Multivariable Calculus

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Lecture - 08
Differentiability-II

Hello friends. So, welcome to lecture series on multivariable calculus. So, we were discussing about differentiability, that what do we mean by differentiability of 2 variable or more than 2 variable functions. We have seen that for differentiability existence of first order partial derivatives at that point is necessary condition, and to put that whether a function is differentiable at a point or not.

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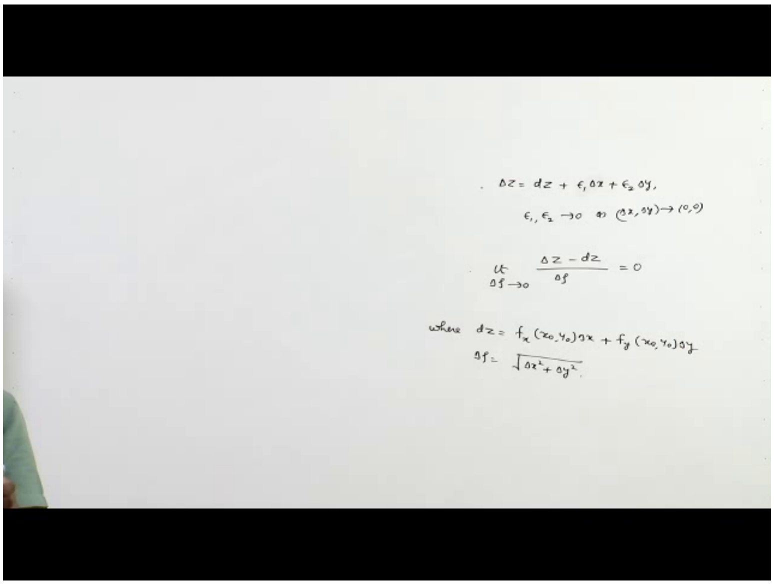

We have to show either delta z is equals to d z plus f x at x naught y naught ok, at f x this is this thing and epsilon 1 delta x plus epsilon 2 delta y. So, epsilon 1 epsilon 2 should tend to 0 as delta x delta y tend to 0 comma 0.

So, these are first way out by which we can showed a function is differentiable at a point x naught y naught. And the second way out is simply take limit delta root tending to 0 delta z minus t z upon delta rho should equal to 0. So, we are we are what is d z, d z is f x at x naught y naught into delta x plus f y at x naught y naught into delta y. and delta rho is under root delta x square plus delta y square. So, we can prove the differentiability of a function f x y by any one of the definition. Either by the by this definition either right this condition and try to showed at epsilon 1 epsilon 2 tend to 0, as delta x delta y tend to 0 0 0, or try to find out this value and try to show that this limit is tending to 0.

Now, we will discuss one more problem based on this show that this function is continuous process is partial derivative at 0 comma 0 but not differentiable at 0 comma 0.

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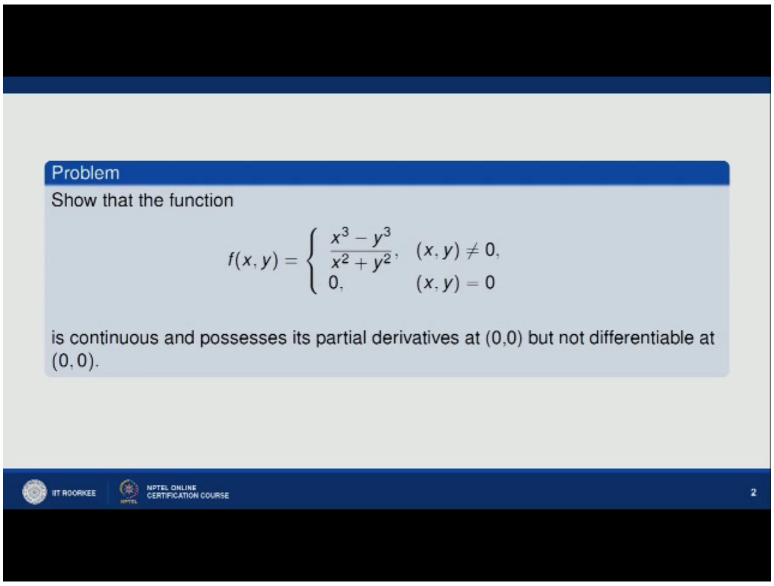

So, how can we solve this problem let us see.

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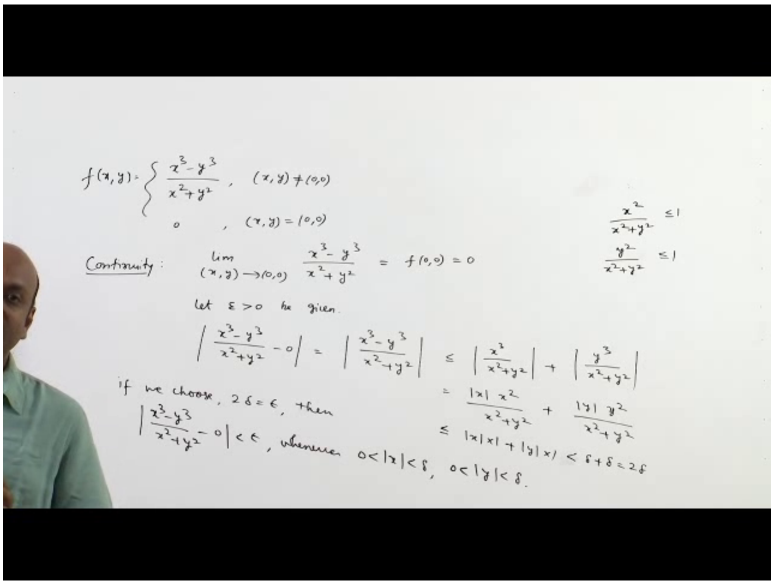

So, f x y is equals to f cube minus y cube upon x square plus y square, x y not equal to 0 comma 0 and 0, when x y is equals to 0 0. So, first part is continuity so, for continuity at 0 comma 0, what we have to show we have to show that limit x y tending to 0 0 x cube minus y cube upon x square plus y square should be equals to f at 0 0 which is 0 f at 0 0 is 0. So, this to prove for continuity. Now how can we show that this limit exists and is equal to 0?

So, you know how to prove that this limit exists and equal to 0, we have to use delta epsilon definition. So, let epsilon greater than 0 be given. Now take mod x cube minus y cube upon x square plus y square minus 0 which is equals to mod of x cube minus y cube upon x square plus y square which is less than equal to mod of x cube upon x square plus y square plus mod of y cube upon x square plus y square which is equal to mod of x into x square upon x square plus y square, plus mod of y into y square upon x square plus y square. Because y square and x square are nonnegative quantities can be taken out from the modulus.

Now, these quantities, now x square upon x square plus y square is always less than equal to 1. Similarly, y square upon x square plus y square is always less than equals to 1. So, these quantities are always less than equal to 1. So, we can take that less than equals to mod x into 1 plus mod y into 1. So, if you take mod x less than delta. So, it is less than so, it is less than delta plus delta that is 2 delta. So, if you choose 2 delta is equals to epsilon, then mod of x cube minus y cube upon x square plus y square minus 0 will be less than epsilon whenever 0 less than mod x less than delta, and 0 less than mod y less than delta. So, in this way we have shown that this function is continuous at 0 comma 0.

Next, we have to show, that the first order partial derivative exist at 0 comma 0; so, f x at 0 comma 0.

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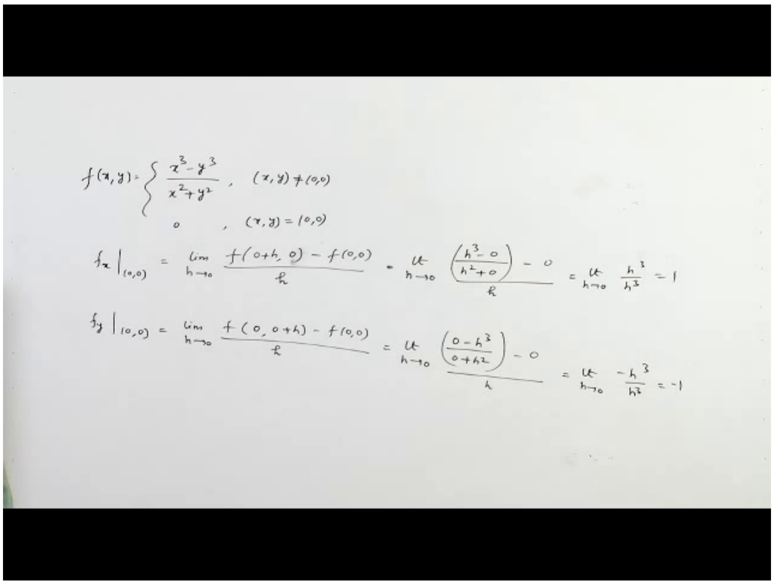

It will be by the definition limit x tending to 0 f of 0 plus h minus f 0 0 upon h. Now this will be equals 2 limit h tending to 0. Now you replace x by h and y by 0 in this definition. So, it will be h cube minus 0 upon h square plus 0 and it is minus 0 upon h because f 0 0 is 0. So, this will be limit h tending to 0 h cube upon h cube and this is 1. Now you compute f y at 0 comma 0. It is limit h tending to 0 f of 0 comma 0 plus h minus f 0 0 upon h. This is limit h tending to 0 f 0 comma h. Now you replace x by x 0 and y by h it is 0 minus h cube upon 0 plus h square minus 0 upon h. So, this will be limit h tending to 0 minus h cube upon h cube which is minus 1.

So, we have shown that is first order partial derivative at 0 comma 0 exists. And the values of f x is one and a f I is minus 1 at 0 comma 0. Now the third part is function is not differentiable at 0 comma 0. It is continues at 0 comma 0, first order partial derivative exist at that point, but function is not differentiable at that point. How can we should a function is not differentiable? Not to show their functions not differentiable at 0 comma 0, we can use any one of the definition. So, try to find out this limit.

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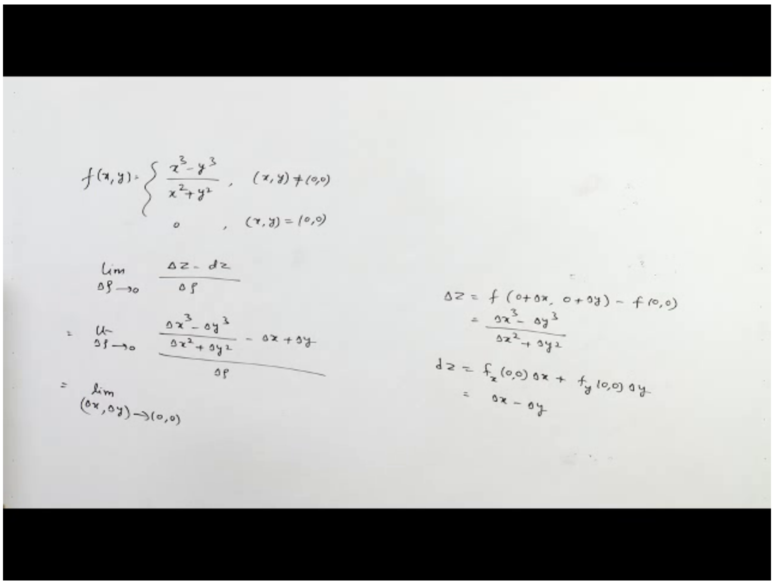

Now, limit delta root tend into 0 delta z minus d z upon delta rho first what is delta z delta z is f of 0 plus delta z delta x 0 plus delta y minus f 0 0, which is delta x cube minus delta y cube upon delta x square plus delta y square. This value is 0. Now what is d z? D z is f x at 0 0 into delta x plus f y at 0 0 into delta y, which is delta x minus delta y, because this is one and this is minus 1.

So, what will be what will be delta z minus d z? So, this will be delta x cube minus delta y cube, upon delta x square plus delta y square minus delta x plus delta y, and whole divided by delta rho. Now we can convert this limit as limit delta x delta y tending to 0 0, because when delta rho tending to 0. And delta rho is under root delta x square plus delta y square, when delta rho tending tend to 0 tend delta x delta y will definitely tend into 0 0.

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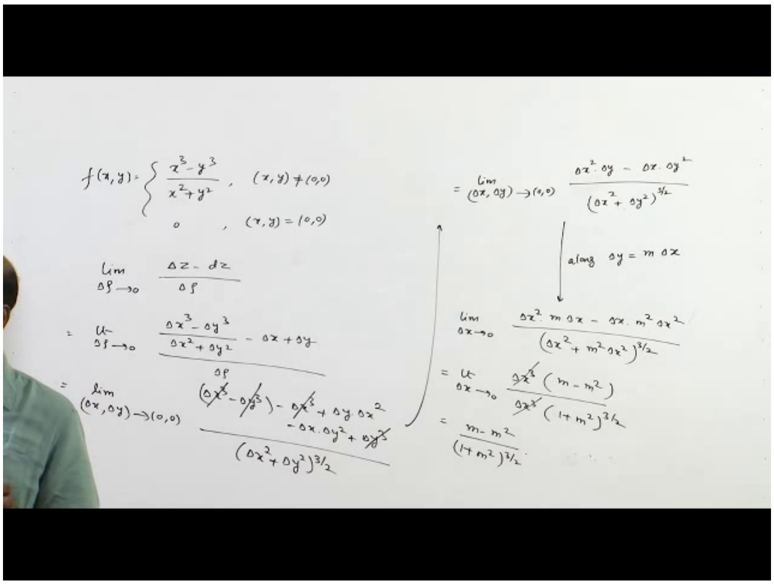

And this will be delta x cube minus delta y cube, minus delta x cube plus delta y into delta x square ok. Then minus delta x into delta y square, and plus delta y cube. And whole divided by now this expression is delta x square plus delta y square the under root of delta x square plus delta y square and this term.

So, this will be delta x square plus delta y square whole raise to power 3 by 2. Now this term is cancel with this term, this term is cancel this term. So, we are having this expression as we are having this expression as is equals to limit, delta x delta y tending to 0 0. This is delta x square into delta y minus delta x into delta y square, upon delta x square plus delta y square whole raise to power 3 by 2. Now function is not differentiable at that point this we have to prove. So, what we have to prove basically we have to prove that this limit does not exist. Because if this is limit exists tend is equal to 0, then function will be differentiable at that point. So, anyhow we have to show that this is path dependent.

So, you can easily see that the power of the numerator is 3 and the power of denominator is also 3. So, if you take a linear function. So, it will be comes out to be dependent on m the parameter m. So, move along move along delta y is equals to m delta x. If you move along this path then it is delta x is 0, it is delta x square into m delta x minus delta x in to m squared delta x square upon delta x square plus m square delta x square whole raise to power 3 by 2. And this will be equals to limit delta x tending to 0, delta x cube m minus m square upon it is delta x cube come out and it is 1 plus m square whole raise to power 3 by 2. So, this will cancel out and this will be m minus m square upon 1 plus m square whole raise to power 3 by 2. So, this limits depends on m, and hence this limit does not exist. And this limit does not exist means function is not differentiable at 0 comma 0.

So, that is how we can show that function is not differentiable. If it is not differentiable; that means, we have to show that this limit does not exist ok. Now let us see some more properties of differentiability.

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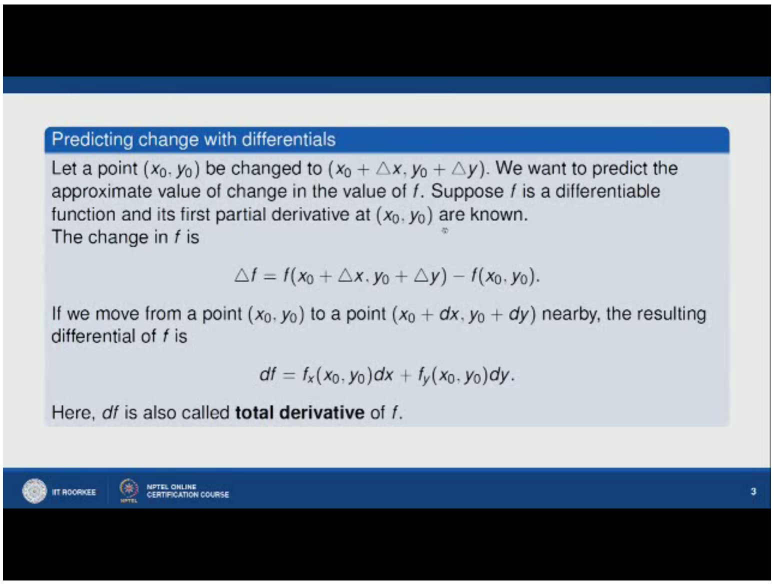

Now, let a point x naught y naught be changed to x naught plus delta x and y naught plus delta y. We want to predict the approximate value of change in the value of f ok, x changing x y changing from x naught y naught to x naught plus delta x y naught plus delta y. So, what will be the approximate change the value of a function that you want to find out? How can we find that? Suppose f is a differentiable function and it is first order partial derivative at x naught y naught are known that changed. The value of a function is given by this expression. We already know this thing. Now if we move from a point x naught y naught to a point x naught plus d x and y naught plus d y nearby the nearby point the resulting differential of f is given by this thing.

So, what is the resulting differential of f? Resulting differential of f is d f is equals to f x x naught y naught d x plus f y at x naught y naught to d y which we are also calling as total differential of f.

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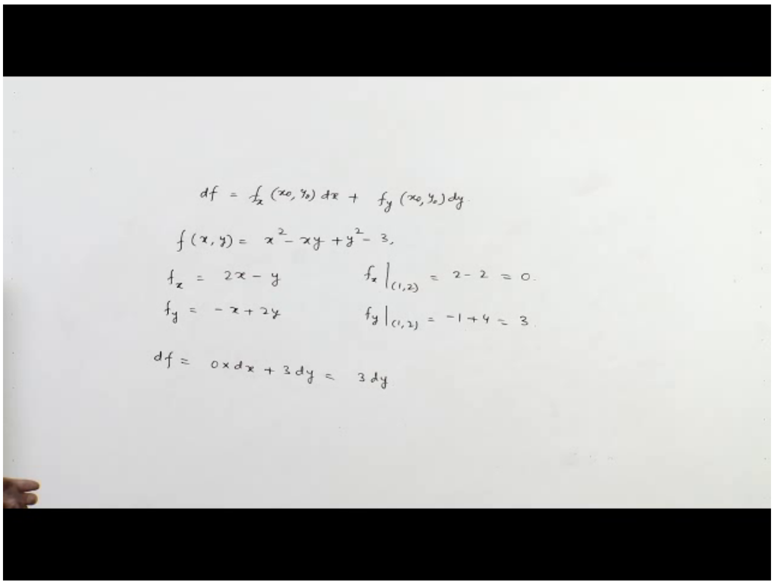

Now let us discuss few examples based on this. Basically, we are finding the approximate change the value of f by this expression. This is the first order derivative at x naught y naught f y f x in f y. And they are change the value of x and change the value of y.

So, let us discuss few examples the first problem.

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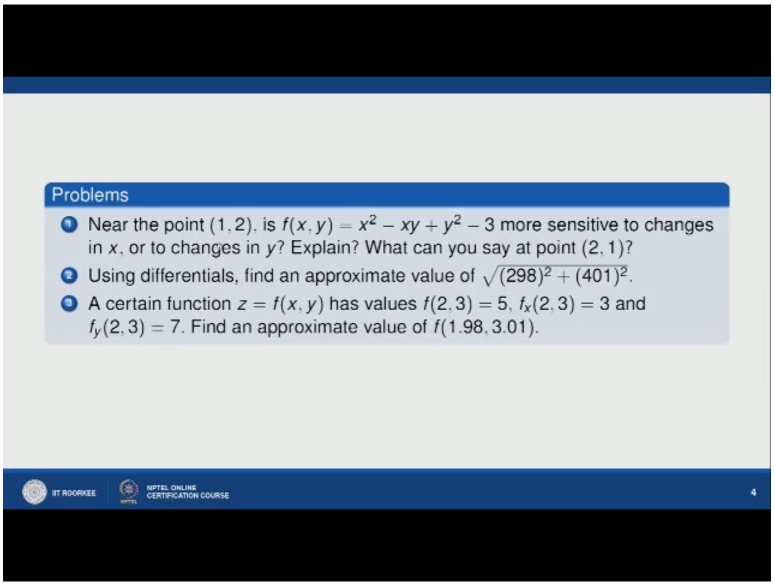

Near the point 1 comma 2 is f x y, which is equal to x square minus x y plus y square minus 3. More sensitive to changes in x or to change in y this way to find out is it more sensitive in changing x for changing in y. And what can we say about this at 0.2 comma 1? So, what is the function? Function here is x square minus x y plus y square minus 3 ok. First find f x, f x is 2 x minus y. And f x at point is 1 comma 2 add 1 comma 2. It is 2 minus 2 which is 0 now f y. F y is minus x plus 2 y. So, what is f y at 1 comma 2? It is minus 1 plus 4, that is 3. So, what will be d f? D f will be f x at x naught y naught f x at 1 comma 2 which is 0 into d x plus 3 into d y which is equal to 3 d y.

So, this means the change in the; if you change the value of x, the change the value of there will be no change the value of f. Because there is no term in the there is no term of d x here ok. If you change the value of x, there will be no change the value of f at a point 1 comma 2. But if we change the value of y, the value of function will change by 3 unit. Suppose we change y by one unit, that the change in the value of f will be 3 units. So, that means, it is more sensitive sensitive to change in the variable y not of x it remains unaffected in the change the value of x. Now if we can check the same sensitivity at a point say 2 comma 1. So, how can we check that now at 2 comma 1 we have to find out.

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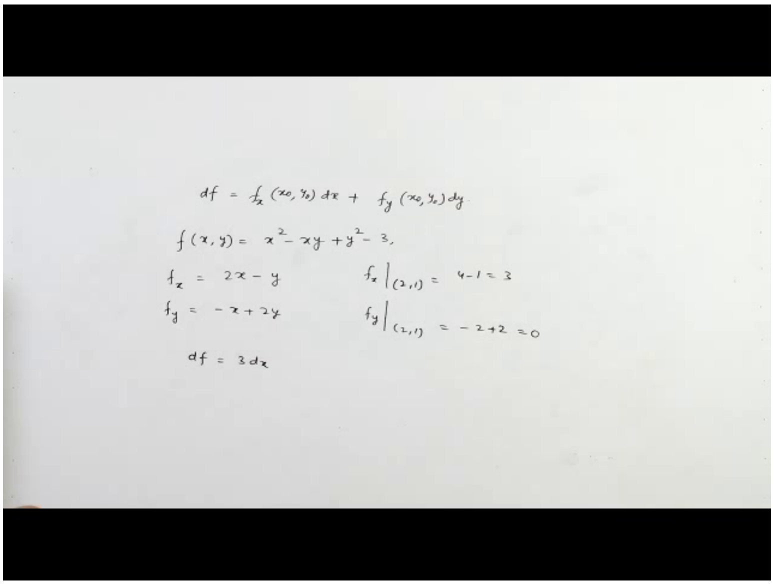

So, at 2 comma 1 it is equals to 4 minus 1 which is 3, and at 2 comma 1 it is minus 2 plus 1 minus 2 plus 2 is 0 .

So, what will be d f? D f will be simply 3 d x because this is 0. So, what we can say now? We can say that this that change the value of a function will remains unaffected, if you change in the y nearby. Nearby means near the 0.2 comma 1. Now if you change the value of x by 1 unit the value of function will change by 3 units; that means, at this point it is more sensitive to the change in the value of x as remains unaffected to change the value of y ok. Now the second problem using differentials, find the approximate value of this term.

So, what is the term? Basically, it is under root of 298 whole square plus 401 whole square.

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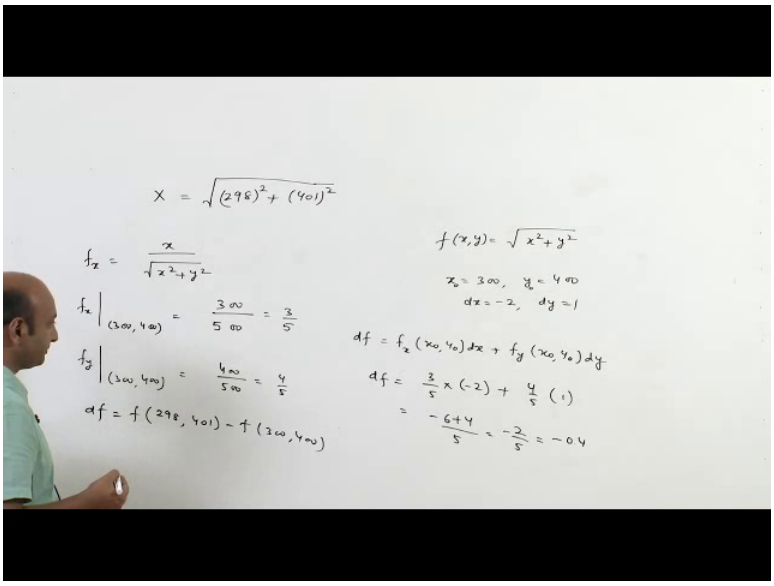

So, we have to find out the approximate value of this x. So, how can you find this? Let us define function as under root x square plus y square, under root of x square y square ok. Take x as 300 and y as 400 take d x as minus 2 and d y as 1 ok. Because this minus 2 will be 298, and this plus this will be 401. The next point, basically we start from this point this is x naught y naught basically this is x naught y naught. And x naught plus d x will be this is point, and y naught plus d x will be this point. So, what will be the approximate in the function? Function value, that will be given by f at f x at x naught y naught into d x plus f y at x naught y naught into d y. What is f x f of this? F x of this it is x upon under root x square y square and f x at f x at 300 400 will be.

So, at this 0.300 and 400. So, when you substitute f x 300, it is a square is 9 square is 9, 16, 25, 5 it is 500. So, it will 3 upon 5 and what will be f y at 300, 400 that will be 400 upon 500 that is 4 upon 5. So, what will be d f? D f will be 3 upon 5 into d x is minus 2 plus f y is 4 upon 5 and d y is one. So, that will be a simply minus 6 plus 4 upon 5, that is minus 2 by 5 that will be minus point 4. Approximate change the value of a function is minus point 4. So, what will the function value? What will a function value? So, d f will be, d f will be f at f at 298 and 401 minus f at 300 and 400. Because f at x naught plus delta x y naught plus delta y, and f minus f y f at x naught y naught.

So, what is function value of this? So, f at 298 and 401 will be d f plus f at 300 and 400.

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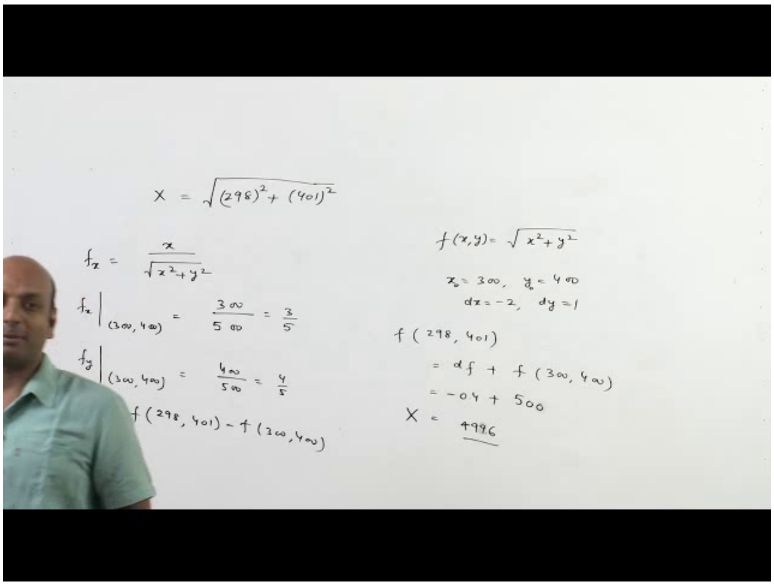

And d f is minus point 4 as we have already computed, and this is simply 500 is substituted here. So, this will be 49.6. So, the approximate the value of this capital x will be 499.6. Now the third problem, a certain function z is equals to f x y has values f at 2 3 is 5 f x is 3 and f y is this find the approximate value of this. So, again we can easily find out in the same concept. So, what are thinks given to us F at 2 comma 3 f x at 2 comma 3 and f y at 2 comma 3.

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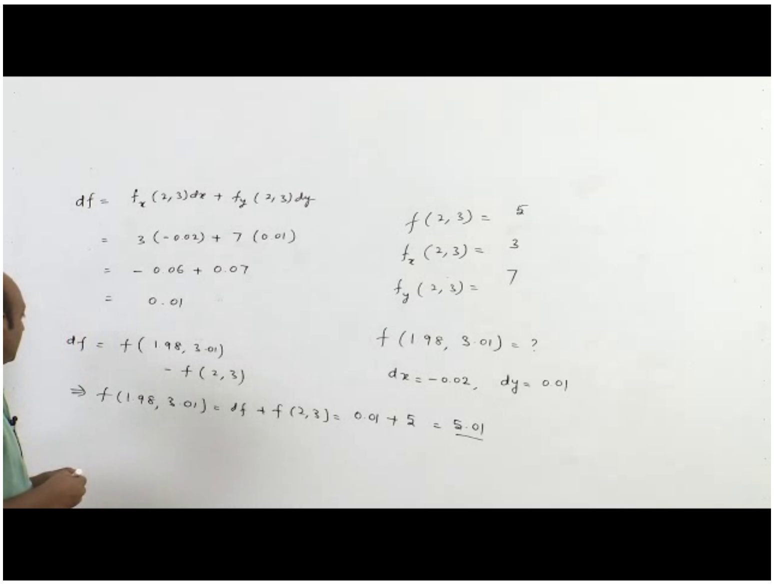

So, what is f at 2 comma 3? Is 5 3 7 now where to find out f at 1.98 and 3.01.

So, we can take d x as minus 0.02 ok, because 2 when you add with this you get 1.98, and d y as 0.01. Now d f the approximate change the value of a function will be will be f x at f x at 2 3 d x plus f y at 2 3 into d y. So, this will be 3 into minus 0.02 plus 7 into 0.01 that will be minus 0.06 plus 0.07 that is 0.01. And the change the value of function is given by f at 1.98 3.01 minus f at 2 3. So, this implies f at 1.98 3.01 will be equals to d f plus f at 2 3. And d f is 0.01 plus f at 2 3 is 5. So, this will be 5 that mean this value is 5.01.

So, the approximate value of a function at this point is 5.01. So, there is one type of problem which can be solved using differential. Now we can also find absolute relative and percentage change in the value of the function. How we can find out? So, if you move from a point x naught.

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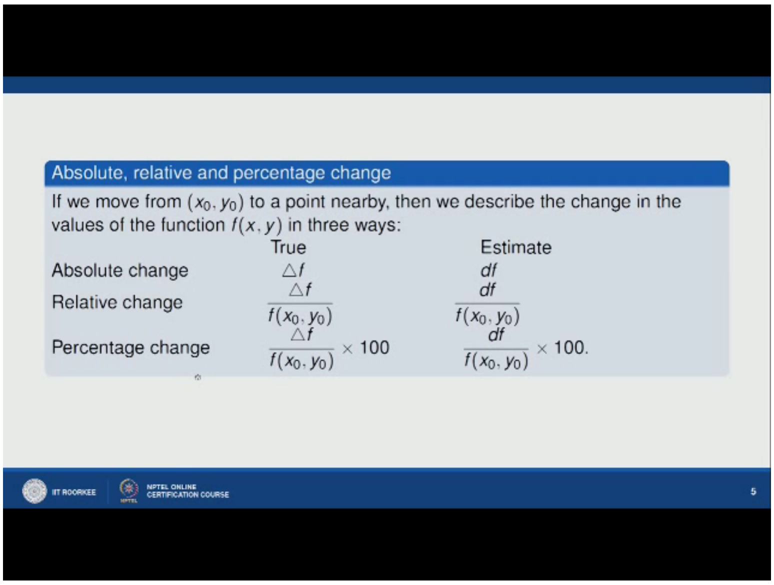
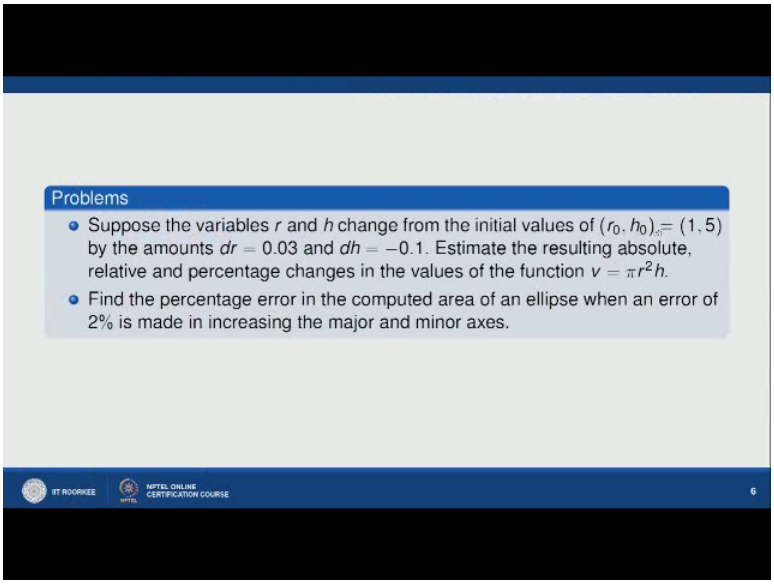


Y naught to a point nearby, then we can describe the change in the value of the function f x y in 3 ways. First is absolute change, which is an estimate is d f true is delta f, relative change with true is delta f divided by f at x naught y naught and estimate is d f upon f at x naught y naught. The percentage change means you will simply multiplied the relative change by 100 ok. So, how can we solve some problems based on this let us see. Suppose a variables r and h, change from the initial values r naught h naught which is 1 comma 5 by the amount d r equals to 0.03, and d h equal to minus 0.1.



Then estimate the resulting absolute error relative and percentage change in the value of function V equal to pi r square h.

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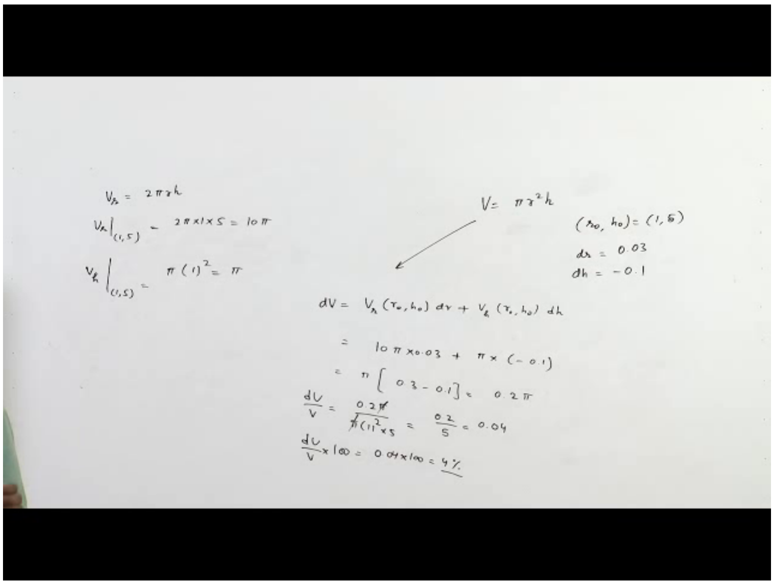

The initial point r naught h naught is 1 comma 5. D r is the change the value of r is 0.03. And d h is d h is minus 0.1 ok. Now we have to find the relative and absolute change in the value of V. How can you find that? For you find dV dV will be. Now here V is a function of r and h ok. In instead of x and y we are having r and h.

So, how can we define d r here d r will be V r at r naught h naught, comma d r plus d r sorry into d r and V h at x naught y naught, r naught r naught h naught into d h. Because here in the instead of x and y we are having r and h as independent variables. So, we have to differentiate partial respect to r and h only, and function is V. So, what is V r? So, first you find V r, V r will be 2 pi r h and V r at 1 comma 5, which is r naught h naught is 2 pi into 1 into 5; which is 10 pi. Again, V h at 1 comma 5 will be; what is V h, V h will be pi r square. So, pi into r square r is 1. So, that is pi. So, this will be 10 pi into d r is 0.03 plus V h is pi into this is minus 0.1. You take pi common. So, this is 0.3 minus 0.1 which is 0.2. So, it is 0.2 pi.

So, what will be? What will be f relative change? Relative will be dV upon V, that is 0.2 pi upon the value of the V at r naught h naught, value the function at x naught y naught ok. So, that will be pi r square into h. So, that will be a simply pi-pi cancel out, it is 0.2 upon 5, that is point 0.04. And the and the percentage error will be percentage change will be these into 100 that is 0.04 into 100, that is 4 percent. So, that is how we can find out absolute relative and percentage change in the value of the function. The next problem find the percentage error in the computed area of an ellipse, when an error of 2 percent is made in increasing the major and minor axis.

So, what what is area of the ellipse is pi a b.

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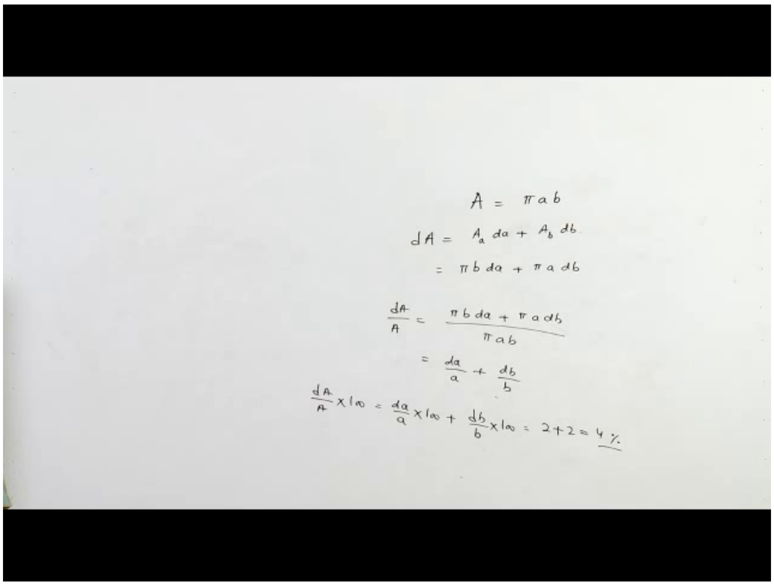

Now what will change in the area d A? will be A a d A plus A b d b. I mean here here, instead of x and y we are having a and b. So, we have to differentiate partial partial respect to a and b only ok. So, here we are we are we get pi b into d a plus pi a into d b now d a upon a will be equals to pi b d a plus pi a d b upon pi a b. And that will be d a upon a plus d b upon d. So, the percentage change will be d upon a into 100 will be equals to d A upon A into 100 plus d b upon b into 100. So, that will be 2 plus 2 that is 4 percentage. So, that is how we can find out the corresponding percentage error in the value of the area of the ellipse. Now there are also some more problems based on this. Let us solve one more problem.

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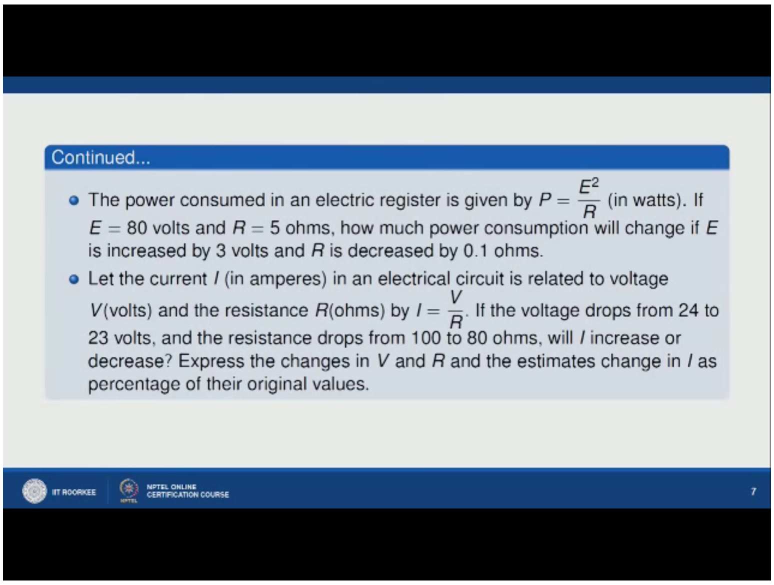

The power consumed in the electric resistance given by P is equals to E square upon R in watts.

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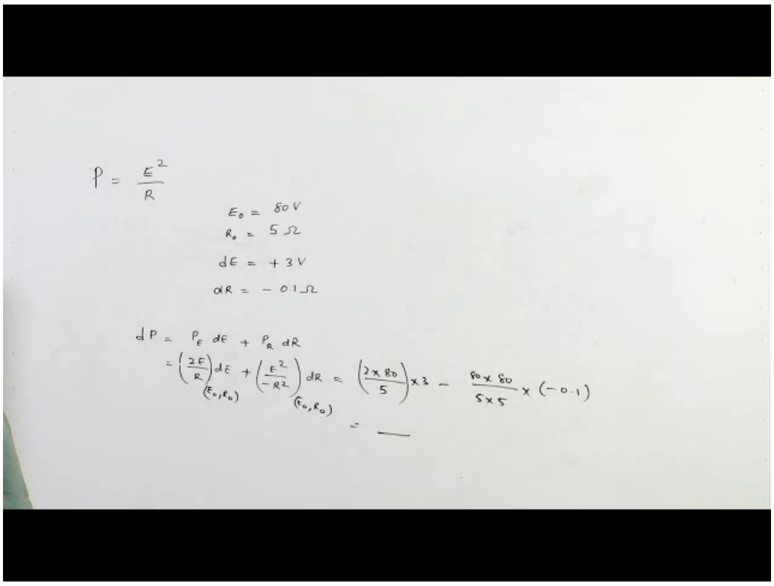

If e equal to 80 volt, and r equals to 5 ohms, how much power consumption will change if E is increased by 3 volt and R is decrees by 0.1 ohm?

So, that we can find out. So, here E naught is E naught is 80 volt. And r naught is 5 ohms ok. D E is it is increased by 3 volt, that is plus 3 volt and d r is decrease by 0.1 ohms that is minus 0.1 ohm. And we have to find the corresponding change in the power, power consumption ok. So, how can you find now? So, here p is given by this expression. So, what will be the relative change the value of P? That will be now p is function of E and R. So, it will b P E into d E plus P R into d R ok. Because here independent variables are E and R.

So, when you differentiate here is respect 2 r. So, it will be 2 e upon r into d e plus it is e square upon minus R square into d R. Now what is E? Because we have to find this value at this value at e naught r naught, these values at E naught R naught. Now what is E naught? E naught is 80. So, it is 2 into 80 upon it is 5 d E is 3 minus this is minus e square is 80 into 80, upon R square is 5 into 5 into and d R is minus 0.1. So, we can simplify this and we can find out the corresponding change in the power consumption. Similarly, we can solve the next problem also, which is given in electrical circuit. Now now the first problem of the next slide is say y equal to u V where u and V are independent variables.

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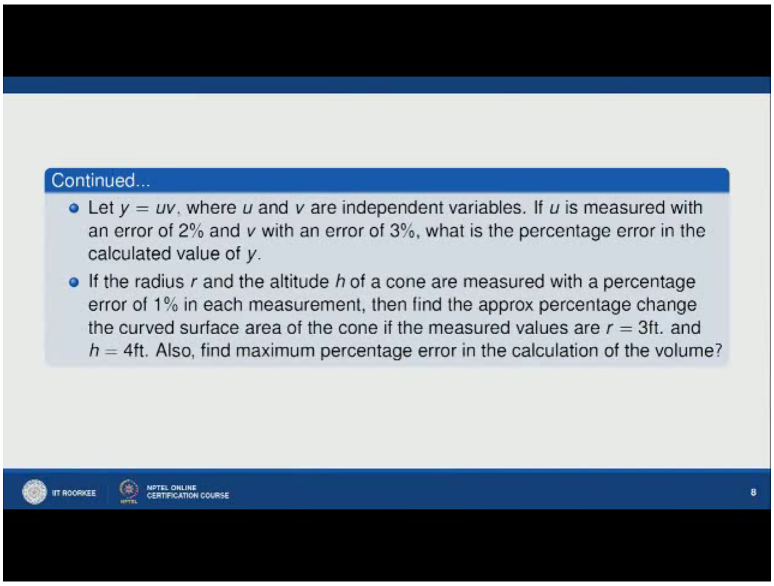

If u is measured with 2 percent error and V is measured with 3 percent error, what is the percentage change the value of y it is very simple problem.

So, here y is equals to u V. So, d y will be equals to basically y u into d u plus y V into dV.

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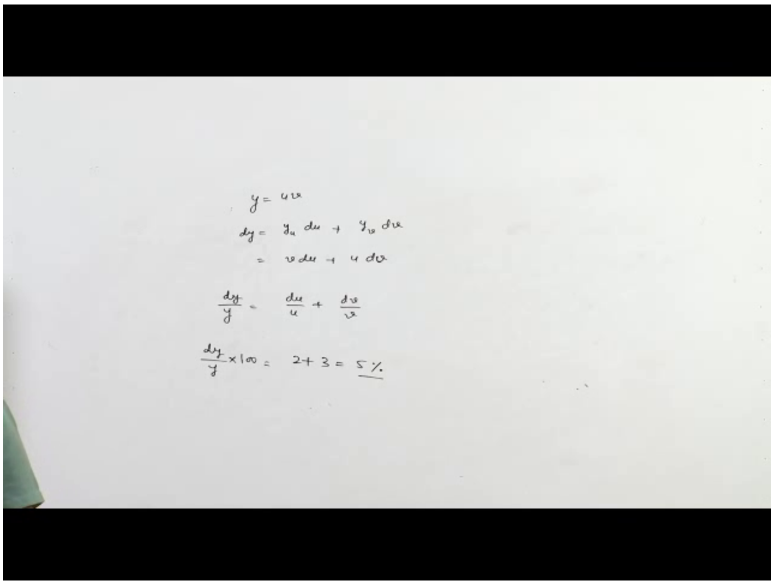

That is V d u plus u dV. And what is d y upon y? That will be d u upon u plus dV upon V. So, d y upon y into 100 that is a percentage change in the value of y will be the percentage change in the value of u plus percentage in the value of V; that is 2 plus 3, that is 5 percent. Because it is given to was that a 2 percent is the percentage change in the value of u, and percentage change the value of V is 3 percent. But at the corresponding change the value of y will be 5 percent. So, that is how we can solve problems of this type.

So, thank you very much.