now take delta t tending to 0 now take delta t tending to 0 both the sides. So, what we will obtain now d w by d t at t naught will be limit delta t tending to 0 delta w by delta t which is equals to which is equals to this is equal to this expression. So, you apply delta t tending to 0 in this expression.

So, what we will obtain it is del f upon del x at p naught or p now delta x upon delta t when delta t tending to 0 is d x upon d t and we are taking at a point t naught plus delta f upon delta y at p into again when delta t tending to 0 delta y upon delta t will tend to d y upon d t at t naught plus when delta t tending to 0 when delta t tending to 0.

This means epsilon 1 and epsilon 2 will tend to 0 because these are the function of delta x and delta y and when delta t tends to 0 epsilon 1 epsilon 2 both will tends to 0. So, plus 0 into d x upon d t at t naught plus 0 into d y by d t at t naught ok. So, this implies this is equals to del f by del x at p into d x by d t at t naught plus del f by del y at p into d y by d t. So, we get back to your same expression again and it is valid for every p and every t naught. So, we get back to this expression. So, this is how we can obtain the proof of the chain rule.

Now, we will solve some problems based on this first now the easy way to remember the chain rule is by tree diagram. So, how can we obtain this or what is tree diagram now you see.

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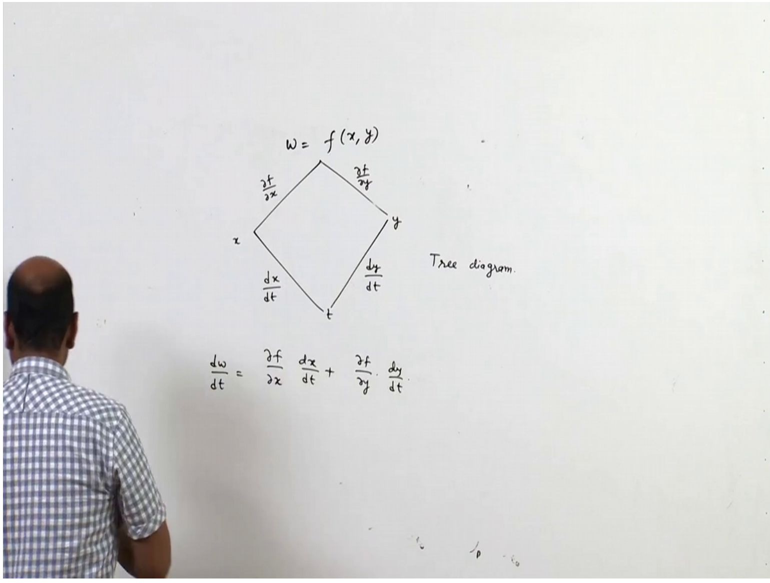

Now you see, as w is a function of x and y. So, this w is a function of x and y and x and y both are the functions of t. So, x is a function of t and y is also a function of t. Now it is f and a function of x it is x. So, it is del f by del x when you take derivative, here it will be a partial derivative del f by del x and from here to here it will be d x by d t ok, again, here will be del f by del y and this will be d y by d t.

Now we have to write say d w by d t from here from this point to this point d w by d t. So, how can we remember that expression now you simply multiply one branch one branch of this is this into this the other branch of this will be this into this and add them. So, simply this will be del f by del x into d x by d t plus del f by del y into d y by d t. So, this is how we can easily remember chain rule this is called t tree diagram ok, in this we call as tree diagram.

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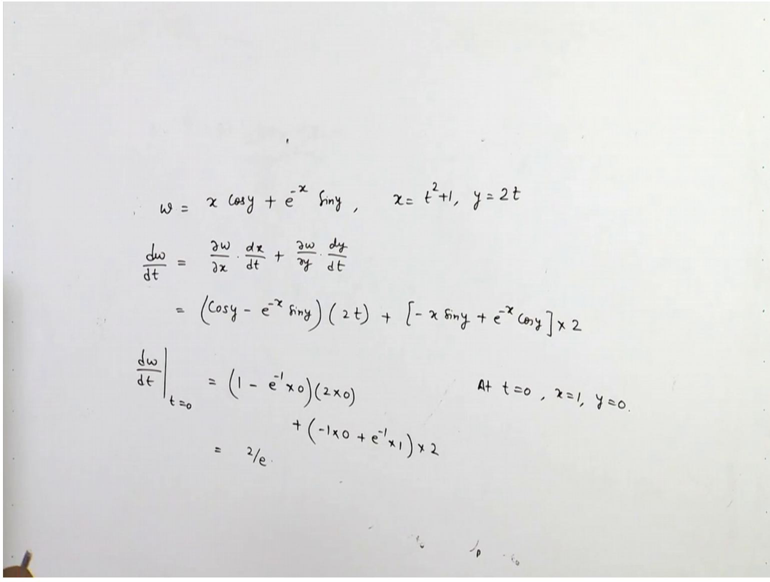

It is x cos y plus e raised to power minus x into sin y. So, here w is a function of x and y and x is t square plus one and y is simply 2 t. So, w is a function of x and y and x and y both are the functions of t. So, we want to compute say d w by d t. So, how can we compute d w by d t, it is del f by del f or del w, you can say you know problem del w by del x into d x by d t plus del w by del y into d y by d t by the chain rule we can say this thing now del w by del x is you simply differentiate partially respect to x keeping other variables constant.

So, it will be cos y minus e raised to power minus x sin y and d x upon d t from here is 2 into t plus del w by del y you differentiate partially this as with respect to y keeping all other variables constant. So, how can we differentiate it is minus x sin y plus e raised to power minus x cos y and d y by d t is 2 ok.

So, we have to compute this value this expression at t equal to 0. So, d w by d t at t equal to 0 will be now when t is 0 x is 1 and when t is 0 y is 0. So, at t equal to 0 x is one and y is 0. So, simply substitute x equal to 1 and y equal to 0. So, cos 0 is 1 1 minus e raised to power minus 1 sin y is sin 0 means 0 and t is 0 that 2 into 0 plus e raise is it is x x is 1. So, it is minus 1 into sin y a sin 0 plus e raised to power minus 1 into cos 0 is 1 into 2 ok. So, this value is equal to. So, this is 0; so, this is 2 upon e ok. So, that is how we can obtain this value.

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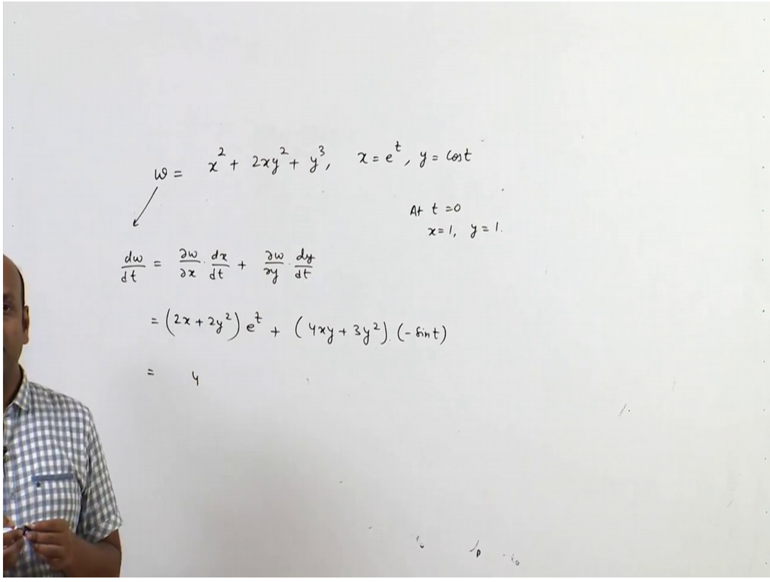

So, now we will try on more problem based on this now w is it is x cube x square sorry plus 2 x y square plus y cube where x is e raised to power t and y is cos t. So, again at t equal to 0 x is 1 and y is also 1 ok. So, what will be d w by d t again by the chain rule it will be del w by del x into d x by d t plus del w by del y into d y by d t. So, that will be equal to del w by del x that is 2 x plus 2 y square into d x by d t is e raised to power t plus del w by del y is 4 x y plus 3 y square into d y by d t is minus sin t. Now when t is 0 this is 0. So, this entire expression is 0 and when t is 0, it is 1. So, x is 1 y is 1. So, it is. So, it is 4 x is 1, y is 1, 2 plus 2; 4 into 1 is 4. So, this value is simply.

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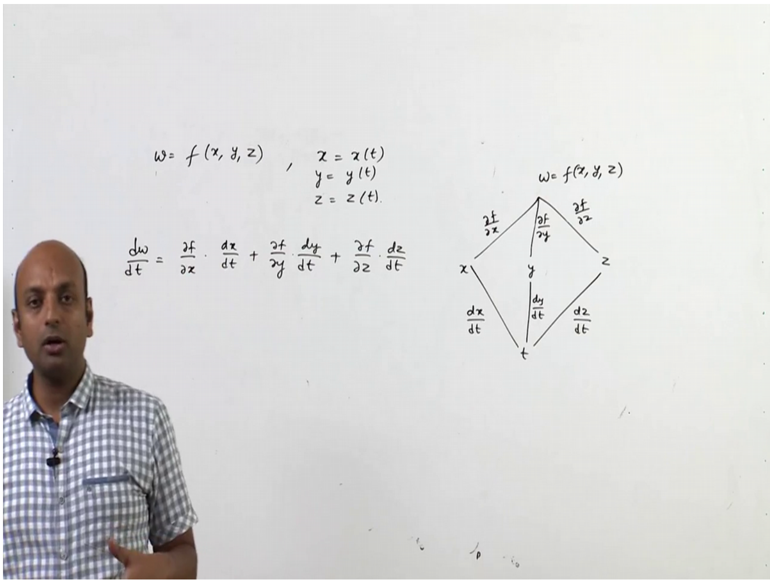

Now, come to functions of 3 variables ok. Now suppose f w is a function of 3 unknowns 3 independent variables x y z and x is a function of t y is a function of t and z is a function of t. Again suppose that after the differentiable functions of in x y and z and x y z are differentiable functions of t, then if you want to find out d w by d t. So, it is nothing, but del f by del x because now function w or f is a function of 3 unknowns x y and z and each is a function of t. So, it is del w del f by del x into d x by d t plus del f by del y into d y by d t plus del f by del z into d z by d t.

This is by the chain rule and the tree diagram also we can see this thing here w is a function of x y z function of x y and z and x y z each is a function of t. So, x is a function of t y is a function of t z is a function of t now from this branch it is del f by del x and from this branch it is d x by d t because only one unknown is there it is del f by del y and it is d y by d t it is del f by del z it is d z by d t.

So, if you want to compute d w by d t. So, it is this into this plus this into this plus this into this which is this expression you take you take the first branch first this into this; this branch this into this the second and the last third branch this into this and add them. So, we will get the chain rule for this function.

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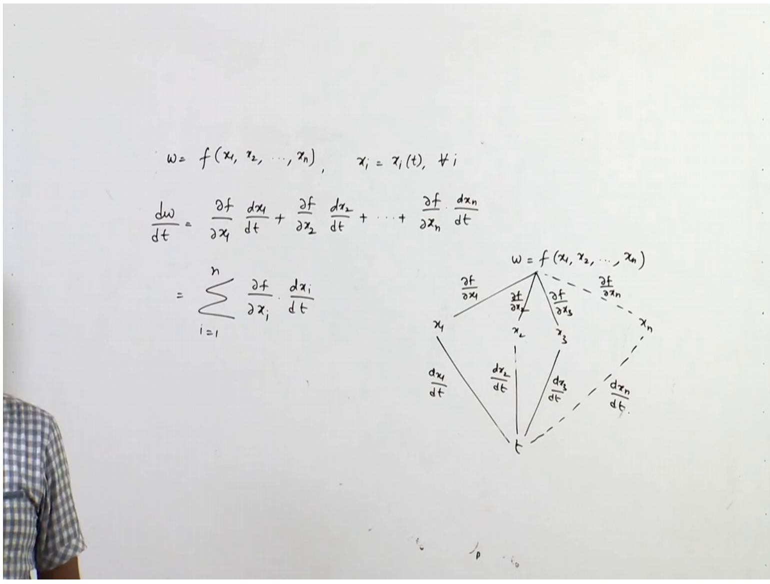

Now let us suppose w is a function of N unknowns and each x i is a function of t and suppose you want to compute d w by d t. So, again by the chain rule how can we write this; this will be del f by del x 1 into d x 1 by d t plus del f by del x 2 into d x 2 by d t plus and so on.

Plus del f by del x i del x n into d x n by d t. So, that is summation i from one to n del f by del x i into d x i y d t. So, that is simply d f d w by d t. So, also for the tree diagram also we can say this it is function of n variables. So, it is x 1 say it is x 2 x 3 and so on up to x n and each is a function of t ok. Now from this branch it is del f by del x 1 it is d x 1 by d t from here it is del f by del x 2 it is d x 2 by d t, it is del f by del x 3 into d x 3 by d t and from the last branch it is del f by del x n into d x n by d t.

So, therefore, d w by d t will be this into this plus this into this plus this into this and so on up to this into this ok. So, that is how we can easily obtain d w by d t applying chain rule for n variables function.

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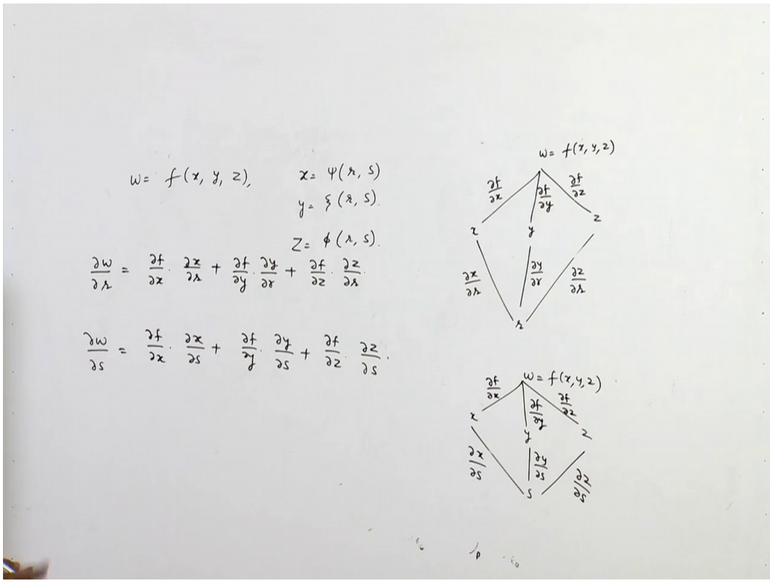

Now suppose now suppose w is a function of x y z and x is a function of x is a function of say r and s y is also some function of r and s instead of one variable now suppose and z also and z is also a function of say phi here also function of r and s. So, in a instead of only one variable now suppose x y z all other function of 2 independent parameters 2 independent variables r and s. Now when you substitute x as a function of r and s y as a function of r and s z is the function of r and s.

So, overall this function will be a function of 2 unknowns 2 variables r and s. So, instead of complete derivative, now we will be having partial derivative. So, basically now we will be having either del w by del r or we or we will be having del w by del s. So, how can we compute del w by del r or del w by del s again we can we can write chain rule for this function this is x y z, it is x it is y it is z and it is a all our functions of r if you want to compute del w by del r. So, write only r in the bottom of the tree and then when you want to compute del w by del s. So, write s in the bottom of the tree x y z it is x it is y it is z and each is a function of s.

Now, here, here, it is del f by del x and it is also del x by del r because now x is a function of 2 unknowns r and s. So, instead of complete derivative; now you will be having partial derivative. So, it is delta x y del r again here it will be del f by del y and it will be del y by del r it is del f by del z and here it is del z by del r. So, what will be del w by del r it will be equal to del f by del x into del x by del r. So, what branches over plus del f by del y into del y by del r plus del f by del z into del z by del r. So, this will be basically del w by del r now similarly if you want to compute del w by del s.

So, that will be it is del f by del x it is del x by del s it is del f by del y it is del y by del s here it is del f by del z and it is del z by del s. So, del w by del s, similarly will be del f by del x into del x by del s plus del f by del y into del y by del s plus del f by del z into del z by del s. So, that is how that is how we can compute del w by del r or del w by del s in first type of problems ok.

Now, we will try to solve some problems based on this first.

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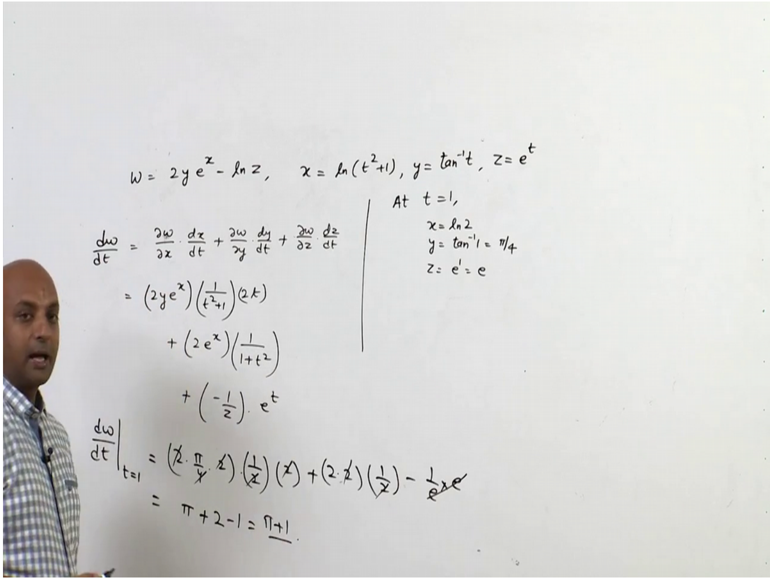

Now let us suppose w is 2 y into e raised to power x minus l n z where x is function of t that is l n t square plus 1 and y is also a function of t which is tan inverse t and z is again a function of t, it is e raised to power t and we want to compute d w by d t at t equal to 1 ok, this is want to find out in the first problem how can we compute this. Now at t equal to 1 x is l n 2 y is tan inverse 1 which is pi by 4 and z is e raised to power 1 that is e ok. Now d w by d t first you write d w by d t, then we will find a t equal to 1 it is del w by it is del w by del x into d x by d t plus del w by del y into d y by d t plus del w by del z into d z by d t this is simply by chain rule ok.

Now del w by del x is 2 y e raised to power x and d x by d t from here is now d x by d t is you suppose this as some function to l n l n this is one upon t square plus 1 into derivative of this which is 2 t plus del w by del y from here is 2 e raised to power x and then it is d y by d t d y by d t is 1 upon 1 plus t square again plus del w by del z partial derivative of this w respect to z no z is here. So, it is 0 and from here it is minus 1 by z into derivative of this respect to t which is e raised to power t.

Now you want to compute d w by d t at t equal to 1. So, x is l n 2 y is pi by 4 and z is e at t equal to 1 you simply substitute it here. So, 2 into pi by 4 into e raised to power x is 2 from here t is 1. So, it is 1 plus 1 is 2 and it is 2 into 1 is 2 plus again e raised to power x is 2. So, it is 2 into 2 it is 1 upon 1 plus 1 that is 1 upon 2 plus. So, it is minus sign; so, minus will come and z is e. So, it is 1 by e into and e raised to power t is e because t is 1.

So, what we will obtain from here this will cancel out this will cancel out this will cancel out and this cancel out. So, we will obtain pi from here and we will obtain plus 2 from here minus 1 that is pi plus 1. So, pi plus 1 will be the final answer ok.

Now, similarly we can try for the second problem.

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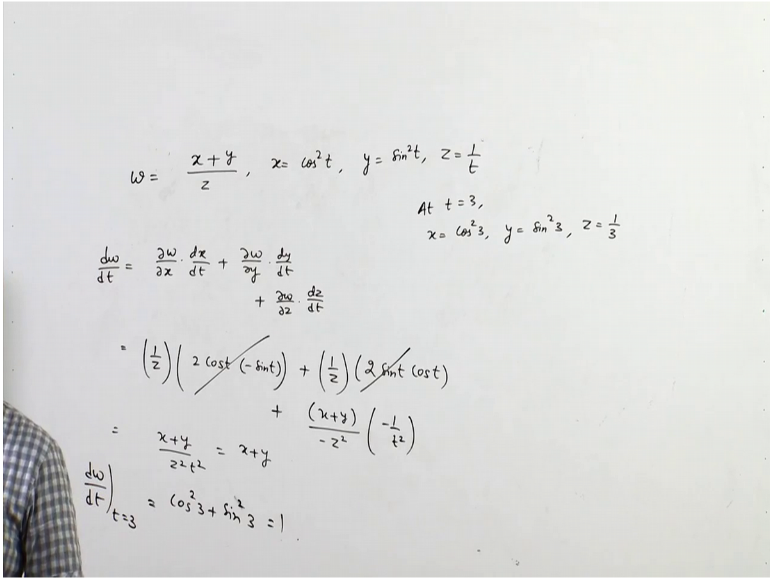

Now w is for a second problem w is x plus y upon z x is cos square t y is sin square t and z is 1 by t and at t equal to 3 now at t equal to 3 x is cos square 3 y is sin square 3 and z is one by 3 ok. Now, you first compute d w by d t, then we will compute d w by d t at t equal to 3. Now d w by d t by again by the chain rule it is del del w by del x into d x by d t plus del w by del y into d y by d t plus del w by del z into d z by d t.

So, this will be del w by del x from here is 1 by z and into d x by d t which is 2 cos t into minus sin t plus del w by del y is again 1 by z into del d y by d t from here is 2 sin t into cos t plus del w by del z is x plus y upon minus z square because 1 by derivative 1 by t is minus 1 by t square into d z by d t d z by d t is minus 1 by t square.

Now, this term and this term cancels out and it will be x plus y upon z square t square now z into t is 1. So, it is equal to x plus y and at t equal to 3, it is cos square 3 plus sin square 3 because it is x plus y and x cos square 3 and y sin square 3. So, it is 1. So, the answer is 1 ok. So, that is how we can simply apply a chain rule to find out derivative respect to t or some other variables. So, this I have already discussed, now come to some more problems based on this.

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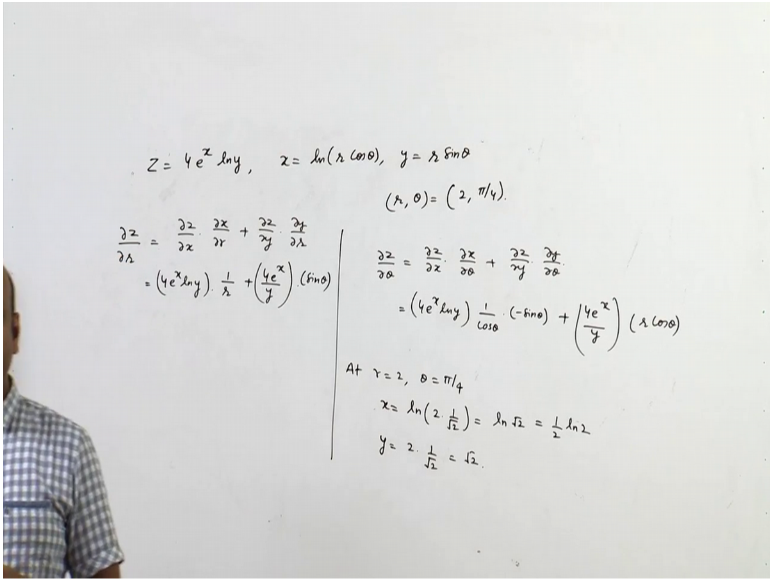

So, we will compute say we will take first problem z is 4 e raised to power x and l n y and x is l n r cos theta and y is r sin theta and the point is point is 2 comma pi by 4 ok. So, in the first problem we have to find out del w del z by del r or del z by del theta at 2 comma pi by 4.

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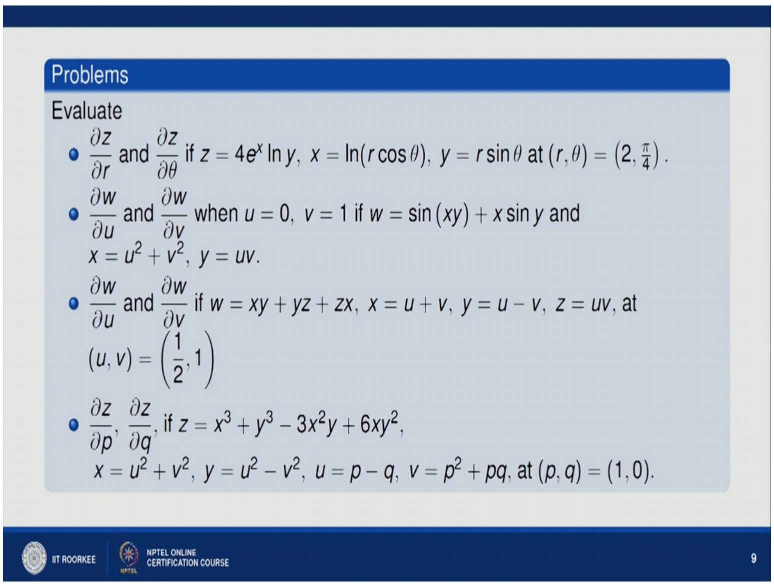

So, how can we compute; now here z is a function of x and y and x and y are again functions of 2 variables r and theta ok. So, how can we find del z by del r suppose. So, either we can make use of t tree diagram or we can try directly it is a there is a function of x and y. So, it is del z by del x into del x by del r plus del z by del y into del y by del r.

And similarly if you want to find out del z by del theta. So, it is del z by del x into del x by del theta plus del z by del y into del y by delta theta similarly you can write this by the chain rule. So, what is del z by del x it is 4 e raised to power x into l n y into del x by del r is one by r plus del z by del y it is 4 e raised to power x upon y into del y by del r is sin theta.

So, so this is expression now this is del z by del x it is for e raised to power x l n y del x by del theta from here it is 1 by cos theta and it is minus sin theta plus del z by del z by del y is 4 e raised to power x upon y and del y by del theta is r cos theta. So, now, you can easily compute these values at r at r equal to 2 and theta equal to pi by 4 because at r equal to 2 and theta equal to pi by 4 x is x is l n 2 into 1 by under root 2 which is l n under root 2 that is 1 by 2 log 2 and y is equal to r sin theta r is 2 theta is this. So, it is 2 into one by under root 2 which is under root 2. So, you can simply substitute x equal to this y equal to this r equal to this and theta equal to this and this and this expression. So, you can easily find out del z by del r or del z by del theta in the first topic.

Similarly, we can solve second and third problem also let us try the last problem del z by del p and del z by del q ok.

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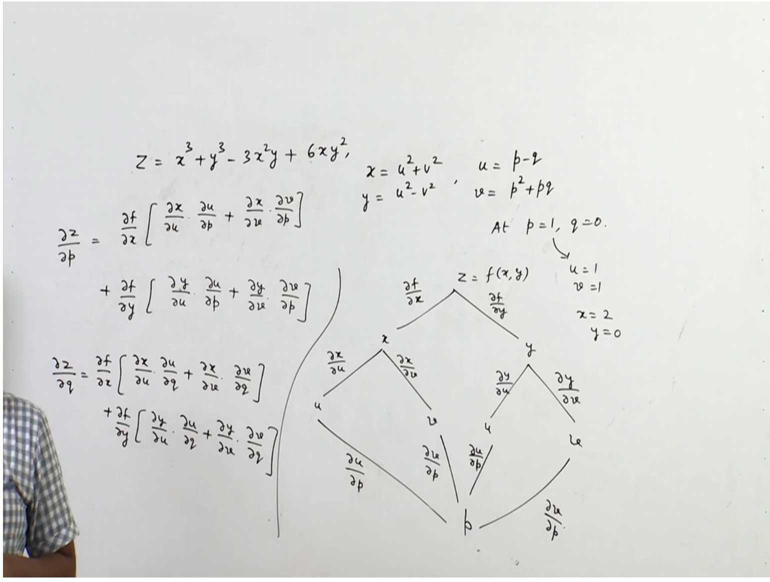

Now here in the last problem what we are having we are having z is equals to x cube plus y cube plus it is minus it is 3 x square y plus 6 x y square. Now x is equal to as a function of u and v which is u square plus v square and y is again a function of u and v which is u square minus v square now u and v are again function of 2 variables p and q which are given by u is p minus q and it is p square plus q square p square plus p q sorry. So, when you substitute x as u square plus v square and y as u square minus v square and again u as p minus q and v as p square plus p q.

So, depth z will be a function of p and q. So, we have to compute del z by del p and del z by del q in this problem at p equal to 1 and q equal to 0. So, how can you find this how can we first write chain rule for this expression. So, let us see here z is a function of x and y we will first try to write down the chain rule for del z by del p and similarly we can compute del z by del q now z is a function of x and y x is a function of u and v.

Here also it is a function of u and v and all our functions of p and you want to compute del z by del p we want to compute del z by del p the first expression del z by del p ok. So, this is basically del f by del x this is del x by del u this is del u by del p this is del x by del v this is del v by del p this is del f by del y this is del y by del u this is del u by del p this is del y by del v and this is del v by del p.

Now, how can we write del z by del p now del z by del p will be given as now first you come to this point and then you come to this point ok. So, for this point this into this plus this into this and whatever we are obtaining at this point multiply this with this. So, then the first branch will be over and similarly for the second branch and add them we will get del z by del p. So, what we would del z by del p it will be del f by del x into whatever we are obtaining from here this is this into this plus this into this that is del x by del u into del u by del p plus del x by del v into del v by del p.

This first branch and from a second branch it is del f by del y into del x del y del y by del u into del u by del p plus del y by del v into del v by del p. So, this is by the second branch. So, this is expression for del z by del p you can see from here also that f is a function of x and y and x and y are the functions of u and v which are in ton which in turn other functions of p and q. So, del z by del x into del x by del u with p and del x by del v with del v by del p because x and y both of the functions of u and v and u and v both of the functions of p if you are talking about del z by del p.

Similarly, for del z by del y the same expression here with y. So, is it is easy to remember if you if we draw a chain or if we draw a t a tree diagram for the same by that tree diagram, we can easily write down such expressions. Now similarly we can compute del z by del q it is del f by del x into del x by del u into del u by del q plus del x by del v into del v by del q plus del f by del y into del y by del u into del u by del q plus del y by del v into del v by del q. Similarly simply replace p by q we will get back to this expression.

Now, you can you can find out all these values from these expressions substitute it over here and at p equal to 1 q equal to 0 first u is 1 and v is 1 when u is 1, q is 1, x 2 and y is 0. So, you simply substitute all these points in this expression after solving this. So, we can compute del z by del p or del z by del q at p equal to 1 and q equal to 0. So, this is how we can solve some problems based on chain rule so.

Thank you very much.