Multivariable Calculus
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Lecture - 06
Partial Derivatives-II

Hello friends. So, welcome to lecture series on multivariable calculus. So, in the last lecture we have seen what is mean by a partial derivative of multivariable functions ok. So now, we will see second or higher order partial derivatives. So, what do you mean by this let us see. So, again suppose z equal to f x y.

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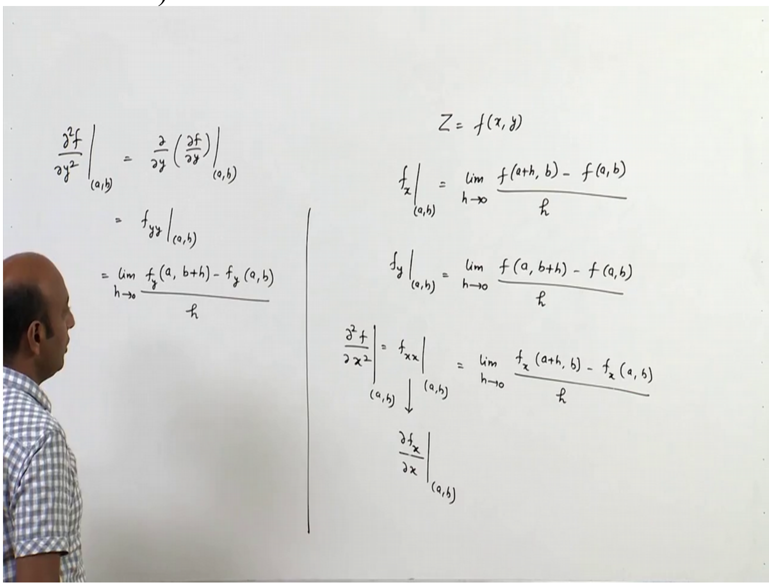

Where x and y are independent variables and z is a dependent variable. Now, we have seen we have already seen what do you mean by f x at a point say a comma b it is simply del f upon del x, which is limit h tending to 0 f of a plus h comma b minus f a b upon h. And similarly, f y at a comma b is equals to limit h tending to 0 f of a comma b plus h minus f a b upon h.

Now, suppose we define del square of a point del x square. Similarly, as we define second order derivative for single variable functions, f double dash x here, we define second order partial derivative of f respect to x, it is also denoted by f x x. So, at a point suppose we are defining it at a point a comma b, at a point a comma b it is simply limit h tending to 0; f x of a plus h comma b minus f x of a comma b upon h because, what does it mean basically? It means del f x upon del x or at a comma b it means del del by del x of f x; that means, now you have to take f x a new function and you have to apply the definition of del upon del x in f x. So, we have applied definition of del by del x in f x as f x a plus h minus b; f x of a comma b minus upon h, we simply replace f by f x in this definition so that we can get del square f upon del x square.

Now, similarly if we define in del square f upon del y square, that is second order partial derivative of f respect to y at say a comma b, which is can be defined as del by del y of del f upon del y at a comma b or also denoted by f y y at a comma b is simply given by limit h tending to 0; f of f y of because now, you have to apply the definition of del of del upon del y on f y ok, and del upon and f definition of f y is this. So, you have to simply replace f by f y here so; that means, f y of a comma b plus h minus f y of a comma b upon h.

So, this is how we can define second order partial derivative of f respect to x or respect to y. Now, come to mix order partial derivatives mixed order mixed second order partial derivative of f respect to x or y. Now, suppose you want to define del square f upon del x del y at say a comma b; it is del by del x of del f upon del y at a comma b or we can say, it is f del of f y of del f upon del x at a comma b.

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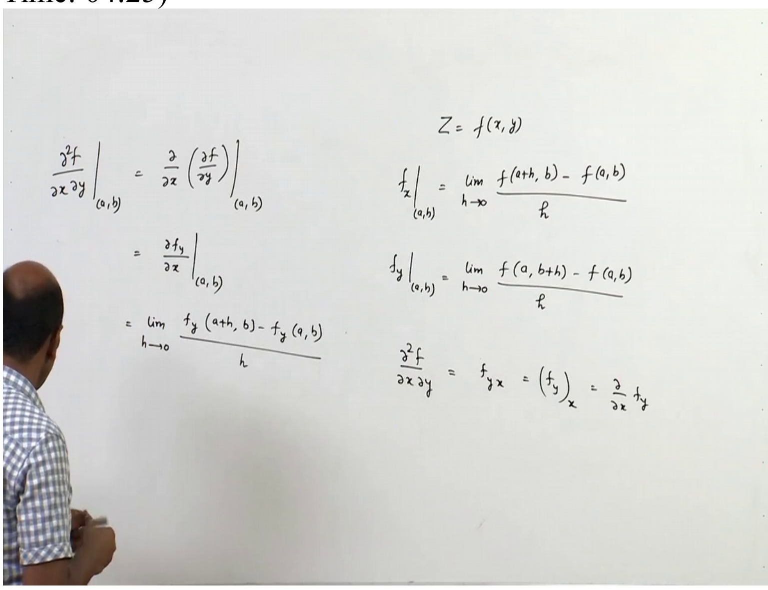

So, this is by definition we have to replace f by f y here, within this definition because, it is del f upon del x and instead of f we are now having f y. So, that will be limit h tending to 0 f y of a plus h comma b minus f y of a comma b upon h.

Now, this mixed order second order mixed partial derivative can also be represented it as f of y x because, it is what? It is f y respect to x; that means, del upon del x of f y; that means, I differentiate partially this function respect to x, that is del upon del x of f y which is same, which is same which is here. So, we can also represent this partial derivative as f y x.

Now, the second one is del square f upon del by del x, which is at a comma b which is also represented as del by del y of del f upon del x at say a comma b or it is f of x y at a comma b.

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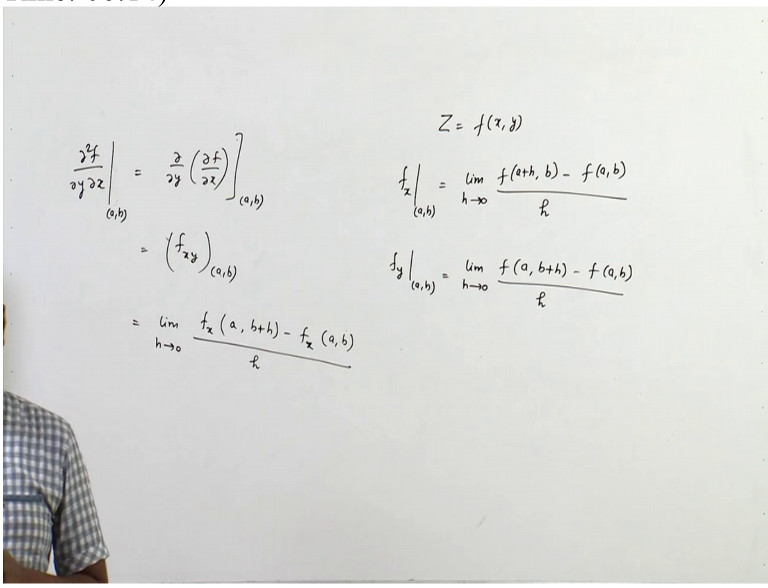

It is by the definition we can write it like this, it is limit x tending to 0. Now, you have to you have to apply the definition of del upon del y on f x. So, it is f x at a comma b plus h minus f of f x at a comma b upon h. So, this is how, we can define f x y or f y x.

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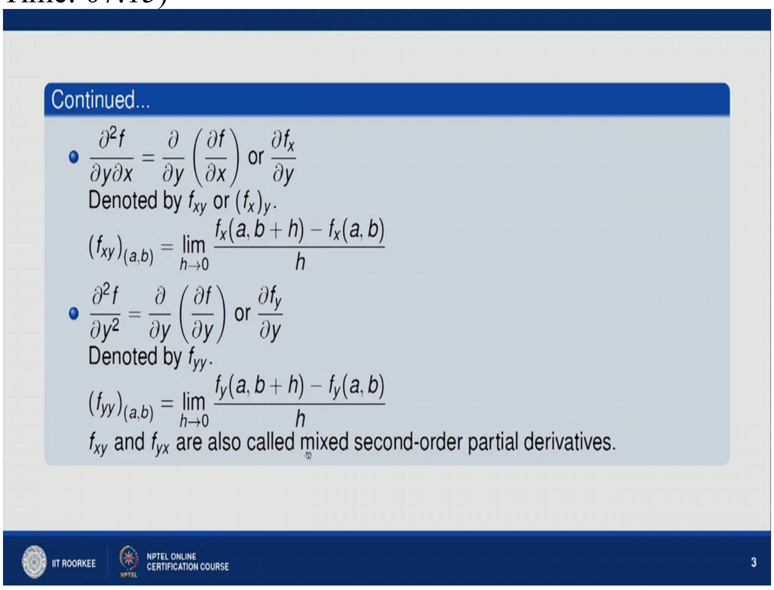

So, f x y or f y x are also called mixed second order partial derivative. Now, let us try some problems based on this. So, these are all second order partial derivatives respect to x or y.

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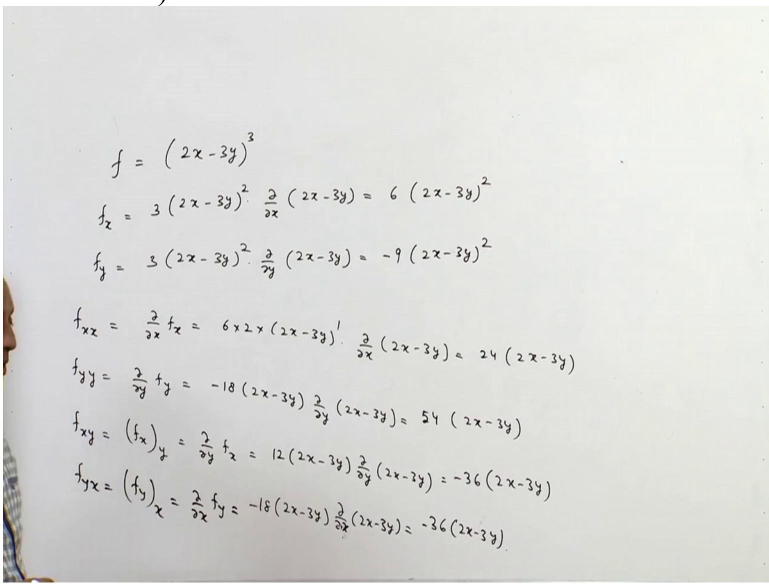

Now, suppose f is 2 x minus 3 y whole cube. Now, what is f x? it is it is t cube t cube is 3 t square and del by del x of t again which is which is 2 here, and 2 into 3 is 6 a 6 times 2 x minus 3 y whole square, and again and f y is similarly it is 3 t square t is 2 x minus 3 y and it is del by del y of t again, which is minus 3 minus 3 into 3 is minus 9 minus 9 2 x minus 3 y whole square. Now, suppose you want to compute f x x f x is means, del by del x of f x.

Now, del by del x of f x is you have to differentiate f x respect to x again partially. So, when you differentiate this partial respect to again. So, what we will obtain this is again 6 into t square, which is 6 into 2 2 into 2 x minus 3 y this for 1 into del by del x of t again, which is 2, 6 into 4 is 24, 24 2 x minus 3 y. Now, if you want to compute f y y, which is del by del y of f y and it is differentiate this function respect partial respect to y.

So, it is it is minus 18 2 x minus 3 y and del by del y of t again. So, that will be 54 times 2 x minus 3 y ok. Now, suppose you want to compute f x y, which is f x of y ok; that means, del by del y of f x ok. So, del by del y of f x this is f x the del by del y of this function, that is differentiate this partial respect to y and; that means, it is 12 2 x minus 3 y del by del y of this again respect to y, that is minus 3 into 12 is minus 36 times 2 x minus 3 y.

Now, f y x suppose f y x f y respect to x, that is del by del x of f y; that means, it is f y you have to differentiate this partial respect to x; that means, minus 18 2 x minus 3 y del by del y of or del by del x of this function 2 x minus 3 y, which is 30 minus 36 times 2 x minus 3 y. So, that is how we can compute second order partial derivatives respect to x or y ok. Now, see this again say second problem.

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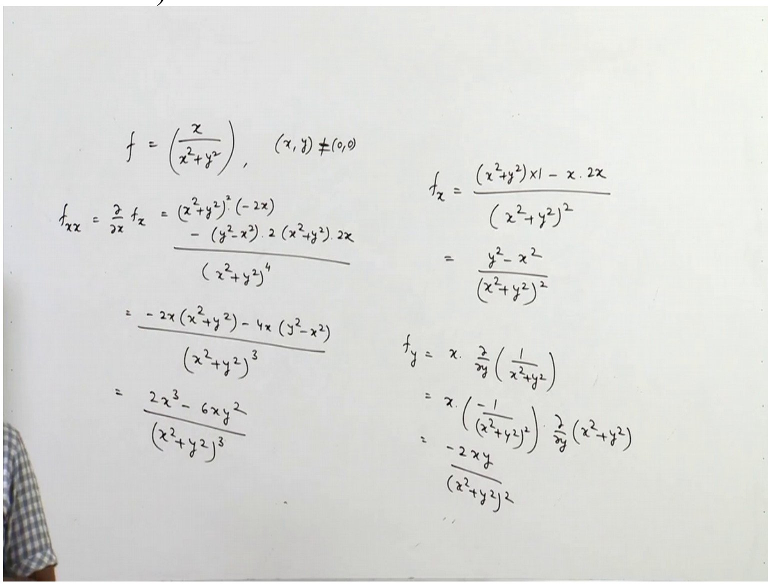

So, what a second problem now, it is f equal to x upon x square plus y square and x y should not equal to 0 0 because, there is a problem in 0 0. So, we are excluding 0 0 from the domain. Now, what is f x? First you compute f x f x is. Now, x is a numerator x is a denominator also. So, we have to apply quotient rule, that is denominator derivative of numerator minus numerator derivative of denominator, that is 2 x upon denominator whole square.

So, this will be equals to y square minus x square upon x square plus y square whole square ok. Now, what is f y? Now, it is x constant for this partial derivative of f is respect to y. So, we take x into del by del y of 1 upon x square plus y square. So, it is x into 1 by t, that is minus 1 by t square. So, it is minus 1 upon t square then t again, that is del by del y of t and this is 2 x y upon x square plus y square whole square and with negative sign. So, this is del f upon del y.

Now, you want to compute second order partial derivatives again we can use this thing f x x, which is del by del x of f x we have to differentiate this again respect to x ok. So, it is denominator as it is derivative of numerator, which is minus 2 x minus numerator derivative of denominator it is 2 times x square plus y square and it again upon denominator whole square ok.

So, this will be minus 2 x x square plus y square and it is x square y square 1 is cancelled out it is minus 2 x of square y square and it is minus 4 x y square minus x square upon x square plus y square whole cube ok, and it is it is equals to it is minus 2 x cube it is plus 4 x cube. So, it is 2 x cube and it is minus 2 x y square, it is minus 4 x that is minus 6 x y square upon x square plus y square whole square whole cube. Similarly, we can find f y y also.

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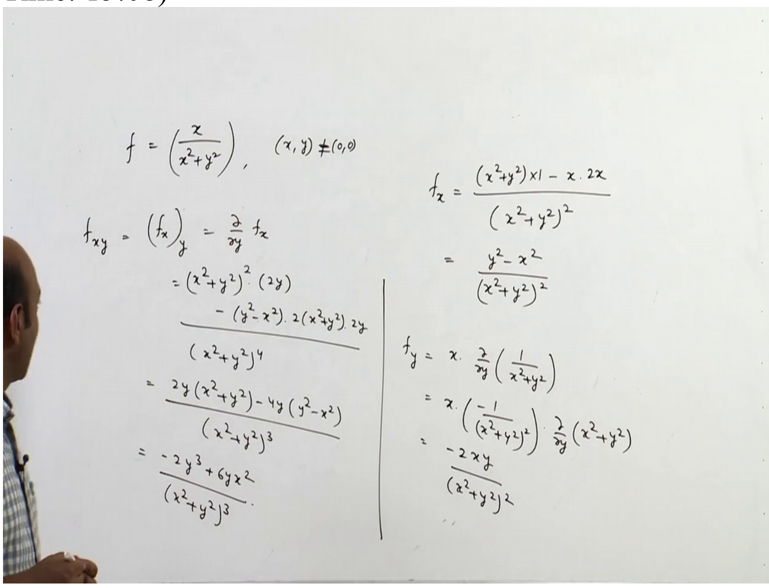

Now, let us compute one more term here, it is one of the mixed derivative anyone say we compute f x y f x y is f x y that is, del by del y of f x that is simply means, you have to differentiate this partial respect to y. So, again denominator as it is derivative of numerator minus numerator derivative of denominator 2 t t again that is 2 y upon denominator whole square. So, this is further equal to cancel a, so square y square from numerator and denominator.

So, we will obtain 2 y into x square plus y square and it is minus 4 y y square minus x square upon x square plus y square whole cube. So, this is 2 y cube minus 4 y cube, that is minus 2 y cube and it is 2 y x square it is minus plus 4 y x square, that is plus 6 y x square upon x square plus y square whole cube. And similarly, we can find f x f y x. So, that is how we can find f x x f y y f x y or f y x.

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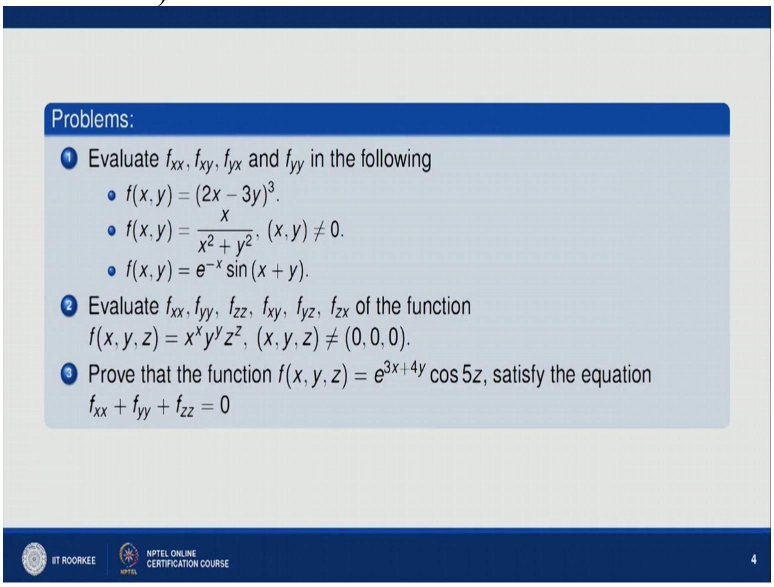

Now, let us solve one problem of 3 variables. Now, suppose f x y z is x raised to power x, y raised to power y, z raised to powers z. We are x y and z are non-negative.

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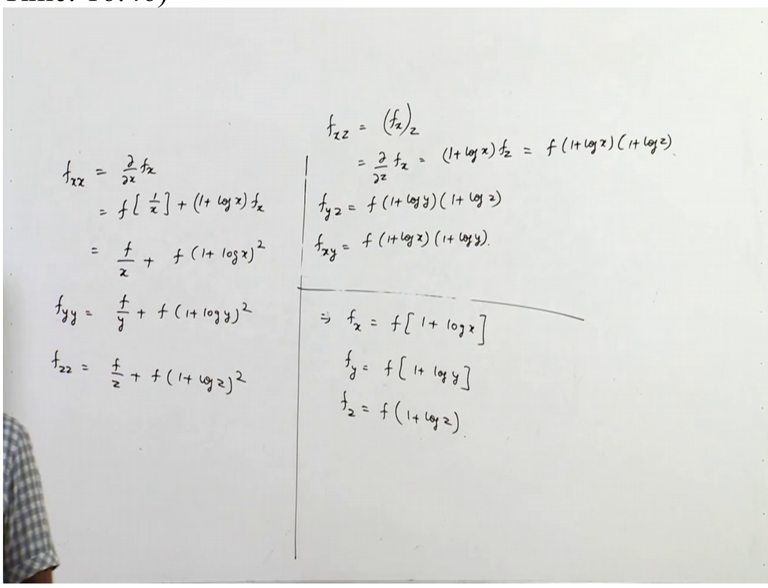

Now, you have to find here you have to find f x x all second order partial derivatives for this function. So, how will you compute f x. So, you have to take log first. So, log of f will be equals to x log x plus y log y plus z log z. Now, differentiate this equation partially respect to x. So, log t that is 1 upon t and t again, that is f x. Now here, both are function of x. So, you apply product rule first as it is derivative second plus second as it is derivative of first and these two are independent of x. So, derivative will be 0 partial derivative will be 0.

So, this implies f x will be f into 1 plus log x of course, it is base e. Now, similarly f y will be f into 1 plus log y and f z will be f into 1 plus log z. Now, suppose you want to compute x x x; f x x x. So, how can you compute this? It is del by del x of f x which is so, so you have to differentiate this again respect to x. So, this is also a function of x and this is also a function of x. So, you have to apply product rule. So, f first as it is, derivative of second plus second as it is, derivative of first f x and what is f x? what is f x? F x is this.

So, it is f into 1 plus log x. So, this is f x x. Now, similarly f y y will be because, it is symmetrical in x y and z. So, we can simply write f y y as f upon y f 1 plus log y whole square f z z as f upon z plus f plus 1 plus log z whole square ok. Now, suppose you want to find out f suppose x f x z that is f x at z, that is del by del z of f x.
Now, what is f x f x is this and you have to differentiate this f x respect to z partially. Now, this is independent of z because, x y z are independent of each other independent variables ok.

So, it is 1 plus log x you can take outside and this is the function of x y z. So, partial derivative of respect to z will be f z and f z is f 1 plus log z. So, it is f 1 plus log z log x into 1 plus log z. So, again using a symmetry we can simply say that, f y z will be f into 1 plus log y into 1 plus log z and similarly, f x y will be f 1 plus log x into 1 plus log y ok. So, that is how we can find out all other all mixed second order partial derivative for this function. Now, the next problem proved at the function this satisfy this equation. So, again this is a simple problem let us see how can we.

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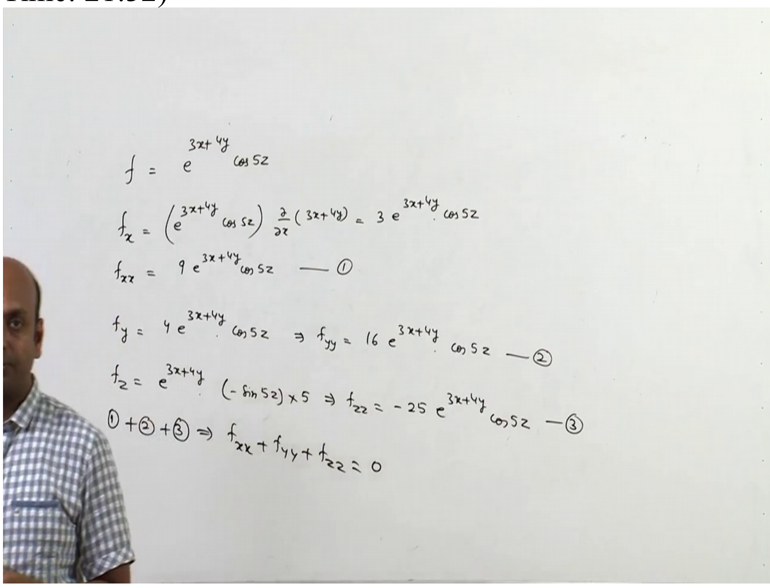

So, show this what is f here? F is e raised to power 3 x plus 4 y and cos 5 z ok. Now, what is f x now, it is e raised to power t. So, e raised to power t is e raised to power t derivative cos 5 z remain as it is and t again that is del by del x of t again that is 3 x plus 4 y, which is 3 times e raised to power 3 x plus 4 y into cos 5 z ok. Now, what is f x x? the derivative of this again partial derivative of this again respect to x. So, that will be 9 e raised to power 3 x plus 4 y cos 5 z this is one.

Now, compute f y f y is 4 e raised to power 3 x plus 4 y into cos 5 z we can directly see, derivative of e raised to power t is e raised to power t and t again, which is 4 cos 5 z remain as it is. So, it is 4 e raised to power 3 x plus 4 y into cos 5 z. Now, this implies f y y will be now, differentiate this again partially respect to y. So, what we would obtain this is 16 e raised to power 3 x plus 4 y cos 5 z.

So, this is second equation. Now, what is f z? f z is now, this is free from z remain as it is cos t is minus sin t and t again, which is 5 and this implies f z z will be now, differentiate this again partial respect to z. So, what we will obtain this is 5 into 5 is 20 minus 25 it is e raised to power 3 x plus 4 y into cos 5 z because, derivative of sin is cos.

So, this is third equation. Now, when you add when you add 1 2 and 3. So, what we would obtain we will obtain f x x plus f y y plus f z z in the left-hand side and the right-hand side this is same and 16 plus 9 is 25 minus 25 is 0. So, this is equal to 0. So, hence we have proved. So, this equation is basically called Laplace equation, this equation we call as Laplace equation. So, we can say that this function e raised to power 3 x plus 4 y into cos 5 z satisfy a Laplace equation now, what can we say about mixed second order partial derivatives, that is f x y or f y x are they always same or there should be some condition on function f. So, that we can say, that they are equal at some point.

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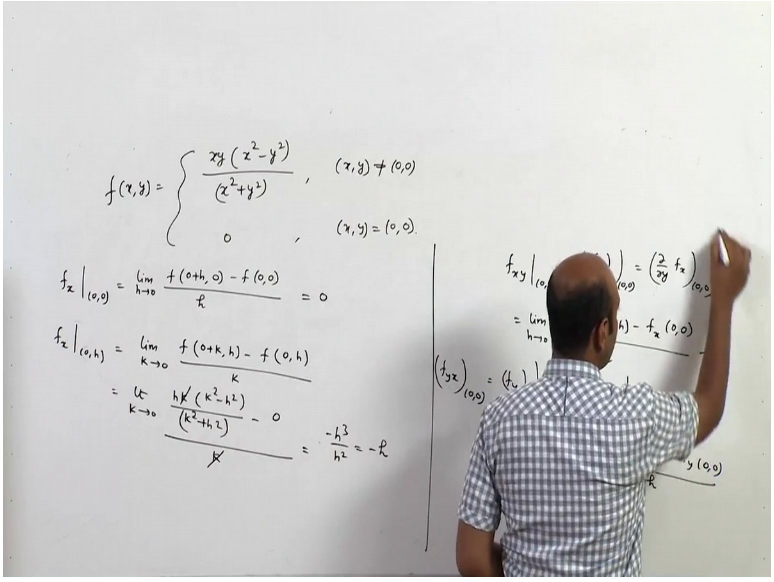

So, let us discuss this example example is f x y is equals to x y x square minus y square upon x square plus y square x y not equal to 0 0 and 0 when x y is equal to 0 0 ok, let us discuss this example. So, things will be clear after setting this example. So, let us compute f x y at origin and f y x at origin for this particular problem ok.

So, what is f x y at origin? It is f x at origin is f x at y at origin, that is del by del y of f x at origin and that is limit h tending to 0. So, what it is it is limit h tending to 0 f x at 0 comma 0 plus h minus f x at 0 comma 0 upon h. So, this is f x y at 0 comma 0. Now similarly, what is f y x at 0 comma 0 it is f y at x 0 comma 0, which is del by del x of f y at 0 comma 0; that means, it is limit h tending to 0 f y at 0 plus h comma 0 minus f y at 0 comma 0 whole divided by h.

So, first we will compute f x at 0 comma h f x at 0 comma 0 and f y at this then, we can compute f x y or f y x 0 comma 0. So, what is f x at f x at 0 comma 0 it is limit h tending 0 f at 0 plus h comma 0 minus f at 0 comma 0 upon h ok. So, it is now, you take x as h y as 0 when you take x as h y as 0 y as 0. So, this will be simply 0 and f 0 0 is 0. So, 0 minus 0 is 0. So, this value is simply 0. Now, you compute f x at 0 comma h we we need now, we need f x at 0 comma h ok. Now, this is say now take limit k tending to 0 because, h is already here.

So, it is f at 0 comma k h minus f at f at 0 comma h upon k ok, at 0 comma h. Now, you replace x by k and y by h. So, what we will obtain limit k tending to 0 it is h k into k square minus h square upon k square plus h square minus 0 comma h is when you substitute x as 0 y as h. So, it is 0 whole divided by k. So, this k will cancel with this k and when you take k tending to 0 here.

So, it will be it will be, it will be minus when you take a tend to 0 it is minus h cube upon h square, which is minus h. So, so what we can say about f x y at 0 comma 0 now you can substitute it over here, f x at 0 comma h is minus h you can substitute this value as minus h f x at 0 comma 0 is 0. So, it is 0 upon h.

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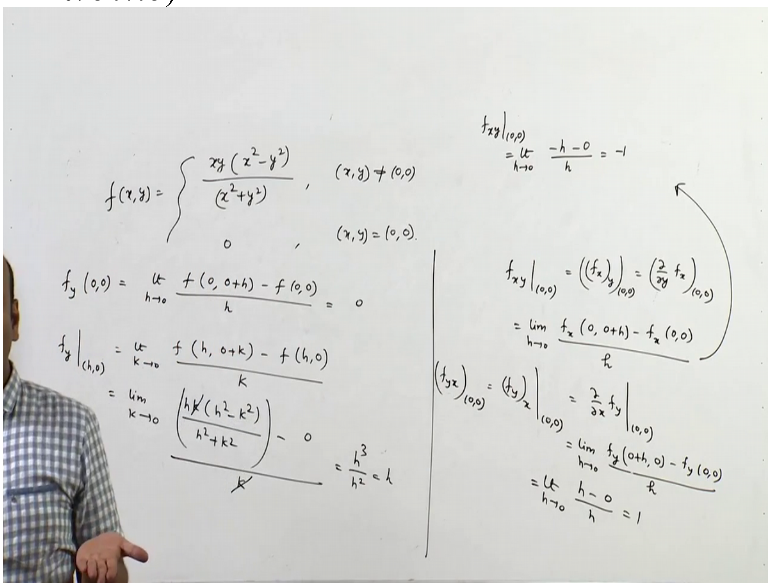

So, this expression will be limitation to 0 minus h minus 0 upon h, which is minus 1. So, so we can say, that this value at 0 comma 0 is minus 1. Now similarly, we try to compute f y x at 0 comma 0. Now, what is f y at 0 comma 0 it is limit h tending to 0 f at 0 comma 0 plus h minus f 0 0 upon h now, again when you take x as 0 y as h ok. So, it will be 0 and f 0 0 is 0. So, this value is 0. Now, when you take f y at say 0 comma h h comma 0 you see, we need f y at h comma 0.

So, f y at h comma 0 is limit k tending to 0 f at we need f y; that means, we have to change y only remaining x has remained fixed 0 comma h comma 0 plus k minus f at h comma 0 divided by k. Now, this is limit k tending to 0 now here, you replace x by h and y by k. So, it is h k. So, it is h k into h square minus k square upon h square plus k square this minus f at h comma 0 x is h y is 0 when you take y as 0. So, this expression is 0. So, it is 0 upon k. So, this k will cancel with this k and when you take k tending to 0. So, k will be 0. So, it is h cube upon h square which is h.

So, what can we say about f y x? So, this will be equal to now here limit h tending to 0 what is f y at h comma 0 it is h. So, it is h minus f y at 0 comma 0 is 0. So, it is 0 upon h. So, this value is 1. So, what do we conclude from here. So, we have concluded that, f y x at some point may not be equal to f x y it is 1 it is minus 1 for some function f x y may not be equal to f y x.

So, what is the additional condition required on f. So, that we can say, that the mixed second order partial derivatives are equal ok. So, that condition is basically we have Euler's theorem, which states that if the function f x y and it is partial derivatives f x f y f x y and f y x are defined throughout an open region containing a point a comma b and all are continuous at a comma b then, only we can say that f x y is equals to f y x otherwise they may not be equal ok.

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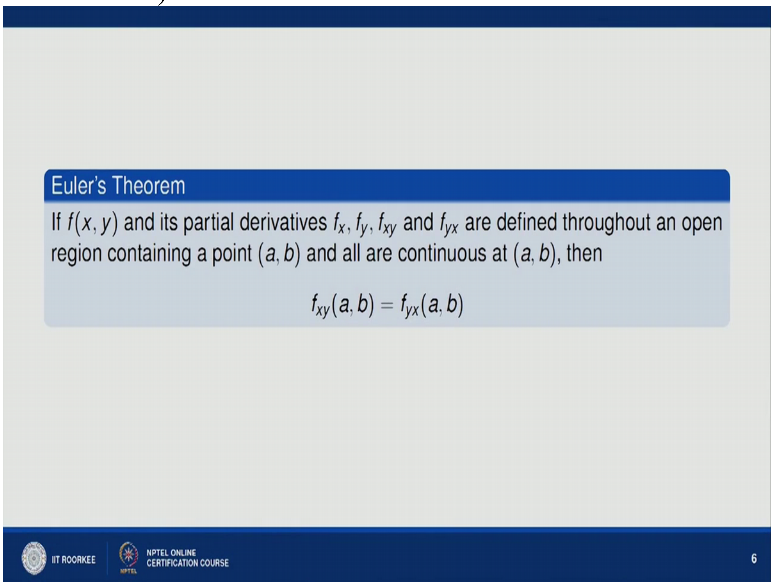

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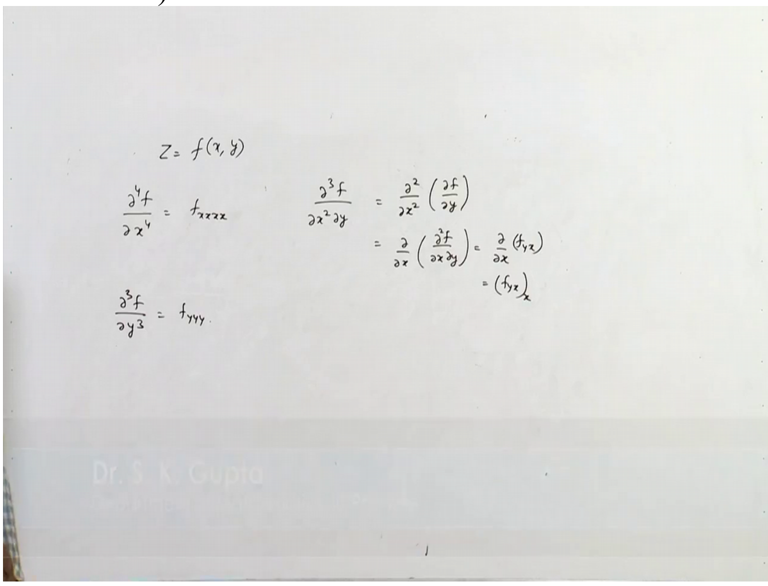

So, this is all about second or partial derivatives. Now, we can define higher order partial derivative also like, suppose z equal to f x y the function of x or y. So, we can define say 4th order partial derivative also with respective to x, which is f x x x x we can also define say del cube f upon del x square del y, which is simply del square upon del x square of del f upon del y or del by del x of del square f upon del x del y ; that means, del by del x of f it is y x or del y x of x.

So, these are all representation of the same expression basically similarly, we may have any higher derivatives suppose del cube f upon del y cube, which is f y y y and so on. So, similarly we can compute these values also for any function f so.

Thank you very much.