Multivariable Calculus
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Lecture - 07
Differentiability- I

Hello friends. So, welcome to lecture series on multivariable calculus. So now, we will deal with differentiability; that when we can say there are several variable function is differentiable. We already know about single variable functions that differentiability of single variable functions. How can I define differentiability of single variable functions? Suppose, suppose, x is changing from.

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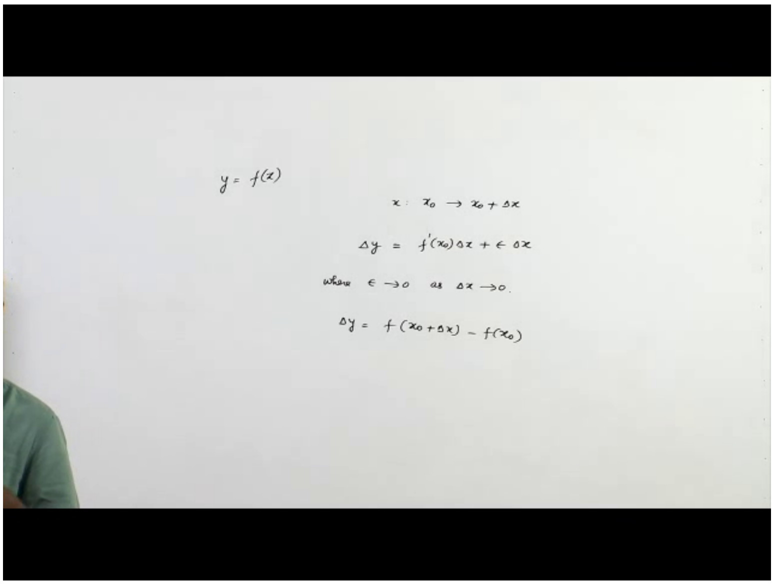

X naught to x naught plus delta x ok then if f is differentiable, then the corresponding change the value of the function ; where y is equals to f x y oh sorry, f x because we are dealing with single variable function here, say y equal to f x.

So, if x is changing from x naught to x naught plus delta x, then the corresponding change in the value of the function will be given by f dash x naught delta x plus epsilon into delta x; where epsilon tending to 0 as delta x tending to 0. So, this is how we define differentiability of a function at a point x naught. A function is differentiable at a point x naught this means, that if x is changing from x naught to x naught plus delta x, then the corresponding change the value of the function will be equal to f dash x naught delta x plus epsilon into delta x; where epsilon tending to 0 as delta x tending to 0. And how we define delta y? Delta y will be simply f of x naught plus delta x minus f x naught. This is the change in the value of the function, when x is changing from x naught to x naught plus delta x.

Now, similarly we can extend this definition for 2 variable functions. How could you find differentiability for 2 variable functions? Now suppose function is given as Z is equals to f x y, function of 2 variables ok..

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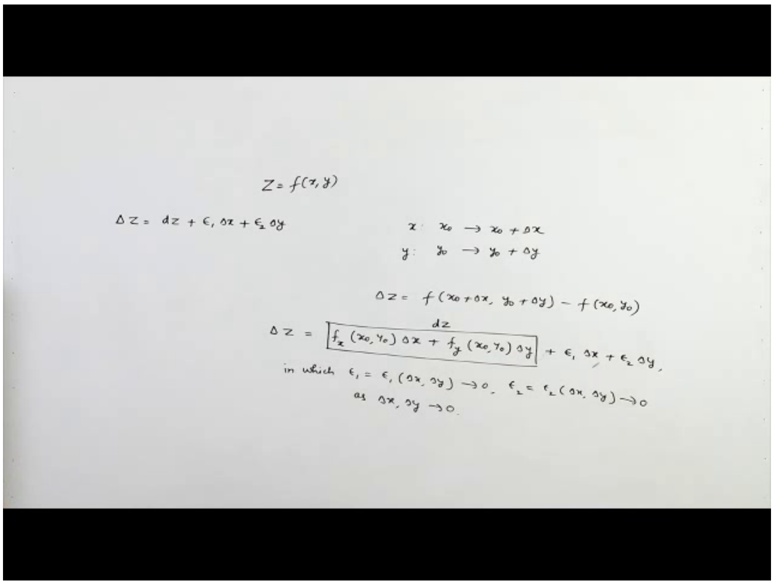

Suppose x is changing from x naught to x naught plus delta x. And y is changing from y naught to y naught plus delta y ok. Now corresponding change, the value of Z, which is delta Z will be nothing but f of x naught plus delta x y naught plus delta y minus f x naught y naught. So, this will be the corresponding change the value of the function; when x is changing from x naught to x naught plus delta x, and y is changing from y naught to y naught plus delta y.

Now, this delta x can be written as f x at x naught y naught delta x plus f y x naught y naught into delta y plus epsilon 1 delta x and plus epsilon 2 into delta y. And in which epsilon 1 ; which is a function of delta x delta y will tend to 0, and epsilon 2 ; which is again a function of delta x delta y will tend to 0 as delta x and delta y will tend to 0. So, here this is called the increment in the value of Z. This will be given by f x at x naught y naught in delta x, plus f y at x naught y naught in delta y plus epsilon 1 delta x epsilon 2 delta y where epsilon 1 and epsilon 2 are the functions of delta x, and delta y and they tend to 0 as delta x delta y tend 0 ok. now if this definition hold, if this condition hold, then we say that a function is differentiable at a point x naught y naught ok.

So, first we are having differentiability for a single variable function which we have discussed.

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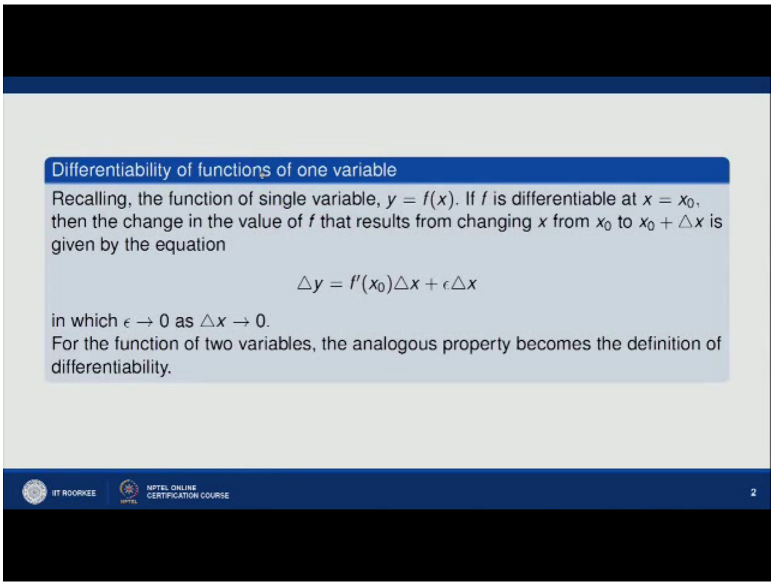

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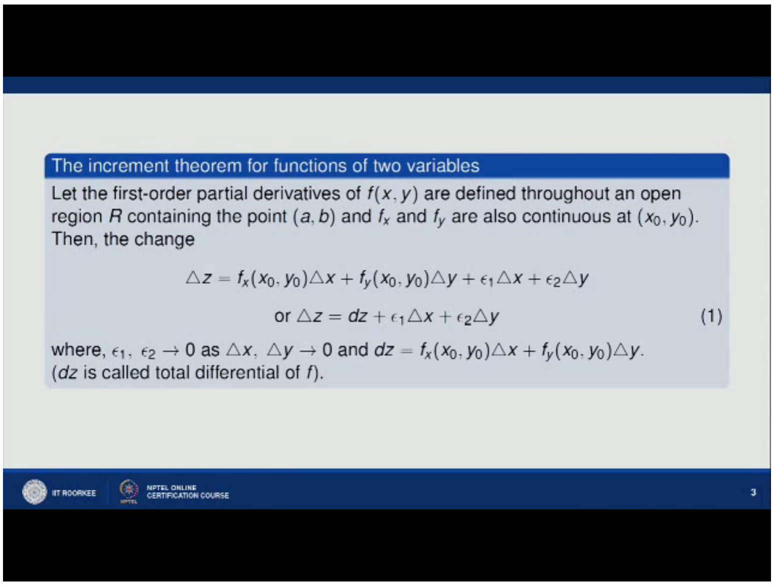

Then the increment theorem of the functions of 2 variables. It is states that if the function if the first order partial derivative of f x y are defined throughout an open throughout an open region are containing the point a b and f x and f are also continuous at x naught y naught, then the change in the value of the function will be given by this expression; where epsilon 1 epsilon 2 are tending to 0 as delta x delta y tending to 0.

Now, this term this term; which we are having here we are we ca [can] we can denote this term by dz, and this dz is also called total differential of f. So, basically, we can write, we can write delta Z as dz plus epsilon 1 delta x plus epsilon 2 delta y ok. This definition has this definition also. Now a function f x y is said to be differentiable at a point x naught y naught..

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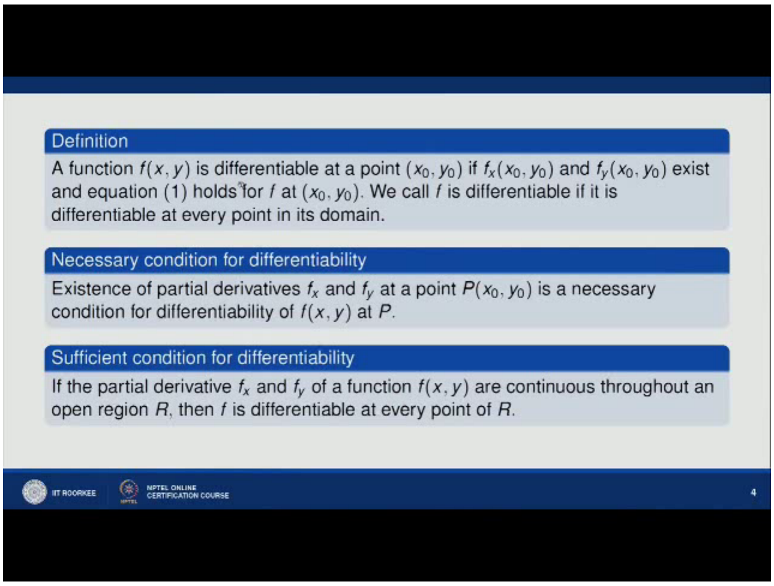

If f x at x naught y naught and f y at x naught y naught exists, and equation one holds for f at x naught y naught; that is, that is this equation. If this equation holds at x naught y naught, and first partial derivative at x naught y naught exists, then we say that the function is differentiable at a point x naught y naught. We call f is differentiable, if it is differentiable at every point in this domain. If this definition holds for every point in it is domain then we say the function is differentiable.

So, what is the necessary condition for differentiability. Of course, if this partial derivative exists at a point, then this is a necessary condition for the differentiability. So, existence of partial derivative f x and f y at a point P is a necessary condition for the existence of differentiability at a point P. And what you are sufficient condition? Sufficient condition is if the partial derivative f x and f y of f x y are continuous throughout an open region R then function is differentiable at every point of R. So, for sufficient condition, we must have continuity throughout an open region of first order partial derivative f x and f y of the function f x only. Then we can say the function is always differentiable ok.

Now, first we solve these 2 problems, then we come to problems related to differentiability. Now we are having first problem say, say Z is equals to tan inverse y by x ; where y by x where x y is not equal to 0 0..

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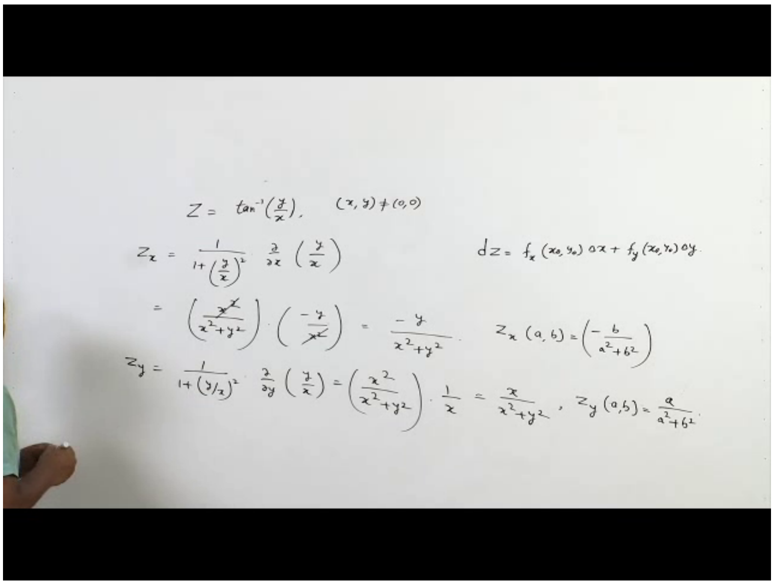

And we have to find total differentiability total differential of this function. So, what do you mean by total differential? Total differential is simply dz of f x at x naught y naught into delta x plus f y at x naught y naught into delta y. So, this is differentiability or I mean total differential. So, let us find the total differential at a point say a comma b. So, what is f x of this function f x or Z x ok, here we are having Z. So, we can call it Z x, it is 1 upon 1 plus y upon x whole square into del by del x of y by x.

So, what it is? It is equals to x square upon x square plus y square into minus y by x square, which is equal to minus y upon x square plus y square. So, Z x has at a comma b, a general point a comma b will be minus b upon a square plus b square. Now similarly Z y will be 1 upon 1 plus y by x whole square into del by del y of y by x, which is equal to x square plus upon x square plus y square in to; the partial derivative of this with respect to y which is 1 by x, and this will be x upon x square plus y square. So, this value so, Z y at a comma b will be, will be a upon a square plus b square.

So, what will be total differential of this function? Total differential of this function will be given by will be given by dz, which is f x at x naught y naught or a comma b which is which is minus b upon a square plus b square times delta x plus, a upon a square plus b square times delta y..

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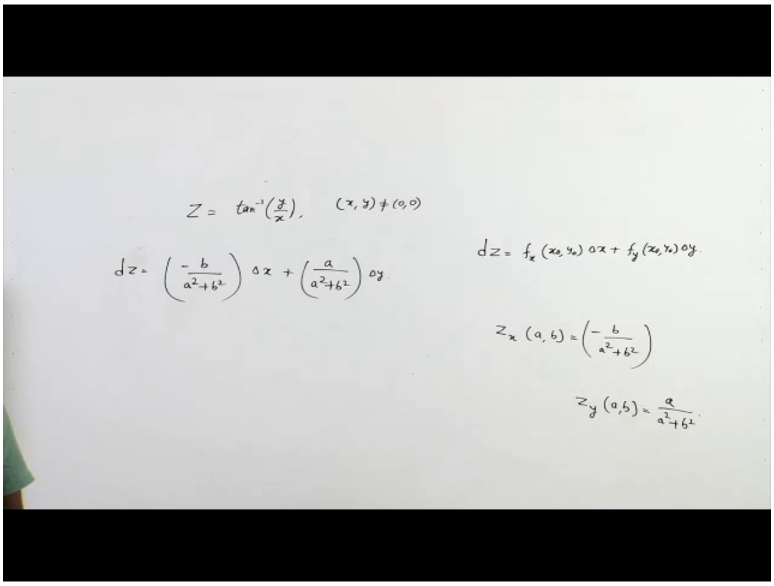

So, this will be the total differential of this function at a point a comma b ok. Similarly for a second problem, the second problem is basically u is equals to x square plus y square plus Z square whole raise to power minus 3 by 2.

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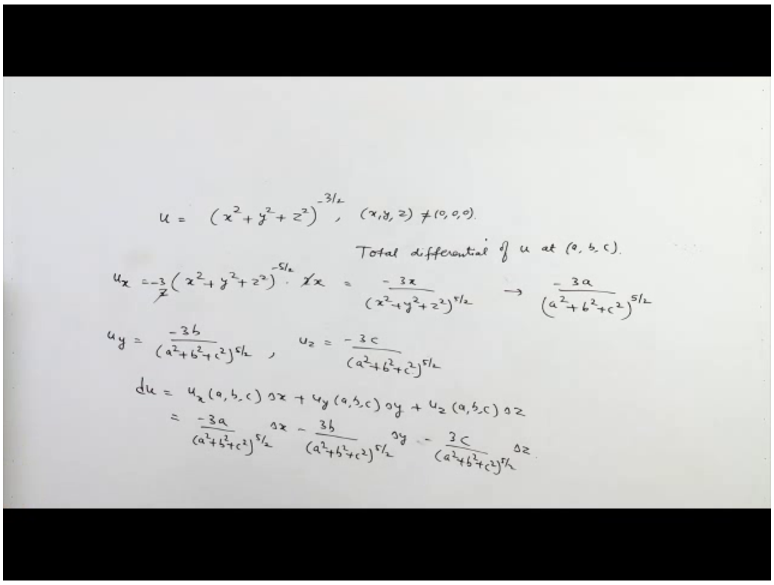

And x y Z should not equal to 0 0 0. So, what we have to find? We have to find total differential of u. Say at a point a comma a b c, say at a point a b c.

So, what will be u x? U x will be x square plus y square plus Z square, minus 3 by 2 and it is minus 5 by 2, and it is into 2 x. So, 2 2 cancels out. It is minus 3 x upon x square plus y square plus Z square, whole raise to power 5 by 2 ok. And this value at a b c is minus 3 a upon a square plus b square plus c square whole raise to power 5 by 2. Now similarly, by the symmetry we can easily write u y as minus 3 b upon a square plus b square plus c square whole raise to power 5 by 2. And similarly, u Z will be minus 3 c upon a square plus b square plus c square whole raise to power 5 by 2.

So, what will be total differential of u? So, total differential of u, which is d u is given by u x at a b c into delta x plus u y at a b c into delta y, plus u Z at a b c into delta Z. So, this will be equals to a minus 3 a upon a square plus b square plus c square whole raise to power 5 by 2 into delta x minus 3 b upon a square plus b square plus c square whole raise to power 5 by 2 into delta y, plus minus 3 c upon a square plus b square plus c square whole raise to power 5 by 2 into delta Z.

So, this is how we can find out total differential of a function f at a point a b c or a b ok. Now how can we show data function is differentiable? So, let us discuss few examples based on this.

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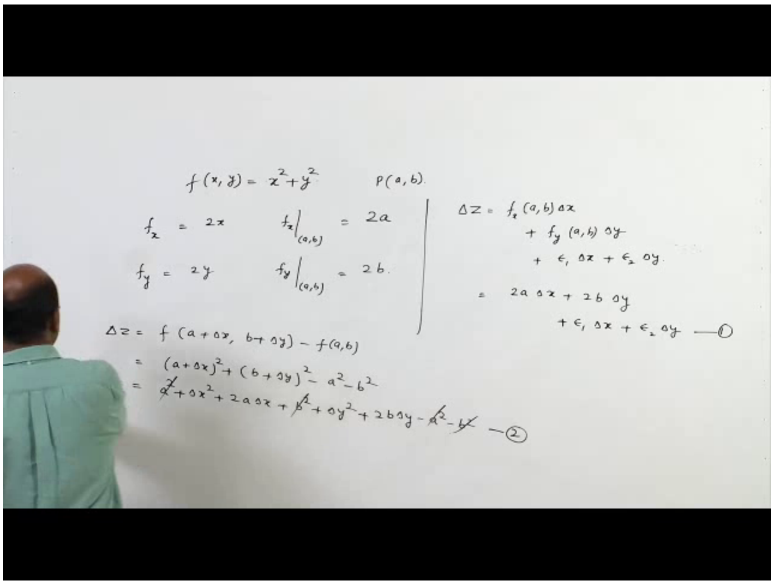

So, first example is f x y is equals to x square plus y square. We have to show that this function is differentiable at any point a comma b. So, for differentiability, first is existence of partial derivative, first order partial derivative at a comma b is required, and sank it is the equation one should hold. Equation me one means that delta Z is equals to f x at a comma b delta x plus f y at a comma b delta y. Epsilon 1 delta x plus epsilon 2 delta y, and epsilon 1 epsilon 2 should tan to 0 as delta x delta y tan to 0.

So, if these 2 conditions hold, then we say the function is differentiable at a point a comma b. So, first we will find f x at a comma b. So, this f x in f y will definitely exist for this function. So, f x will be 2 x and f x at a comma b will be simply 2 a. Similarly, f y will be simply 2 b, and f y at a comma b will be simply to b ok. it has 2 y sorry. Now so, partial first order partial derivative exists at a comma b. Now what is the definition? What is the equation 1? That is delta Z is equals to f x at a comma b into delta x plus f y at a comma b into delta y plus epsilon 1 delta x plus epsilon 2 delta y.

So, we have to write a this equation, try to find out delta one sorry epsilon 1 of epsilon 2 which are the functions of delta x and delta y, and try to show that epsilon 1 and epsilon 2 are tending to 0 as delta s delta y tend to 0. If they tend to 0 as delta x delta y tend to 0, then this function f will be differentiable at a point a comma b.

So, let us try to find out these values. So, it is clear that f x is 2 a. So, it is 2 a into delta x plus it is 2 b into delta y plus epsilon 1 into delta x plus epsilon 2 into delta y. Now what will be delta Z? Delta Z is a increment in the value of Z or the change in the value of the function. When x is changing from x naught to x naught plus delta x, and y is changing from y naught to y naught plus delta y. Here instead of x naught y naught we are having a comma b ok. So, what will be delta? Delta Z delta Z will be f of a plus delta x b plus delta y minus f a b. Instead of x naught y naught we are having a comma b. So, what is this value now? This from here it is a plus delta x whole square plus b plus delta y whole square minus a square minus b square.

So, when you open the bracket it is a square plus delta x whole square plus 2 a delta x plus b square plus delta y square plus 2 b delta y minus a square minus b square. So, these terms will cancel out. So, this would be delta Z. Now you compare these 2-equation ok. This is also equal to delta Z, and this is also equal to delta Z. So, both must be equal. So, suppose it is equation number 1 and suppose it is equation number 2. So, from 1 and 2 what we have obtained?

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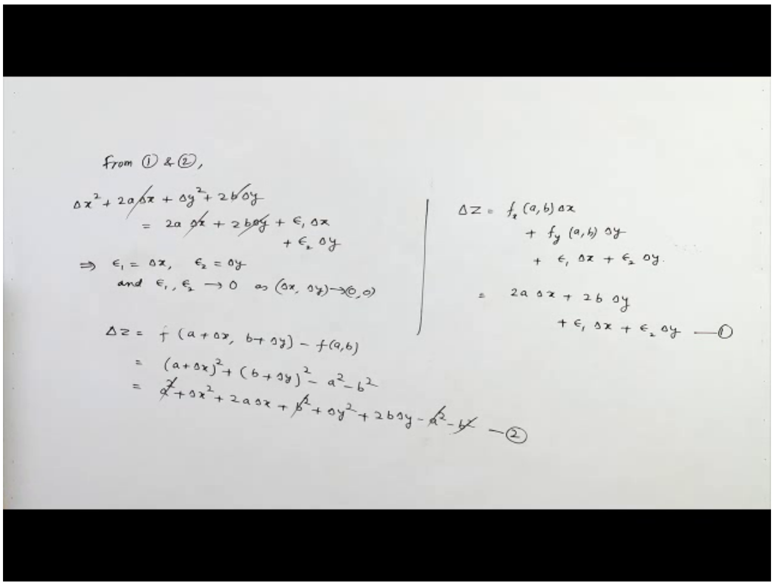

From one and 2 what we have obtained? We have obtained there delta x square plus 2 a delta x plus delta y square plus 2 b delta y is equal to 2 a delta x plus 2 b delta y plus epsilon 1 delta x plus epsilon 2 delta y.

Now, these terms again cancels out ok. So, if you compare both the side so, we can obtain this implies epsilon 1 is delta x and epsilon 2 is delta y. and definitely epsilon 1 epsilon 2 will tend to 0 as delta x delta y tending to 0. So, we can say the function this is differentiable at a point a comma b ok. And epsilon 1 epsilon 2 will tend to 0 as delta x delta y tending to 0 0 0. So, this implies there a function, this is differentiable at a point a comma b. So, that is how we can show that a function is differentiable. We have to find out 2 things basically. First, we have to find out the first order partial derivative at that point. The point where we are checking their function is differentiable or not. And second we have to find this expression delta Z equal to this expression. And try to show that epsilon 1 epsilon 2 which are the functions of delta x and delta y should tan to 0 as delta x delta y tan in to 0 0, your function is differentiable ok.

Now, let us try to show for a second example. Now what a second example?

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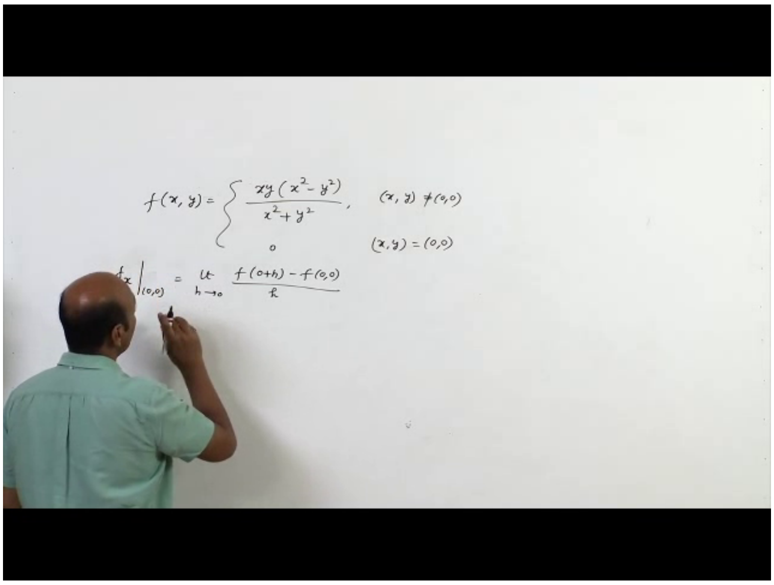

F x y is equal to x y x square minus y square x square plus y square. So, x y not equal to 0 0, and 0 when x y is equals to 0 0. And again at 0 comma 0, we have to show that this function is differentiable. So, first we have to find, first order partial derivative at 0 comma 0, a first partial derivative does not exist. So, there is no question of differentiability, because for differentiability first set of partial derivatives 2 types this exists. It is a necessary condition for differentiability ok. If function if first order partial derivatives does not exist fall this means, we clearly directly we can say whether the function is not differentiable at that point. So, first you find f x at a comma at 0 comma 0.

So, what will be f x at 0 comma 0? It will be limit h tending to 0 f of 0 plus h minus f 0 0 upon h ok. It is a 2-variable function sorry. So, so it will be 0 minus f of 0 0.

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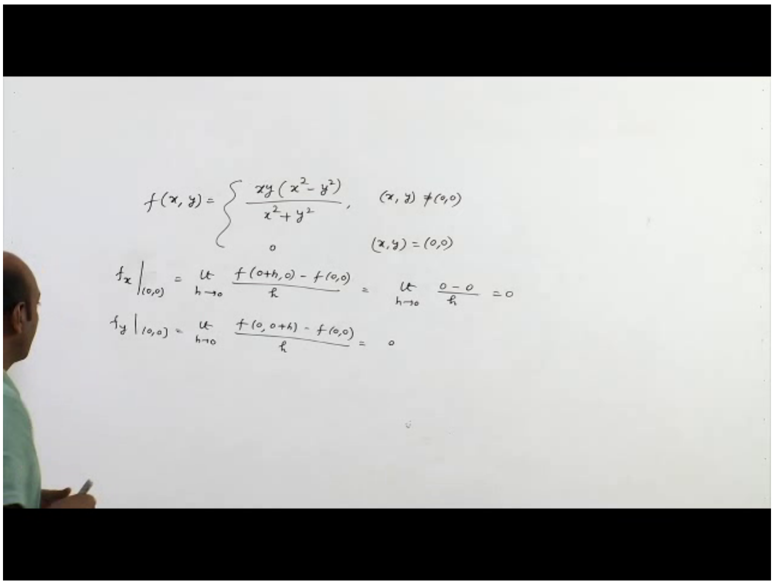

Now when you replace x by h and y by a 0 here. So, what will obtain? It is 0 minus 0 0 is 0 and upon h it is 0. Now what is f y at 0 comma 0? Limit h into 0 f of 0 comma 0 plus h minus f 0 0 upon h. This is by the definition of f y. Now you replace x by 0 and y by h. So, this value will be 0, 0 minus 0 because f at 0 comma 0 is 0 is 0. So, this means first order partial derivative at 0 comma 0 exists, and the value of both are 0 ok.

Now, now first now the expression is delta Z equal to F x at 0 comma 0 into delta x plus f y at 0 comma 0 into delta y plus epsilon 1 delta x plus epsilon 2 delta y.

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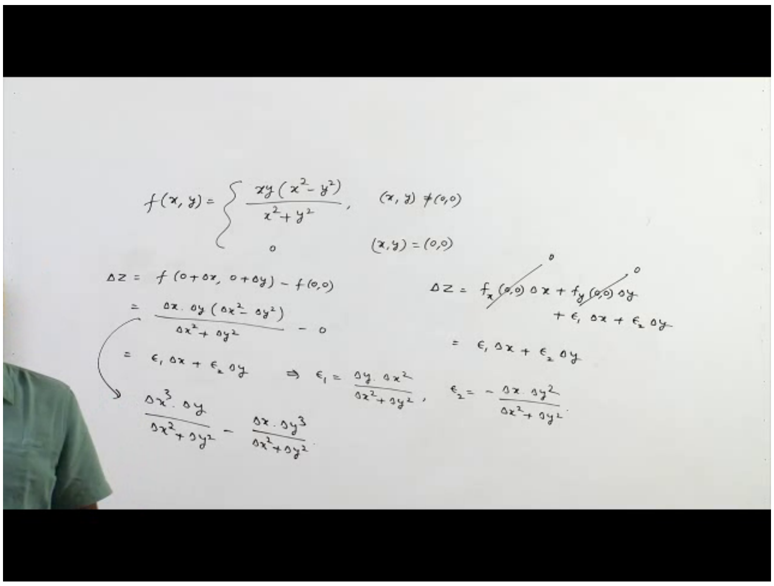

Now this term is 0 and this term is 0. So, this term will not be there in this expression. Because f x at 0 comma 0 is 0, and f y at 0 comma 0 is 0. So, this will be equal to epsilon 1 delta x plus epsilon 2 delta y. Now let us compute delta Z. So, what will be delta Z again? Delta Z will be f of 0 plus delta x, 0 plus delta y, minus f of 0 comma 0. Now it is delta x delta y delta x square minus delta y square upon delta x square plus delta y square minus 0 ok and this is equals to epsilon 1 delta x plus epsilon 2 delta y. So, if we compare these 2, if we compare these 2 so, we can get epsilon 1 as equal to this into this, or we can take delta coefficient of delta x as epsilon 1, and coefficient of delta y from this expression as epsilon 2.

So, epsilon 1 will be delta y into delta x square upon delta x square plus delta y square. And epsilon 2 will be minus delta x into delta y square upon delta x square plus delta y square ok. If you compare these 2, you take the coefficient of delta x as epsilon 1 and the coefficient of delta y as epsilon 2 ok. So, these expression of epsilon 1 and epsilon 2 is may not be unique. I am taking only one of the expression.

It may not be unique, someone can take some other expression may multiply with this, they can take as this expression as delta x, I mean coefficient delta x. You see when you multiply this with this. So, it will be it will be what? From here, from here it is delta x cube into delta y upon delta x square plus delta y square minus it is delta x into delta y cube upon delta x square plus delta y square.

Now, one can take, one can take coefficient of delta y from here. The co delta y is this upon this and coefficient of delta x from here, negative of this upon this. So, if you take any expression of any one of the expression of epsilon 1 epsilon 2 and try to show that epsilon 1 epsilon 2 tending to 0 as delta x delta y tends to 0 comma 0. Now we obtain this expression.

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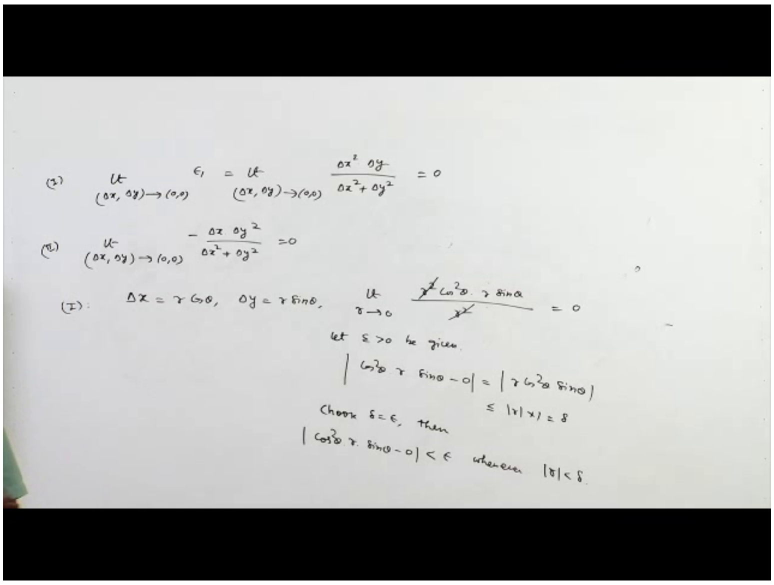

Now only thing now to prove that as limit delta x delta y tending to 0 0, epsilon 1 which is limit delta x delta y tending to 0 0, delta x square into delta y upon delta x square, plus delta y square. This limit and this limit epsilon 2; which is negative of delta x delta y square upon delta x square plus delta y square. This both limit exist or an equal to 0 this is remain to show for differentiability ok.

So, this exists an equal to 0 this to show. Now how can we showed at this limit exists an equal to 0? Let us take first case, I mean the first limit. The easiest way is you convert this in to polar coordinate system. You take delta x as R cos theta, and delta y as R sin theta. So, when you take delta x as R cos theta, and delta y as R sin theta so, this limit when delta x delta y tend to 0 comma 0. So, R will tend to 0. So, we can say that R will tend to 0. And this is R cos squared theta into R sin theta upon R square, R square also cancel out. Now whatever theta may be theta may be? Anything ok, whatever theta may be when R is tending to 0 and 0 multiplied by some finite value this will be equal to 0. And you can show this value is equal to 0 by delta epsilon definition also. You can take you can take let epsilon greater than 0 be given.

Now, when you take mod of cos squared theta into R into sin theta minus 0; which is equal to mod of R into cos square theta into sin theta and it is less than equals to mod R into 1, because mod of this is less than equal to 1 mod of this is less than equal to 1. So, this is less than equal to mod R, and if you take it equal to delta and choose delta equal to epsilon, then mod of cos square theta into R into sin theta minus 0 will be less than epsilon; whenever mod R is less than delta.

So, in this way, we can also show that this limit exists and is equal to 0, by delta epsilon definition. And similarly, for the second part also we can proceed in the same way. So, hence we have shown, that epsilon 1 and epsilon 2 what tending 0 as delta x delta y tending 0 comma 0. So, we can say that this function is differentiable at a point 0 0. So, in this way we can show that the function is differentiable ok.

Now, there is another way also to check whether function is differentiable or not. So, what is that way?.

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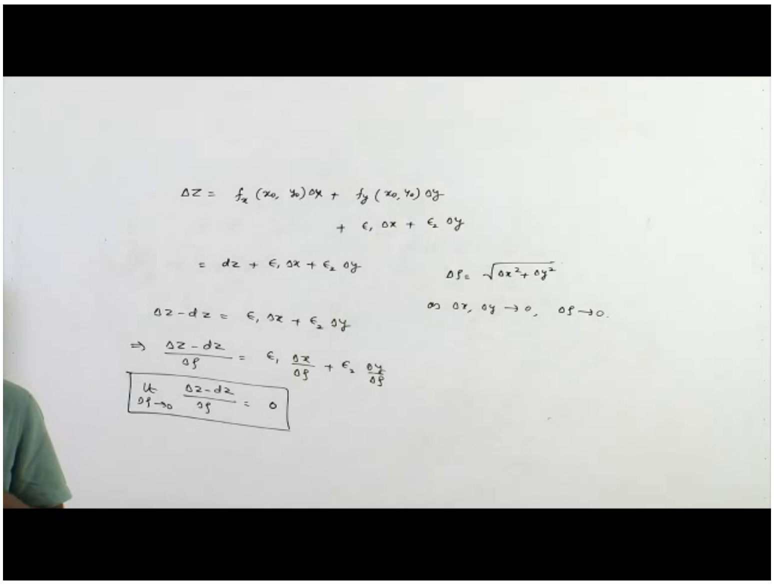

Ah you see that delta Z is equal to f x at x naught y naught into delta y delta x, plus f y at x naught y naught into delta y, plus epsilon 1 into delta x plus epsilon 2 into delta y ok. So, it is it is delta Z delta Z will be f x s naught y naught delta x plus f y x naught y naught delta y and this thing ok. Now you define delta rho as under root delta x square plus delta y square ok. If it you find delta rho as the under-root delta x square plus delta y square. So, definitely as delta x delta y tending to 0. So, delta rho will also tend to 0.

So, this term the first term we can write as dz plus epsilon 1 delta x plus epsilon 2 delta y. So, delta Z minus dz will be equals to epsilon 1 delta x plus epsilon 2 delta y. Now divide the entire equation by delta rho. So, this implies delta Z minus dz upon delta rho will be equals to epsilon 1 delta x by delta rho plus epsilon 2 delta y upon delta rho ok. Now you take the limit. Limit delta rho tending to 0. When you take the limit here, now it is delta x upon under root delta x square plus delta y square whose value is always less than equal to 1. Whose modulus value is always less than equal to 1. Similarly, delta y upon delta rho delta rho is this term. So, this mod is also less than equal to 1 ok. And as delta rho tending to 0, epsilon 1 epsilon 2 will tend to 0, because if we are assuming function is differentiable ok.

So, epsilon 1 epsilon 2 will tend to 0, and this is a finite quantity which is whose mod is less than equal to 1 this is also finite quantity. So, 0 into something is some finite quantity is 0, 0 into some finite quantity is 0. So, this will be 0. So, if we have shown that this value tend to 0 as delta rho tend to 0, then also we can say that the function is differentiable at a point x naught y naught.

So, basically, we are we are having 2 approaches to show that a function is differentiable at a point x naught y naught, how? The first approach is you find epsilon 1 epsilon 2 from the basic definition of differentiability. And try to show that epsilon 1 epsilon 2 10 into 0 as delta is delta tending to 0 comma 0. Or the second way out is you find out this limit, and if this limit it is tending to 0 as delta rho 10 into 0, then we can show the function is differentiable at a point x naught y naught ok. Say we have just proved these 2 problems.

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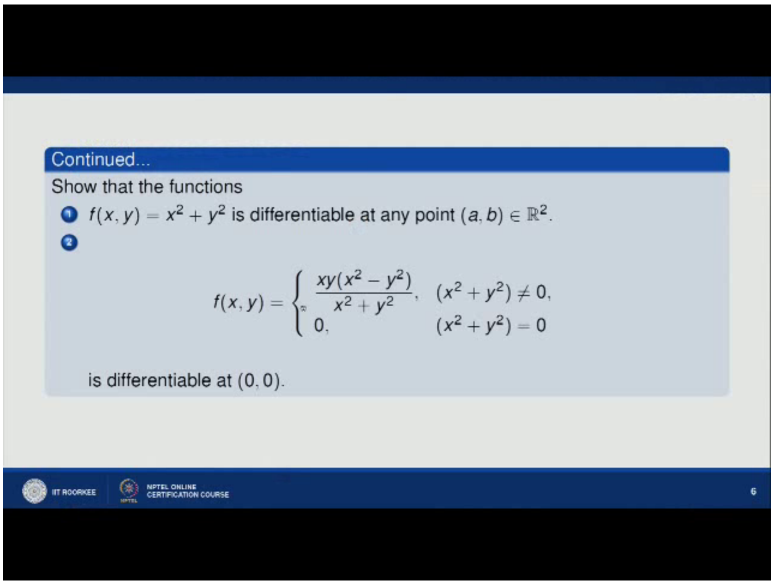

Say we take the second problem, and try to show that this function is differentiable from this definition ok.

So, how can we show for this problem, you see what are the function is x y x square minus y square upon x square plus y square, and x y is not equal to 0 comma 0..

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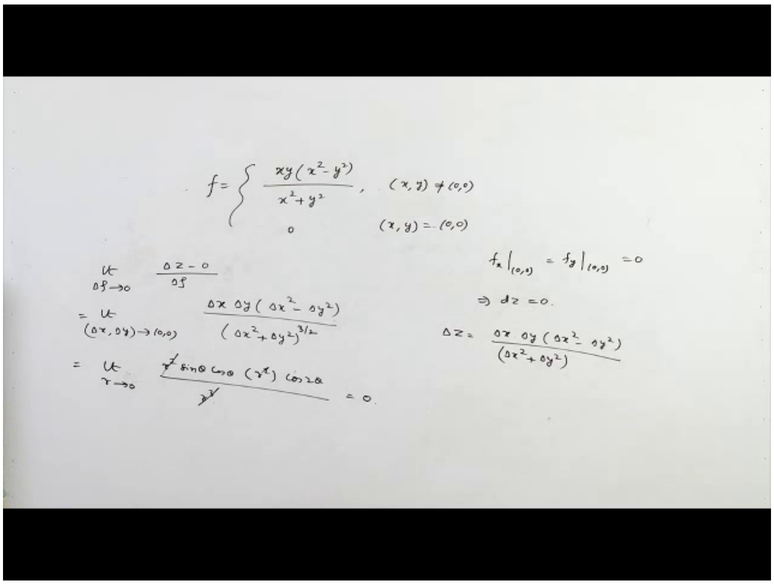

And 0 when x y is equal to 0 comma 0 ok. So now, we have to show that this is differentiable by this definition. We have already find f x and f y at 0 comma 0 for this for this problem. And we have shown that this is equal to 0 ok.

So, dz will be 0. What is delta Z? Delta Z we have already computed for this problem delta Z will be delta x delta y, delta x square minus delta y square upon delta x square plus delta y square. Now try to find out this limit. So, what is this limit now? It is limit delta rho tending to 0 delta Z minus 0 upon delta rho which is equals to; now it is delta x delta y tending to 0 comma 0. Because if delta rho tending to 0; this means, delta x delta y both will tend to 0. Because delta rho is nothing but under root of delta x square plus delta y square is equal to and delta rho is delta x delta y, delta x square minus delta y square upon delta x square plus delta y square, this term and from this term we are having 3 by 2.

Now, we have we have to show that this tending to 0 as delta x delta y tending to 0 comma 0. So, again we can convert this into polar coordinate system. So, in that case in epsilon 1 epsilon 2 definition we have to prove for the 2 limits, we have to prove for the epsilon 1 also and for the epsilon 2 also, that both will tend to 0 as delta x delta y tends to 0 comma 0. And here we have to show that, only this will term to tend to 0 as delta x delta y tends to 0 comma 0.

So, we can convert the polar coordinate system take delta x as R cos theta and delta y as R sin theta. So, what we will obtain? This is equals to limit R tending to 0, this is R square sin theta cos theta. This is again R square this is cos 2 theta, how hold is to power it is R square R cube into this is one basically. And this will tend to this is this cancels out. And for any theta this is always a bounded function. So, when R is tending to 0, this will tend to 0. And we can show this also by delta epsilon definition. So, hence we can show that this function is differentiable using the second definition of differentiability. So, in the last lecture we will deal some more property of differentiability.

So, thank you very much.