Multivariable Calculus
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Lecture - 05
Partial Derivatives-1

Hello friends, welcome to a lecture series on multivariable calculus so, today we will deal with partial derivatives. So, what is mean by a partial derivatives and how we can use this some solving some problems let us see.

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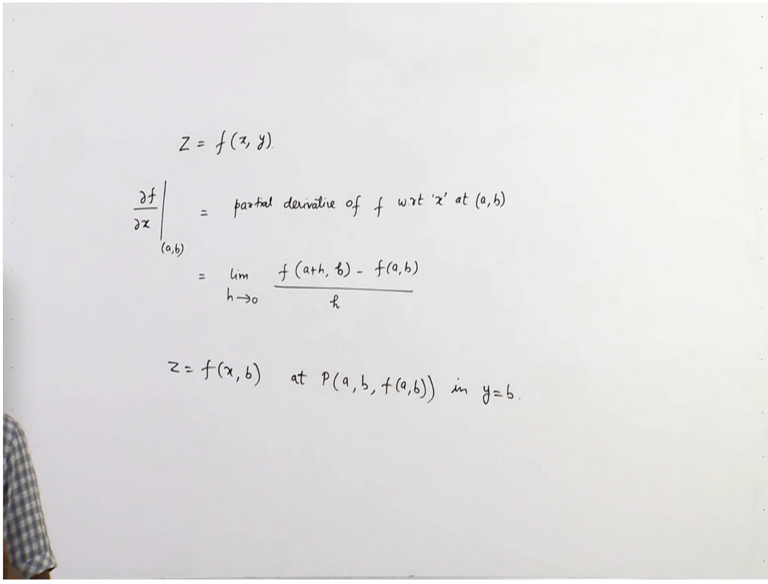

So, we start with one variable problems function of one variable we know that if your function is one variables say y equals to f x where x belongs to R say then if function is differentiable at a point say x equal to a, then we say that d y by d x at x equals to a or is equal to f dash a is equal to limit h tending to 0 f a plus h minus f a upon h. So, this is how we defined derivative of a function at a point say x equal to a, it is simply limit h tends to 0 f a plus h minus f a upon h.

Now, what does it mean what does it give derivative it simply gives slope of a tangent at a point x equal to a, if we have a curve x y equals to f x then at x equal to a it simply give slope of the tangent. It also represent rate of change of f along x axis rate at which the function changes along x axis. So, this we have already deal with we have already seen; what do we mean by function of derivative of function of single variables. Now, in multi variable we have more than one independent variables so, we cannot define derivative like this ok.

So, here we have function of variables more than one so, what we have here. Here we have a concept of partial derivatives so, what do you mean by partial derivative let us see. Suppose we have a function of x and y two variable function here x and y are independent variables and z is the dependent variable now we define del f upon del x it is simply partial derivative of f respect to x say at x equal to a. So, what does it mean it is partial derivative of f with respect to x at x equal to a ok.

Now, here when we deal with partial derivative of f respect to x; that means, we are treating all other variables constant. So, we define this as limit h tending to 0 f of a plus h. We are defining partial derivatives say at a point a comma b so, at a comma b because here we are having two variables a and b I mean x and y. So, it is limit h tends to 0 f of a plus h b minus f a b upon h so, this is partial derivative of f respect to x at a comma b. So, simply we are changing only in x component y will remain same; that means, we are fixing y and changing only respect to x. Now what does it give? It simply gives rate of change of f along x axis.

Or rate of change of f along I cap direction or we can define it like this z equal to f x y is a surface z equal to f x y is some surface. Now, y equal to b is a plane which cut the surface into a curve now the slope of the tangent for this curve z equals to f x b because y is equal to b at the point a b f a b in the plane y equals to b. So, it is simply gives slope of the tangent for the curve this at a point this in the plane y equal to b because z equal to f x y is a surface when y equal to b cuts it will give a curve and the point is this. At this point the partial derivative of f with respect to x simply gives slope of the tangent of this curve in the plane this ok.

So, this is how we can defined del f upon del x or partial derivative of f respect to x geometrically. Similarly, when we talk about partial derivative of f respect to y so, with respect to y how can we define?

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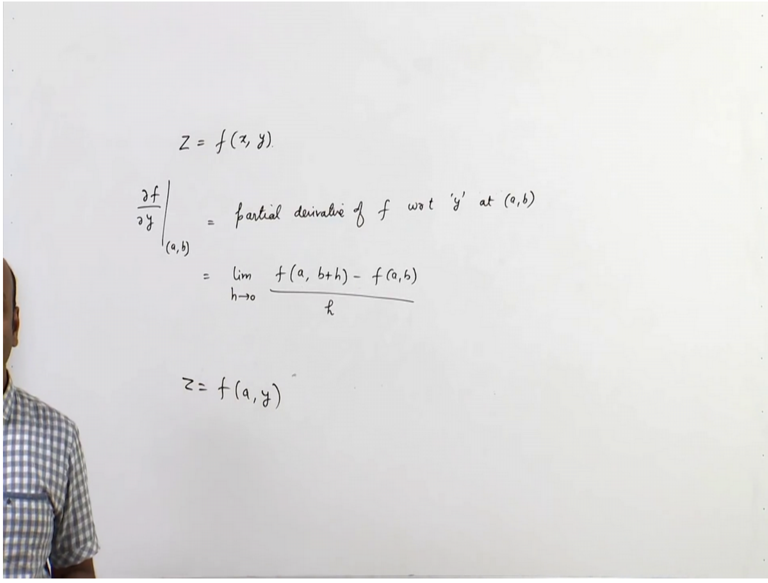

Now del f upon del y at a comma b which is partial derivative of f with respect to variable y at a comma b is simply limit h tending to 0 f of a b plus h minus f a b upon h. Now, here when we are differentiating partially f respect to y we are differentiating only with respect to y keeping all other variables constant. So, here a is constant we are fixing a we are changing only in the y component b plus h to b upon h.

So, this is how we can define rate of change we can define partial derivative of f respect to y again geometrically if we are talking about partial derivative of f respect to y it simply gives rate of change of f along y axis or rate of change of f along j cap direction or z equals to f x y is a surface and x equal to a is a plane. When x equal to a cuts the surface it will give a curve, which curve it will give this curve and the slope of the tangent for this curve at a point p in the plane x equal to a will represent del f upon del y at a comma b. So, this is how we can define first order partial derivative of two variable functions respect to x and y.

Now, we will see some problems based on this let us see the first problem is it is f equal to 3 x cube plus 4 x square y square minus 6 y cube, now suppose you want to find del f upon del x at any point x comma y.

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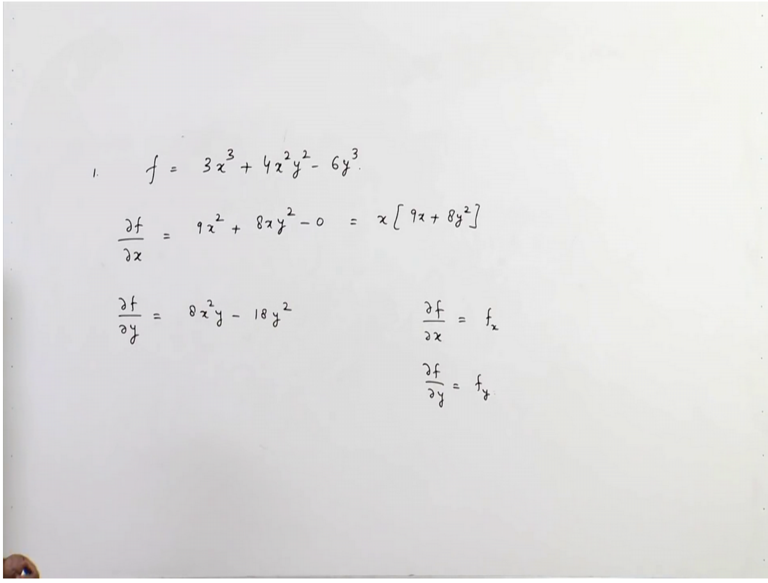

So, del f upon del x is simply partial derivative of f is respect to x; that means, you have to simply differentiate this function respect to x keeping all other variables constant. So, derivative of this respect to x is simply 9 x square plus you have to keep y constant then this is 8 x y square minus 0 because we are treating this as a constant. So, this is x times 9 x plus 8 y square so, this is del f upon del x.

Now, similarly if you want to find out del f upon del y it is nothing, but now you have to differentiate f with respect to y partially; that means, you have to take x as a constant. So, this is 0 and this is 8 x square y minus it is 18 y square so, this is del f upon del y. Now, del f upon del x is sometimes also represented as f x, f x means del f upon del x now del f upon del y is also represented as f y.

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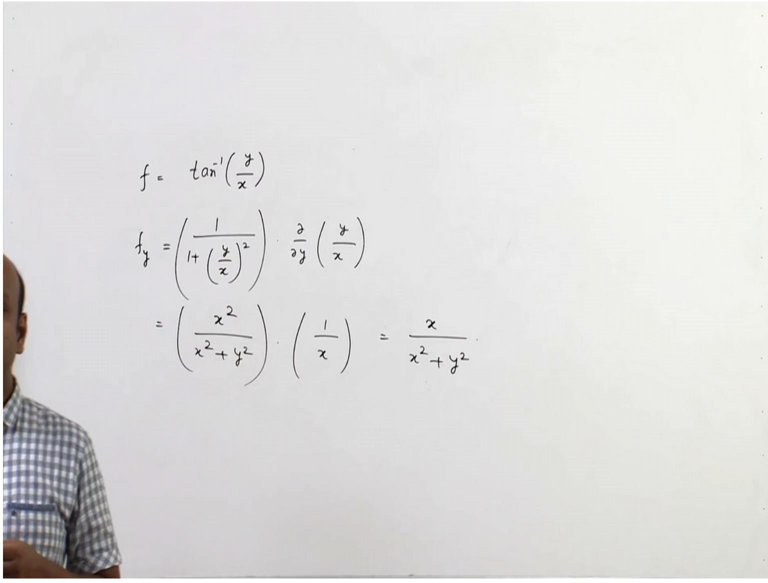

Now, the second problem the next problem now, f is tan inverse y upon x now what is f x, f x simply means del f upon del x. Now, if you want to compute this it is tan inverse something t tan inverse t is derivative of tan inverse t is 1 upon 1 plus t square and then del by del x of t which is y upon x this is simply equal to x square upon x square plus y square into, now partial derivative of this respect to x is you have to take y as a constant it is minus y upon x square because 1 upon x is minus 1 upon x square derivative of one upon x is minus 1 upon x square. So, it is minus y upon x square plus y square so, this is del f upon del x.

Now, del f upon del y is f y which is equals to again tan inverse t it is 1 upon 1 plus t square and then del by del y of t again and it is equals to x square upon x square plus y square. Now, del upon del y of y upon x is simply 1 upon x and this is nothing, but x upon x square plus y square. So, this is how we can find out del f upon del x or del f upon del y for some function of two variables ok.

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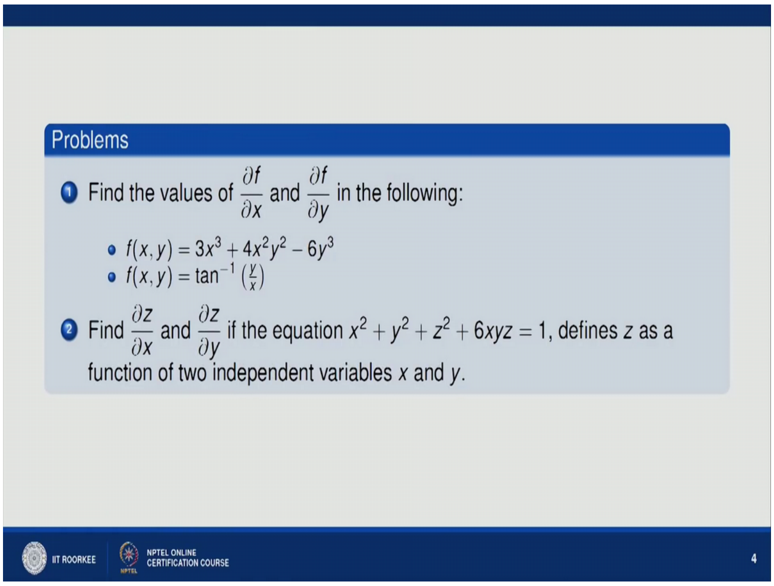

Now, let us solve second problem find z x and z y, if the equation, this defines z as a function of two independent variables x and y so, what is the function.

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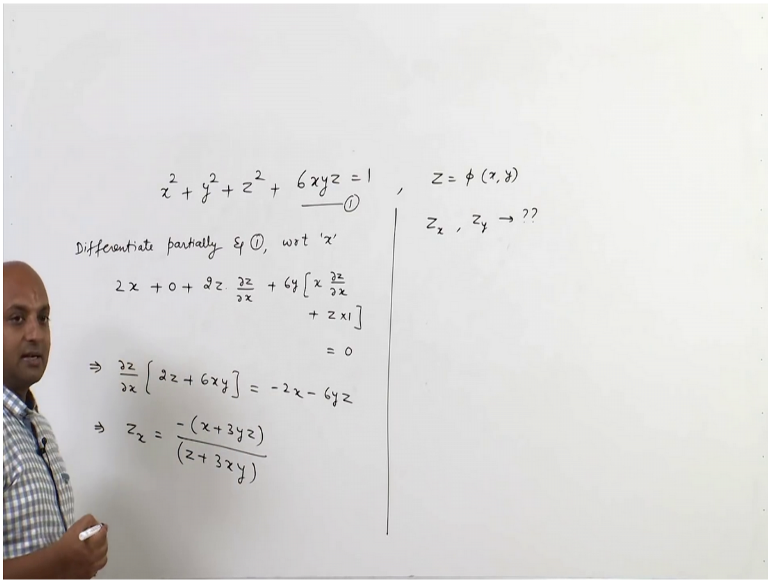

Now, the function is equation is x square plus y square plus z square plus 6 x y z equal to 1 and here z is a function of x and y, x and y are independent variables and z depends on x or y, x and y. Now, how can we compute z x, how we can compute z x and z y this we have to find out.

Now, we simply differentiate partially with respect to x so, differentiate partially say this equation is equation 1, equation 1 with respect to x. So, what we will obtain you see x square is 2 x when you differentiate partial with respect to x y square is because x and y are independent. Since, x and y are independent so, partial derivative of y respect to x will be 0. Now z is a function which depends on x and y. So, partial derivative of z respect to x will not be 0 it is t square means 2 t that is 2 z into del z upon del x plus six times.

Now, when we differentiate partial with respect to x we will take y as constant because y is independent of x so, we can take y also common, now inside bracket we are having x and z and both are the function of x. So, we have to apply product rule that is first as it is derivative of second that is del z upon del x plus second as it is derivative of first and derivative of 1 will be 0 respect to x. So, what we will obtain from here now let us collect del z upon del x this implies del z upon del x let us collect these terms 2 z plus 6 x y which is equal to the remaining terms put it on the right hand side. It is minus 2 x and it is minus 6 y z. So, what we are having this implies z x is equals to you will cancel 2 from both the sides so, it is minus of x plus 3 y z upon z plus 3 x y.

So, this is del z upon del x now if you want to find out del z upon del y so, similarly we can proceed for del z upon del y.

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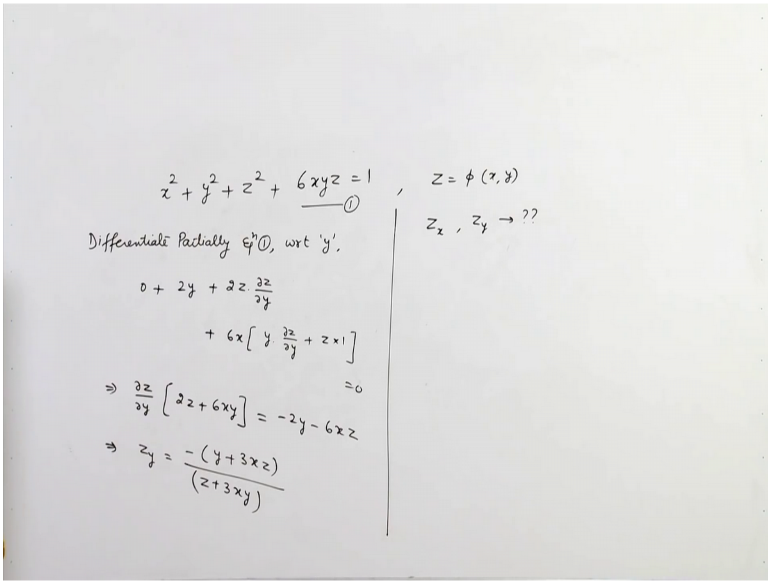

Now, again differentiate partially equation 1 with respect to y so, what we will obtain now x is independent of y. So, partial derivative of f x with respect to y will be zero so, this is 0 plus y square the derivative partial derivative of y respect to y is 2 y. Of course, del y upon del y is 1 plus 2 z into del z upon del y because z is a function of x and y, z depends on y plus now 6 x you can take common. Now, y into z and both are the function of y so, you have to apply product rule here. So, it is y into first as it is derivative of second it is del z upon del y plus z into 1 which is equals to 0.

So, this implies now collect terms of del z upon del y which is 2 z plus 6 x y equal to put all the remaining terms on the right hand side that is minus 2 y minus 6 x z cancel 2 from both the sides again. So, this implies z y will be equal to minus y plus 3 x z upon z plus x y 3 x y so, this is del z upon del y. So, that is how if some implicit equations are given to us we can easily compute del z upon del x or del z upon del y if z is a function of x and y.

Now, we will solve some more problems where f is a function of three variables here the same definition will applicable for three variables also.

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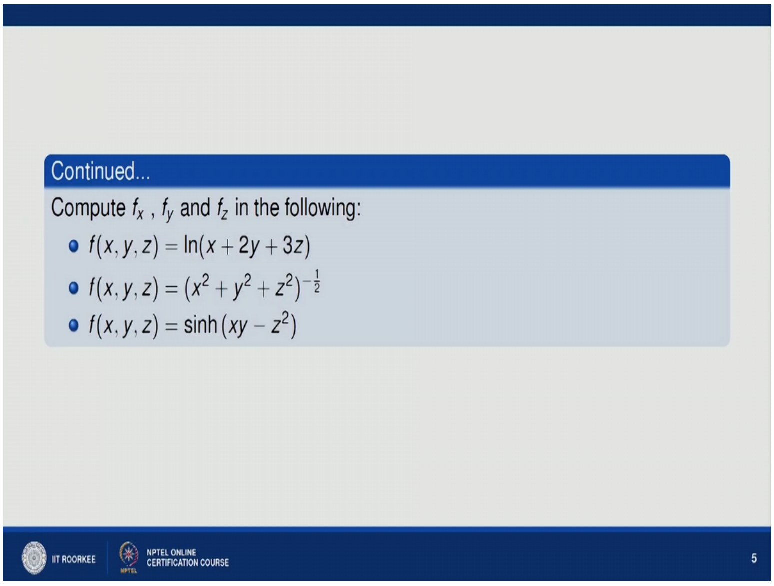

If we have function of three variables suppose you want to compute suppose you are having w is equals to f x y z.

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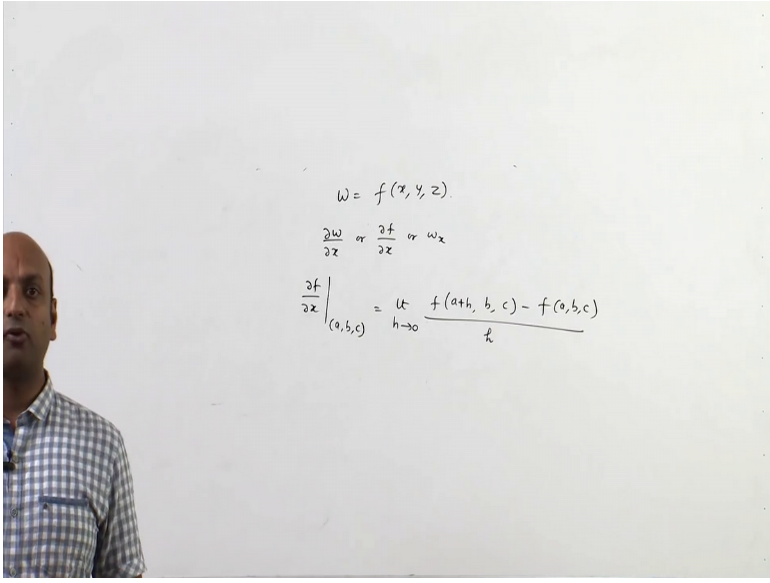

Now, here x y z are independent variables and w is a dependent variable now if you want to compute del w by del x or we can say del f upon del x or we can say w x. So, this is simply means this simply means we are treating x as a variable all other variables as constant. Suppose you want to define del f upon del x at some point say a b c so, how can I define this it is limit h tending to 0, f of a plus h b c minus f a b c upon h. The same definition which we have used in for two variable functions the same will be followed for three variables or more than three variables. Similarly, we can define del f upon del y or del f upon del z.

Now, we will solve some problems based on this now suppose f is f of x y z is now first problem it is l n x plus 3 y plus 6 z.

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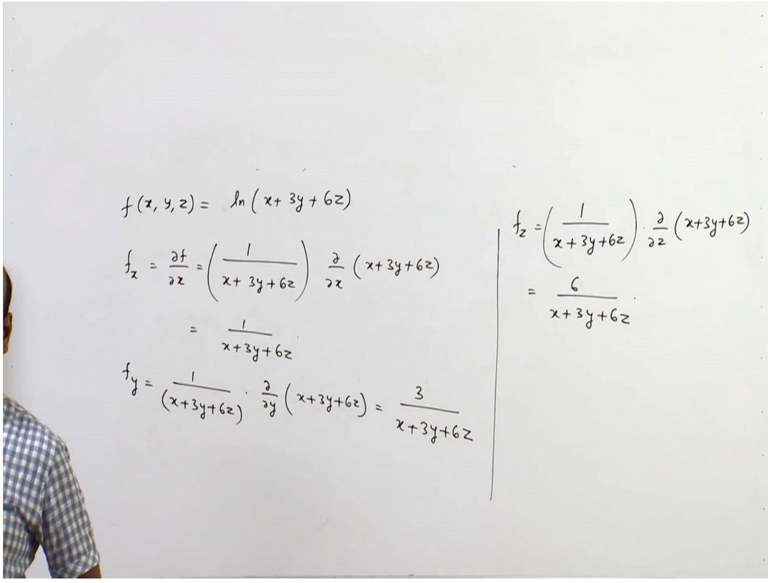

So, what will be f x, f x is del f upon del x and it is it is log t, log t is derivative of log t is 1 upon t that is 1 upon t is x plus 3 y plus 6 z and then del upon del x of t again this is equals to 1 upon x plus 3 y plus 6 z because y and z are independent of x. So, derivative partial derivative of y or z with respect to x will be 0. Now, similarly f y will be again 1 upon t that is 1 upon x plus 3 y plus 6 z into del by del y of x plus 3 y plus 6 z which is 3 upon x plus 3 y plus 6 z.

Similarly, if you want to compute f z it is again log t derivative log t is 1 upon t it is 1 upon x plus 3 y plus 6 z into del by del z of x plus 3 y plus 61 z which, is 6 upon x plus 3 y plus 6 z. So, this is how we can find out del f upon del x del f upon del y or del f upon del z let us try one more problem based on this so, things should be more clear.

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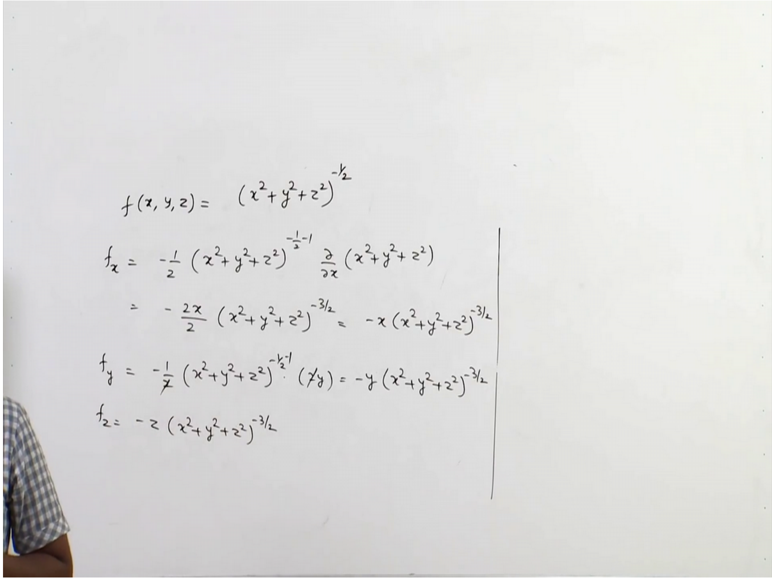

So, it is x square plus y square plus z square whole is to power minus 1 by 2 so, f x will be now it is t raise to power minus 1 by 2 that will be derivative of that will be minus 1 by 2 t which is x square plus y square plus z square and t minus 1 and del by del x of t again. That is x square plus y square plus z square which is equal to it is minus 2 x upon 2 it derivative of this is 2 x and it is x square plus y square plus z square whole raise to power minus 3 by 2, which is minus x into x square plus y square plus z square whole raise to power minus 3 by 2.

But similarly if you find del f upon del y it is again minus 1 by 2 x square plus y square plus z square whole raise to power minus 1 by 2 and del by del y of this term which is 2 y because x and z are independent of y. So, 2 2 cancel out and it is minus y into x square minus 1 again so, it is x square plus y square plus z square and whole raise to power minus 3 by 2 and similarly f z will be minus z x square plus y square plus z square whole raise to power minus 3 by 2 so, this is how we can compute f x f y or f z.

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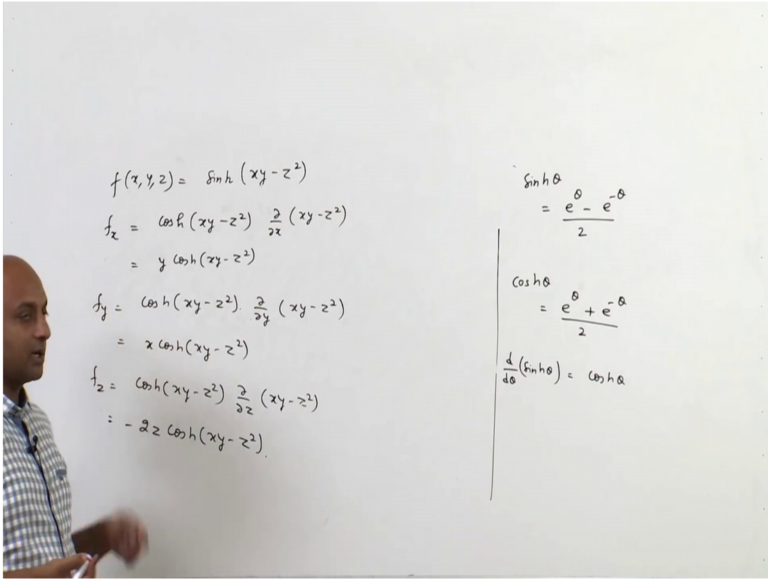

Now, suppose f x y z is sin hyperbolic x y minus z square, what is sin hyperbolic theta? Sin hyperbolic theta is e raise to power theta minus e raise to minus theta upon 2 and cos hyperbolic theta is e raise to power theta plus e raise to minus theta upon 2. So, we can easily see that d by d theta of sin hyperbolic theta is cos hyperbolic theta that we can easily see if we differentiate this respect to theta it is e raise to power theta, it will become plus e raise to power minus theta upon 2 which is cos hyperbolic theta that we can easily see.

So, if you take f x which is del f upon del x now sin hyperbolic theta is cos hyperbolic theta the derivative of sin hyperbolic theta is cos hyperbolic theta and del by del x of theta again it is x y minus z square which is y into cos hyperbolic x y minus z square. Now, f y is similarly, it is cos hyperbolic x y minus z square into del by del y of x y minus z square which is again x times cos hyperbolic x y minus z square and f z will be cos hyperbolic x y minus z square into del by del z of x y minus z square and derivative of this term respect to z partially this will be 0 then this will be minus 2 z times cos hyperbolic x y minus x square. So, this is how we can compute f x f y or f z so, these are some simple illustrations.

Now, what is the relation between continuity and partial derivatives we have already seen,what do you mean by continuity of several variable functions and we have also seen partial derivatives of several variable functions. Now, in single variable function your function is differentiable at a point so, this implies function will be continuous at that point we already know this result that if a function is of single variable and function is differentiable at a point so, this implies function is continuous at a point. A function is continuous at a point then the function may or may not be differentiable at that point. Now, what is a result in multivariable function let us see.

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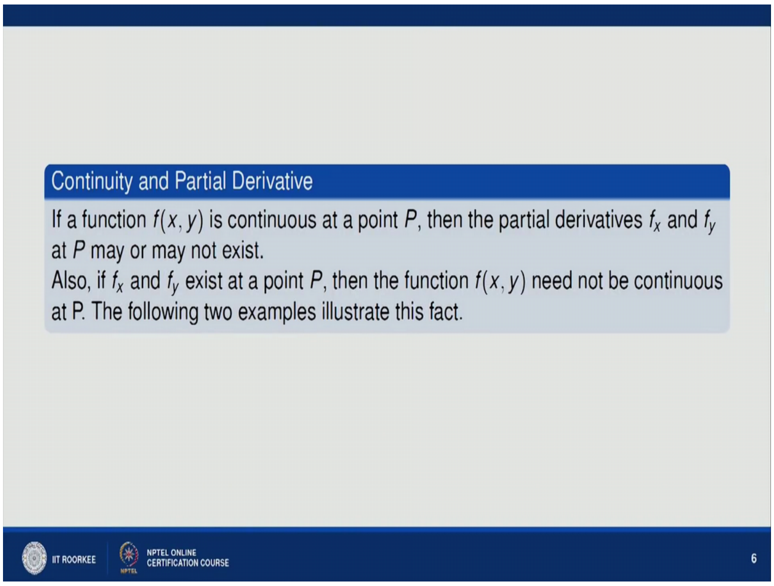

So, if a function f x y is continuous here function f x y is continuous at a point say P, then the partial derivative f x and f y at P may or may not exist. Also, if f x and f y exist at a point, then the function f x y need not be continue at a point. So, your function partially derivative of a function f x and f y exists at a point then it also does not ensure that a function is continuous at that point.

In single variable function differentiability implies continuity, in several variable function is a function is a partial derivative exist a point it does not mean that a function is also continuous at that point how we can say this. So, we have counter examples let us see.

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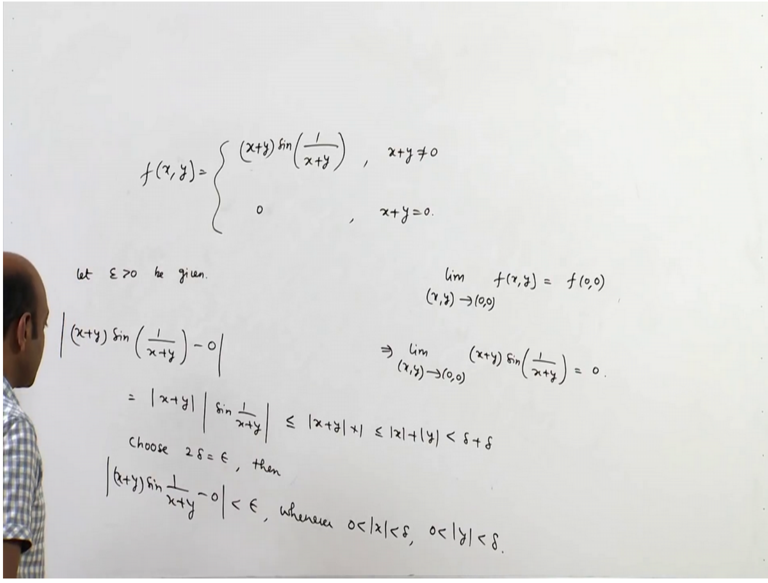

The first example for example is f x y was equal to it is x plus y sin of 1 upon x plus y x plus y should not equal to 0 and 0 when x plus y is equal to 0. Now, first you will see your function is continuous at a point then its partial derivative does not exist at that point. This example which is continuous at a point 0 comma 0, but partial derivative f x and f y does not exist at that point. So, for continuity what we have to show if the if we are saying that this function is continuous at origin so, what we have to prove. We have to prove that limit x y tending to 0, 0 f x y should be equals to f 0, 0 or this means limit x y tending to 0, 0. Here f x y is x plus y sin 1 upon x plus y is equal to f 0, 0 is 0 plus 0 is 0 means 0.

So, this we have to show if we are saying that a function is continuous at a point 0 comma 0 or origin this means we have to prove this equation. Now, in order to prove this we have to use delta epsilon definition the only option is using delta epsilon definition. So, we will take again let epsilon greater than 0 be given. Now take mod of x plus y sin 1 upon x plus y minus 0 f x y minus l, this is equal to mod of x plus y into mod of sin 1 upon x plus y now, sin theta its mod is always less than or equal to 1 for any theta.

So, this will be less than or equals to mod x plus y into 1 and this is less than or equals to mod x plus mod y mod a plus b is always less than or equal to mod a plus mod b and if we take it less than delta plus delta. So, if we choose 2 delta is equal to epsilon, then mod of x plus y into sin of 1 upon x plus y minus 0 is less than epsilon whenever 0 less than mod x delta less than delta and 0 less than mod y less than delta.

So, we have shown the existence of such delta for every epsilon greater than 0 for which this inequality hold. Hence, we can say that the limit of this function is 0 or this equation holds this means function is continuous at origin. So, hence we can say that this function is continuous, now we have to find out its first order partial derivatives. Now, since function we cannot find its partial derivative as such we have to use the definition of limit because we have the function in the split form 0 at 0, 0 function has different value I mean 0 value.

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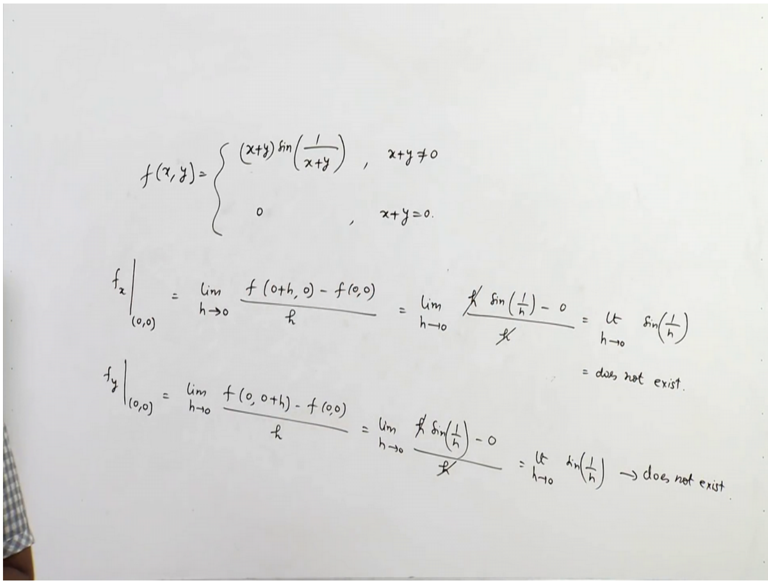

So, we have to use f x at 0, 0 we have to use definition of limit h tending to 0 f of 0 plus h 0 minus f 0, 0 upon h by the definition of partial derivative of f with respect of x.

Now, this is limit h tending to 0 now f at h comma 0 we will use this thing which is h plus 0 is h into sin 1 upon h minus f 0, 0 is 0 upon h, h h cancels out. So, this is limit h tends to 0 sin 1 by h and this is does not exist because it is not a finite value it I mean its not a unique value it lying between minus 1 and plus 1, but it is not unique. So, limit does not exist.

Similarly, if you if you find f y at 0 comma 0 so, it is limit h tending to 0 f of 0 comma 0 plus h minus f 0, 0 upon h which is again limit h tending to 0. Now, f 0 comma h you say you replace y by h and x by 0 so, it is again h sin 1 by h minus f 0, 0 is 0 upon h, h h cancels out and it is again limit h tends to 0 sin 1 by h which is does not exist. So, by this example we have seen there are if a function is continuous at a point it does not mean there is a first order partial derivative exist at that point ok.

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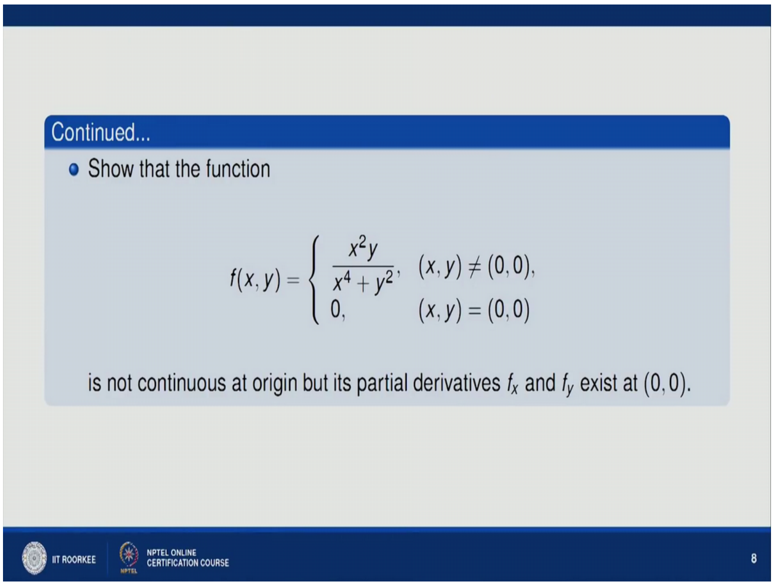

Now, another example now using this example you see the show that the function this is not continuous at origin, but its partial derivative f x and f y exists at origin. So, first we will prove this.

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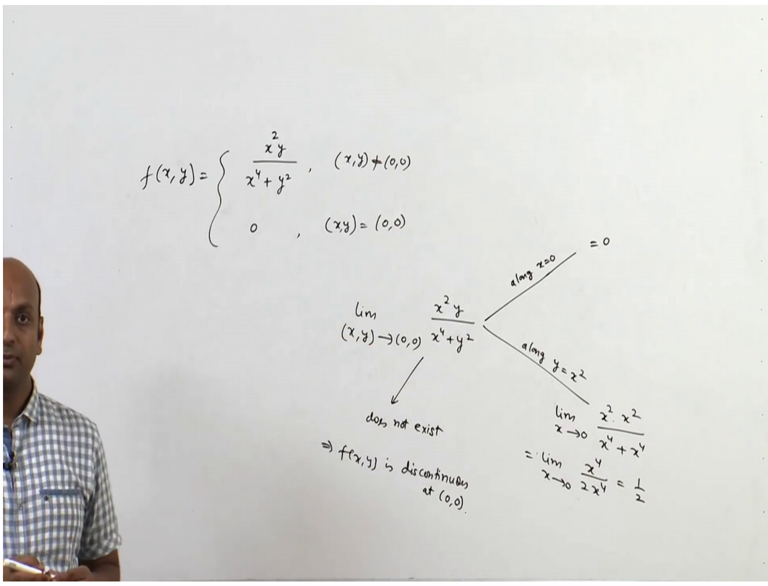

So, it is x raise to power 4 y or it is x square sorry x square y upon x raise to power 4 plus y square when x y is not equal to 0, 0 and it is 0 when x y are 0 comma 0. Now first we have to show that this function is not continuous at origin. So, let us find this limit first limit x y tend into 0, 0 function is x square y upon x raise to power 4 plus y square. Now, to prove that this limit does not exist we have to show that it is path dependent that is from two different paths if we have shown that the value of the limits are different this means limit does not exist.

Now, let us move along say along say x equal to 0; that means, along y axis if you move along y axis then is simply substitute x equal to 0 here which is value is 0 so, it is equal to 0, if you put y x equal to 0 those value is simply 0. Now, you move along say y equal to x square now if you move along y equal to x square so, this will be limit x tending to 0 it is x square into x square upon x raise to power 4 plus x raise to power 4 which is equal to limit x tending to 0 x raise to power 4 upon 2 x raise to power 4 which is 1 by 2.

So, we have show so, from this path from this path value is 0 and from this path value is 1 by 2 that is from two different paths values are different. This means that this limit does not exist and this implies f x y is discontinuous at origin ok.

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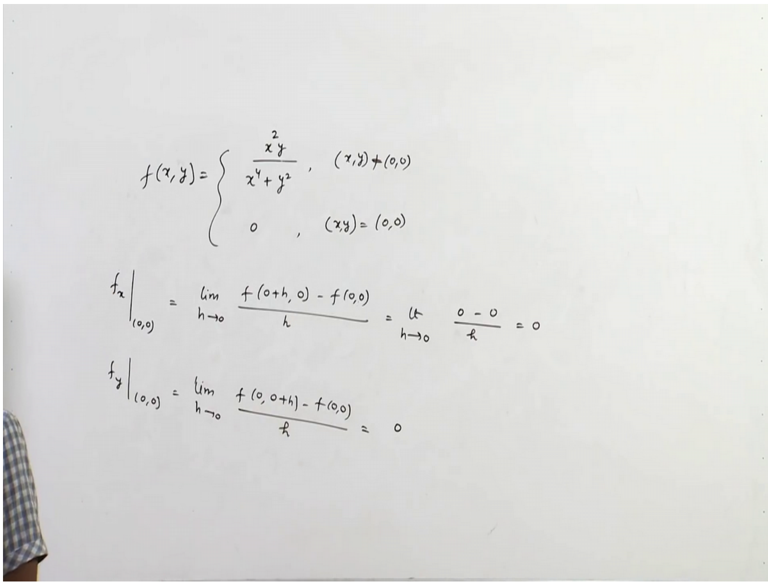

Now, let us try to prove that is first order partial derivative exist so, let us find f x at 0 comma 0. Again we will use the definition of limit it is limit h tends to 0 f of 0 plus h comma 0 minus f 0, 0 upon h which is limit h tends to 0. Now, when you put h comma 0 here x is h y is 0 as x is h y is 0 so, simply this value will be 0. So, it is 0 minus 0 as 0 0 is 0 upon h which is 0 now again when you compute f y at 0 comma 0, it is limit h tends to 0 f of 0 comma 0 plus h minus f 0, 0 upon h which is again equal to limit h tending to 0 it is 0 comma h you replace x by 0 y by h x by 0 y by h. So, this value will be 0, 0 minus 0 upon h it is again 0. So, f x and f y exist at origin which is 0 value is 0, but we have seen that this function is not continuous at origin.

So, existence of partial derivative at a point does not guarantee that a function is continuous at that point also for several variable functions. So, what is the additional condition required when we can say that beside existing of partial derivative what is what are the additional condition which function must have so that we can say that the function is continuous at that point also.

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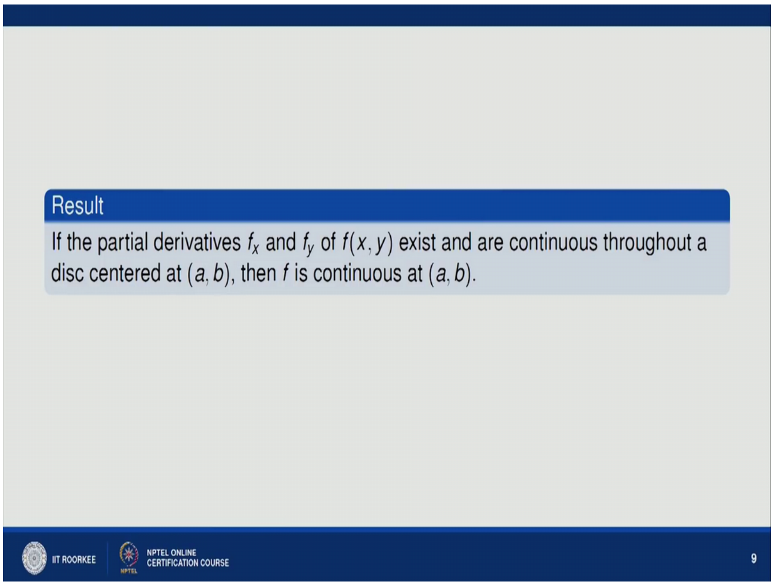

So, that condition is basically if the partial derivative f x and f y of f x y exist and are continuous also throughout a disk centred at a comma b, then function is continuous at that point; that means, beside the existence of partial derivative at a point. We must have continuity of partial derivative throughout a disk centred at that point, then we can say function is continuous at that point.

Thank you very much.