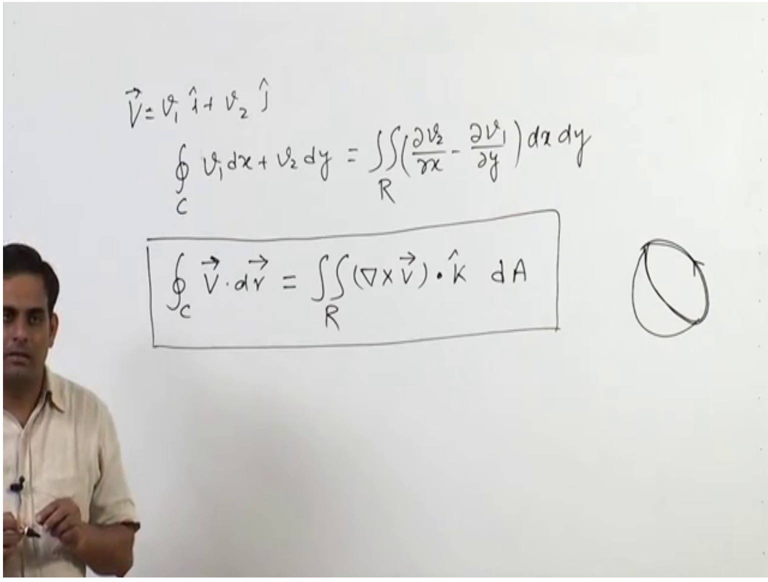
Multivariable Calculus
Dr. Sanjeev Kumar
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 40
Stokes's Theorem

Hello friends. So, this is the last lecture of this course. And in this lecture I will teach Stokes's Theorem that is another important result from the vector calculus. So, stokes's theorem can be seen as a generalization of Green's theorem in three dimensionally space, or better to say that the Green's theorem is a special case of Stokes's theorem for a given plane.

(Refer Slide Time: 01:01)

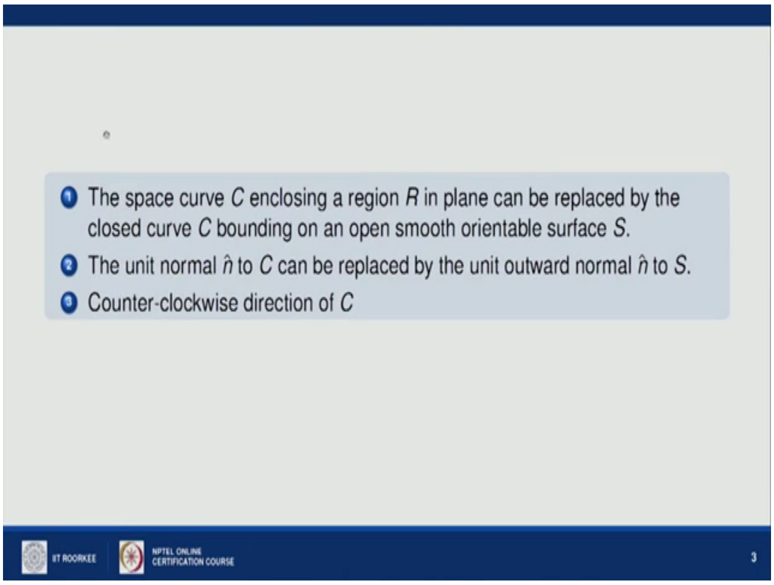


So, let us see in 36 lecture, I told you about Green's theorem and it was like that if you are having a vector function V, which is having component v 1 and v 2; where v 1 and v 2 are functions of x and y. Then the line integral over a closed curve; means the Green's theorem is v 1 dx plus v 2 dy and this integral over a closed curve C, which is bounding a region R equals to R del v 2 over del x minus del v 1 over del y dx d y.

Now, let us write this result in vector form. So, I can write the left hand side, the line integral over the closed curve C and this I can write V dot d r. The right hand side I can write as the curl of V and dot product of it with K dx dy or I can write simply d A because, it is a region in a plane ok. So, it means del v to over del x minus del v 1 over del y is the K component of the curl of V.

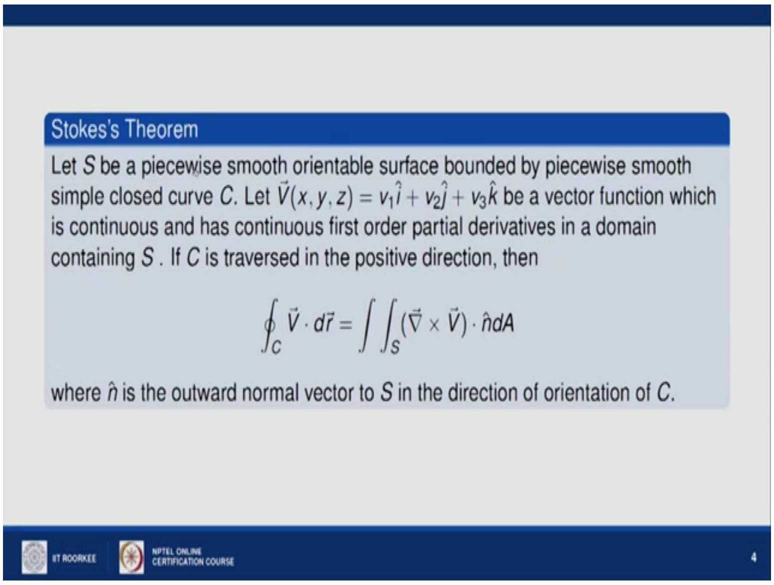
Now, this is the Green's theorem in vector form. Now we will make some generalization on this particular equation and we will write the Stokes's theorem from here. So, replace the closed curve C by a another curve ok C; which is enclosing an open surface S. For example, if you are having a hemisphere; so, hemisphere will be something like this. So, here C will be this curve because this particular curve is enclosing this surface.

(Refer Slide Time: 03:52)



Now, replace this r or line integral by the surface integral or better to say in this way. The generalization are the space curve C enclosing a region R in a plane can be replaced by the closed curve C bounding on an open smooth orientable surface S. The unit normal n to C can be replaced by the unit outward normal n to a surface S; which having curve C on it. Now, counter clockwise direction of C can be seen as the outward direction of the normal n. With these generalization, in the result of Green's theorem in vector form we will state the Stokes's theorem.

(Refer Slide Time: 04:43)

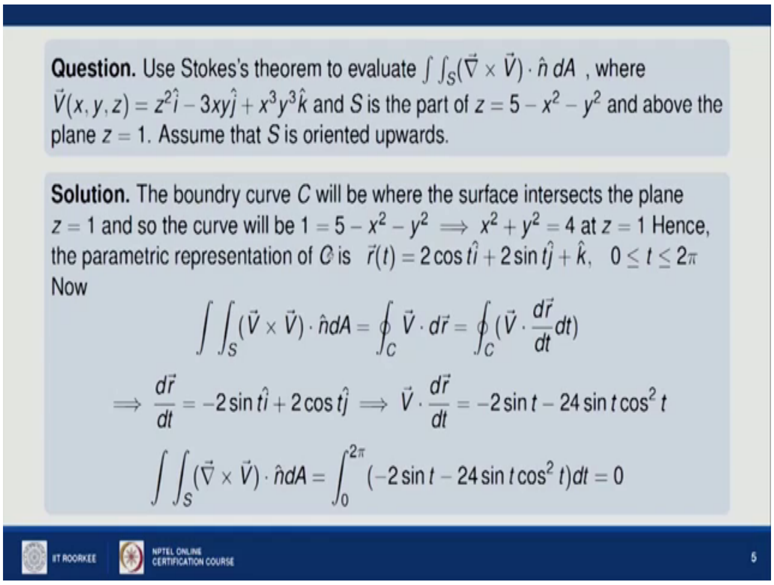


So, let S be a piecewise smooth orientable surface bounded by piecewise smooth simple closed curve C. So, S is the surface which is piecewise smooth, orientable and it is enclosed by a piecewise smooth simple closed curve C. Let V be a vector function having component v 1, in i direction, v 2 in j direction and v 3 in k direction. Here v 1, v 2, v 3 all are functions of x, y, z.

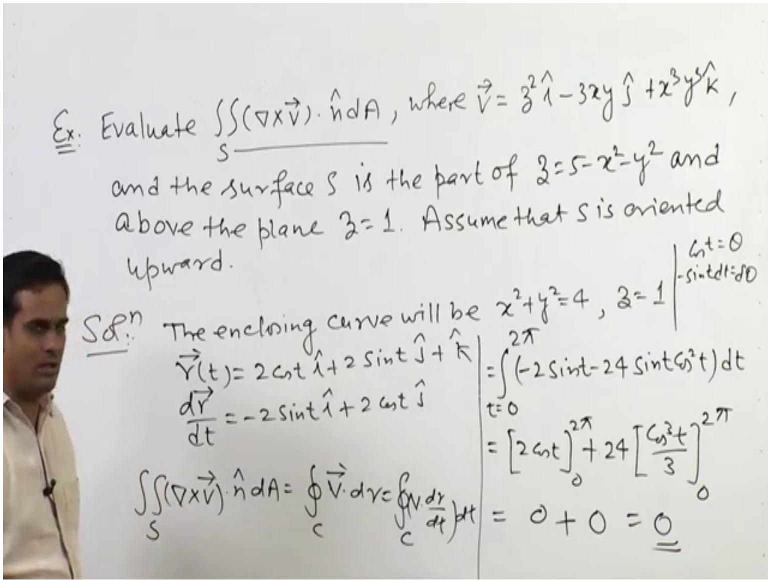
And this vector function V is continuous and has continuous first order partial derivatives in a domain containing S. If C traversed in positive direction, then the line integral C over the closed curve C; V dot d r equals to the surface integral over the surface S curl of V dot product with unit normal vector n into d A.

So, here in this Stokes's theorem, we are having the conversion of a line integral into surface integral or vice versa. So, basically if you want to calculate a surface integral over a surface, you can replace it by a line integral over the closed curve enclosing this surface. So, this is the meaning of Stokes's theorem. Now, we will take few example on this theorem.

(Refer Slide Time: 06:17)



(Refer Slide Time: 06:27)



So, first example is evaluate the surface integral as of a vector curl of V d A, where V is given as z square i minus 3 x y j and finally, plus x cube y cube K and the surface S is the part of z equals to 5 minus x square minus y square and above and above the plane z equals to 1. So, z equals to 1 plane is the lower portion of the surface and upper bound is given by this one.

Assume that S is oriented upward ok. So, first of all we need to find out if we want to apply Stokes's theorem here, we need to find out the curve which is enclosing this surface. So, here that curve will be in this plane; z equals to 1. And what will be this curve, since the curve is in this plane over this surface? So, 1 equals to 5 minus x square minus y square.

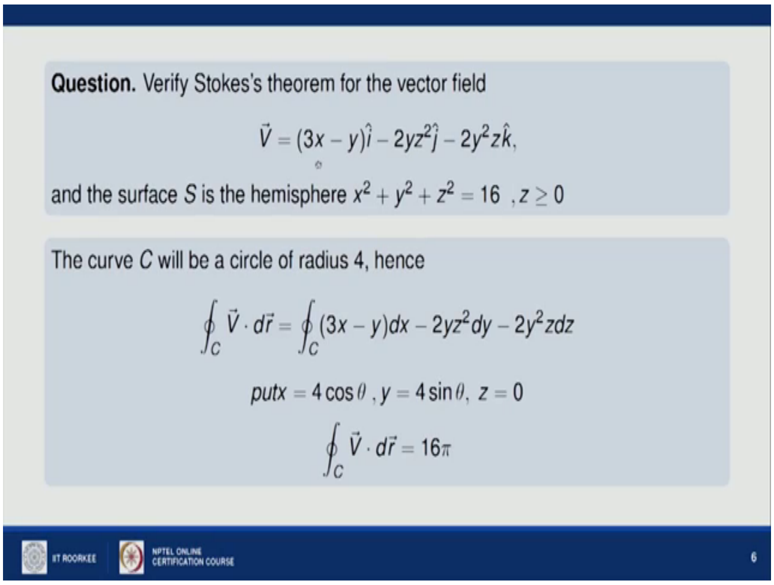
So, from here, the enclosing curve will be x square plus y square equals to 4 and z equals to 1 ok and I told you why I am taking this curve as the enclosing curve. Now, what I am having? The parametric equation of this curve can be written as 2 cos t i plus 2 sin t j plus z is 1; K. So, here d r over d t will become minus 2 sin t i plus 2 cos t j plus ok, k is 0; k component is 0.

Now, so I am having d r over d t. By the Stokes's theorem, I can write the surface integral del cross V dot n d A equals to this line integral. Now, I need to calculate this thing ok. So, here del cross V dot n d A V dot d r ok, my v dot d r can be written as C v dot d r over d t into d t. So, for this I need to write V also in terms of t and then I have to calculate the dot product of it with d r over d t.

So, this comes out to be line integral over the closed curve C and V dot d r over d t becomes minus 2 sin t minus 24; here, I am having x y. So, 12 into 2 and it comes out to be sin t cos square t d t and here t is from 0 to 2 pi; because, curve is a closed curve. So, this equals to 0 to 2 pi. So, this will be when I will solve this; this will be sin t will become cost minus cos t. So, it will be 2 cos t 0 to 2 pi minus 24. If I take cos t as z or cos t as let us say theta. So, I got minus sin t d t as d theta. I am having minus sin t here d t also.

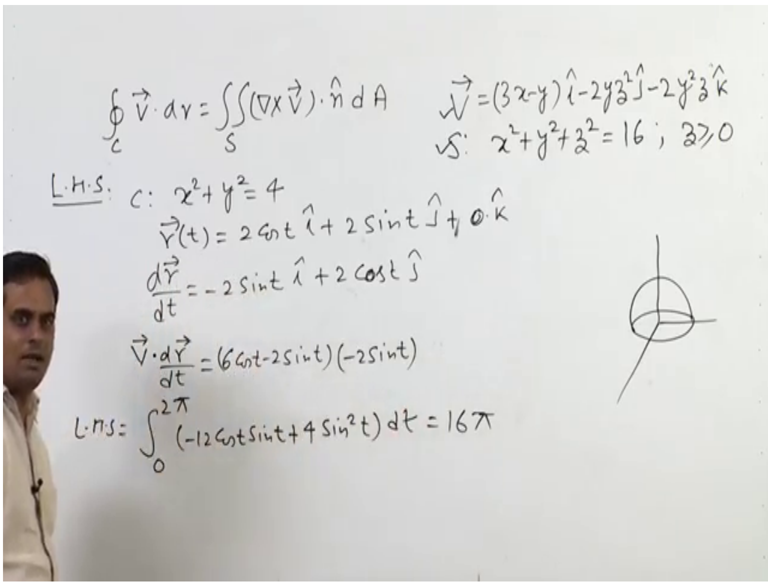
So, that will be replaced by d theta. So, minus minus will become plus, it will be cos cube t over t 3 and then 0 to 2 pi. So, cos 2 pi is 1; 1 minus 1 0. Similarly here, 1 minus 1 0; the answer is 0. So, the value of this integral is 0 and basically if I will do it in a straightforward way without using the Stokes's theorem first I need to calculate del cross V, then I need to calculate a normal vector n over this surface. Here also surface will be having two portions; this and this; I need to calculate the surface integral separately on both the surfaces. So, that will be a lengthy process. However by using the Stokes's theorem, I can do it in this simple way. So, that is one of the application of Stokes's theorem.

(Refer Slide Time: 14:06)



Let us take another example and in this example, we are having verify Stokes's theorem for the vector field, V equals to 3 x minus y i minus 2 y z square j minus 2 y square z k and the surface S is the hemisphere x square plus y square plus z square equals to 16 and it is the hemisphere above the x-y plane as z is greater than equals to 0.

(Refer Slide Time: 14:42)



So, now, is Stokes's theorem is V dot d r equals to del cross V dot n d A. First let us take left hand side. My V is given as 3 x minus y into i minus 2 y z square j minus 2 y square z k. And the surface S is the hemisphere x square plus y square plus z square equals to a square; z is greater than equals to 0. Here a is whereas, let us take something; let us take a equals to 4; so a square is 16.

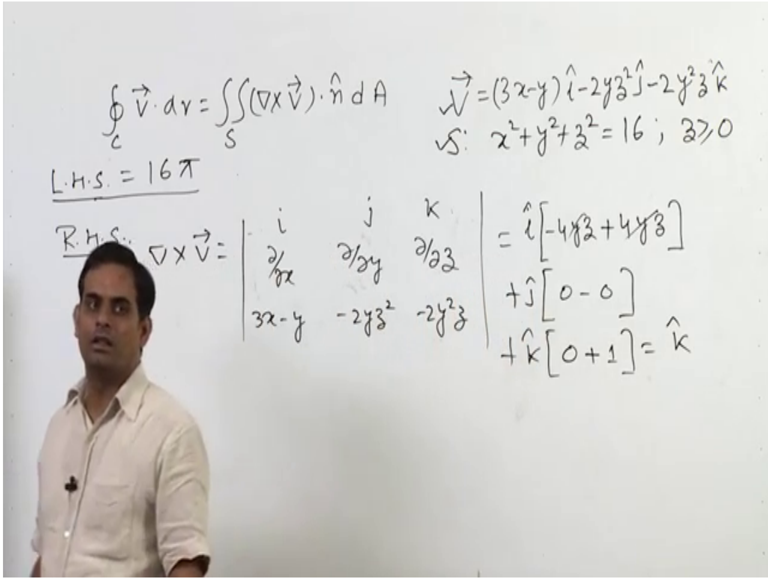
So, now it is a hemisphere above x-y plane. So, what we are having the enclosing curve. So, it is something, so it will be like this. So, enclosing curve will be a circle in x-y plane and circle of radius 4. So, here my C is x square plus y square equals to 4. So as we have done in earlier case r t will become 2 cos t i plus 2 sin t j, that is all because z is 0 there in x-y plane.

So, z is 0; so plus I can write 0 times K; d r over d t will become minus 2 sin t i plus 2 cos t j; V dot d r over d t will become, so, V is 3 x minus i y. So, 3 into 2 6 cos t minus 2 sin t; so, 6 cos t minus 2 sin t. This is the component v 1 of V and dot product with i component of this, so, minus 2 sin t plus there it is 2 y z square z is 0. So, everything is 0. Now this is my V dot d r over d t. So, let us evaluate it; 0 to 2 pi. This will be my closed curve.

So, left hand side is t equals to 0 to 2 pi and then minus 12 cos t sin t minus minus plus 4 sin square t d t. Let us solve it, after solving it you will get it as 16 pi. So, 16 pi is the value of left hand side for this given vector V and this given surface means the line integral over the enclosing curve of the surface S.

Now, we need to evaluate the other side that is right hand side and if the right hand side equals to 16 pi, then the theorem is verified for the given data. So, L.H.S. equals to 16 pi. Now I will calculate right hand side.

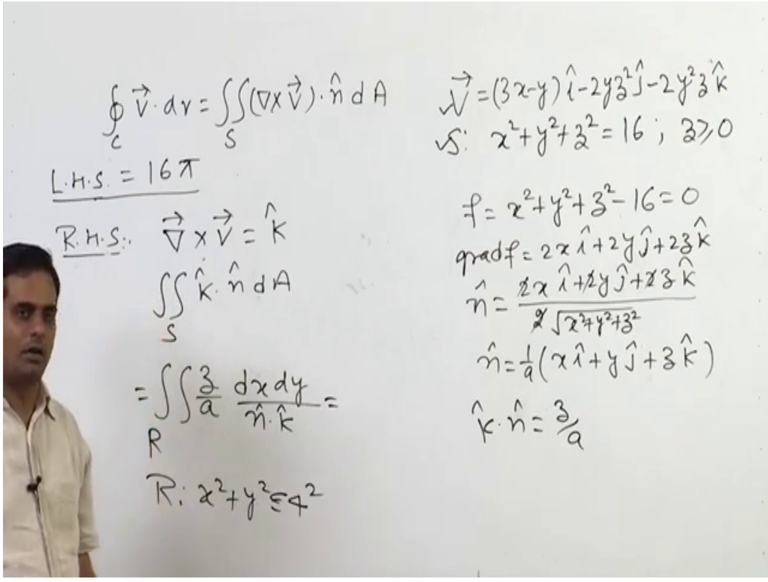
(Refer Slide Time: 19:18)



So, for calculating right hand side, first of all I need del cross V. So, let us calculate del cross V, so, this equals to i, j, k, del by del x, del by del y, del del z and then 3 x minus y, minus 2 y z square, minus 2 y square z. This equals to i del by del y of this; so, minus 4 y z plus del by del minus minus plus del by del z of this; so, plus 4 y z. So, i component is 0.

Similarly, I will try to find out j component. So, j component is del by del z of this. So, 0 minus 0 plus k component; so, k component is del y del x of this. That is 0 minus del by del y of this; so, minus minus plus so, it will become 0 plus 1. So, curl of V is K. So, curl of V equals to K.

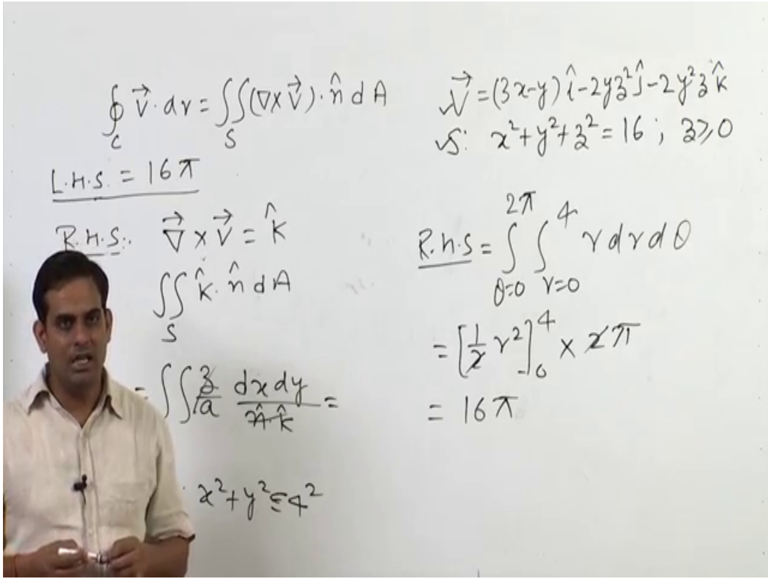
(Refer Slide Time: 20:55)



Now, I need to calculate S K dot n d A. First of all I need to calculate n. So, f is given as x square plus y square plus z square minus 16 equals to 0. So, grade f or del f will become 2 x i plus 2 y j plus 2 z k and unit normal vector to this surface will become 2 x i plus 2 y j plus 2 z k upon square root of this. So, power 2 and square root of x square plus y square plus z square root. So, 2 will be cancelled out. So, unit normal vector n will be 1 upon a x i plus y j plus z k.

So, this is the unit normal vector to surface. Now K dot n will become z upon a. So, this equals to z upon a d A over the surface S. Now I need to project the surface on either on x-y plane y-z plane or z-x plane. Let us project it on x-y plane. So, on the x-y plane d A will become dx dy upon n dot K; this is also n dot K. So, this will become over a region R; where R is so, projection of this on x-y plane. It will be a circle x square plus y square 4 square this disk.

(Refer Slide Time: 23:31)



So, this equals to so, right hand side equals to theta r will move from 0 to 4; theta will go from 0 to 2 pi; r b r d theta; because, these two things will be cancelled out. It will become 1 by 2 r square 0 to 4 into 2 pi. So, 2 will be cancelled out, it comes out to be 16 pi. Left hand side is 16 pi, right hand side is 16 pi. So, both are equal and hence Stokes's theorem is verified.

So, in this lecture we have learn about Stokes's theorem, we have taken few example of Stokes's theorem and we have verified it for a given V and a given surface S. With this I will end this lecture; however, since it is my last lecture. So, for this course so, I need to acknowledge few people, first of all I would like to thank the NPTEL team of IIT Roorkee, all the people here in educational education technology shell, who are the part of this course. Especially I would like to thanks Sharath, who is the person, who is recording this course. I am also thankful to my teaching assistant Farhan, who is a PhD student also under me. So, he is working continuously with me for this course.

Thank you very much.