Multivariable Calculus
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Lecture – 04
Continuity of multivariable functions

Hello, friends. Welcome to a lecture series on multivariable Calculus. So, in the last lecture we have seen that what do we mean by existence of a limit for two variable functions we have seen that limit of two or more than two variable exists this means that it is path independent. Whatever path from whatever path we approach from xy to x naught, y naught the limit if exist is always unique, that is, ah, it is path independent, ok. And if you have to show that the limit does not exist we try to show that from two different path value of the limits are different.

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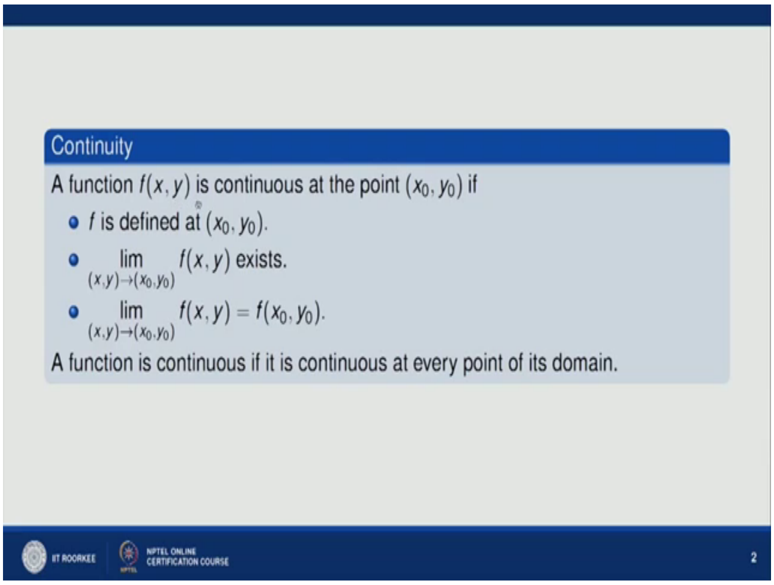

Now, will see what do you mean by a continuity of multivariable functions. So, our function f x y is said to be continuous at a point x naught, y naught, if these three properties hold; that means, function is defined at x naught y naught, limit x, y tending to x naught, y naught f x, y exists and limit x, y tend to x naught, y naught f x, y is equal to f of x naught, y naught and a function is continuous if it is continuous at every points of it is domain.

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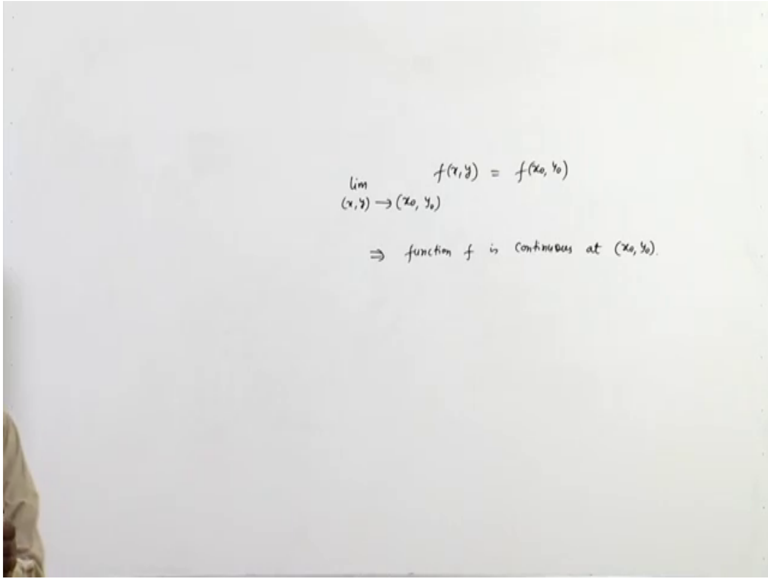

So, if you are talking about continuity at a point. So, this means that limit x, y tending to x naught, y naught f x, y should be equal to f of x naught, y naught, this means this implies function f is continuous at x naught, y naught.

So, this means limits should exist and is unique this means this value of the function should be find x naught, y naught and the value of the limit must be equal to f at x naught, y naught ok, then this means function is continuous at x naught, y naught and if it holds for every x naught, y naught in the domain of f then we say the other function is continuous in the domain of the function f.

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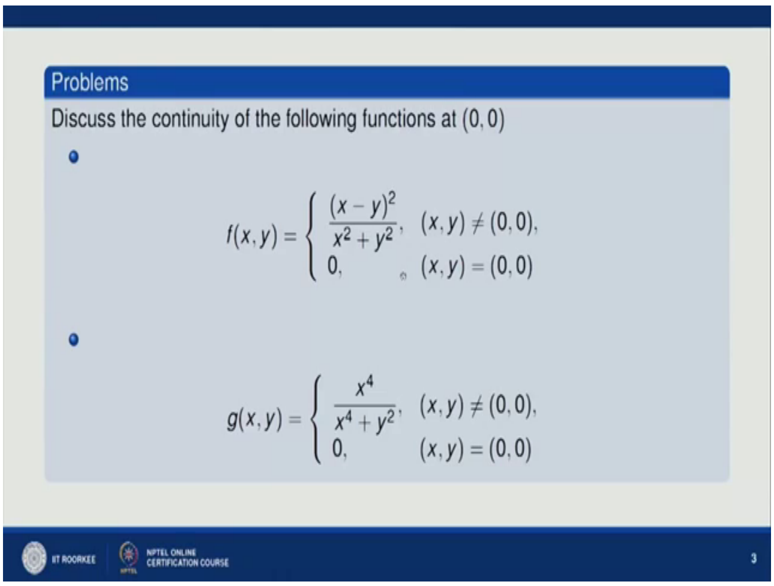

Now, we will see some examples based on this. Discuss a continuity function following function at x naught, y naught.

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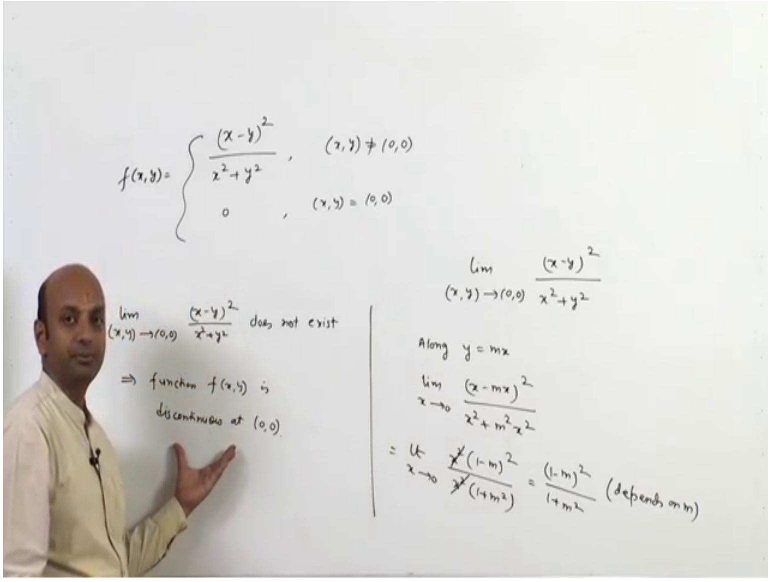

Say, the first function, we are taking the function is, the function is it is x minus y whole square x square plus y square, when x y are not equal to 0, 0 and it is 0, when x, y are equal to 0, 0.

So, function defined at 0, 0 and the value of the function is function value the function is 0, 0, 0. If function is continuous at 0, 0 then this means that this limit must be equal to 0,. So, we if we have if we have shown that the that the value of this limit is equal to 0, which is the value of the function at 0 comma 0 ok, then this means function is continuity at 0 comma 0. So, first we will see whether this limit exists or not then only we can say that this limit is equal to 0 or some other value.

So, if we if we move along say so, if we move along say y equals to m x then this is limit x tending into 0, x minus m x whole square upon x square plus m square x square ok, which is further equal to limit x tend to 0 x square 1 minus m whole square upon x square 1 plus m square and x tend to 0. This cancels out and this is equals to 1 minus m whole square upon 1 plus m square that is depends on m and since it depends on m this means this limit does not exist, this means this limit does not exist and this implies function f this function f x, y is discontinuous at 0, 0 because double limit at 0 comma 0 does not exist, so, we cannot talk about the continuity. So, definitely a function is discontinuous at 0 comma 0.

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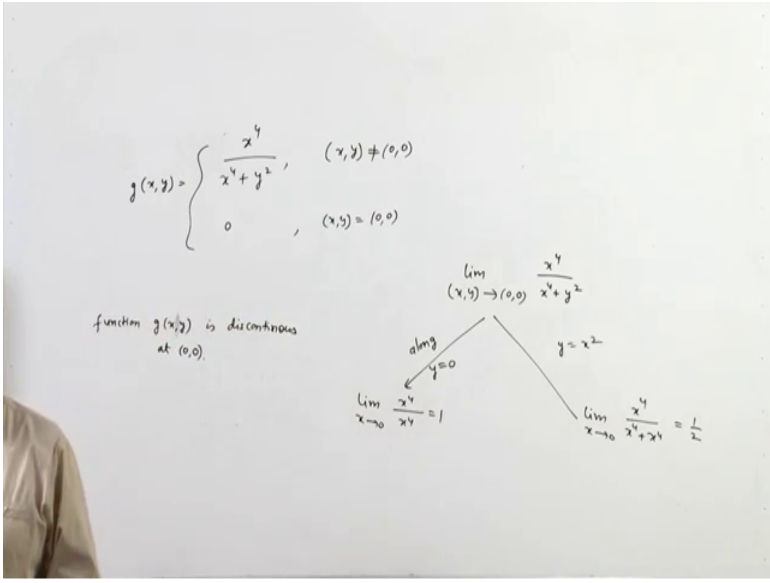

The another example is say g x, y is equals to. It is x is to power 4 upon this quantity when x, y is tending to 0, 0 and it is 0 when x, y are tending to r equal to 0, 0 that is the value of the function at 0, 0 is 0 ok. Now, again we have to check whether this function is continuous at origin or not. So, first we will see whether this limit of this function exists or not.

So, let us move along to different paths or let us first move along y equal to 0, say. So, if we move along y equal to 0, what are value of limit? When you put y equal to 0 here then this is limit x tend to 0 x raised to power 4 over x raised to power 4 which is 1.

Now, when you move along say y equal to x square if we move along y equal to x square then it is limit x into 0 x raised to power 4 upon x raised to power 4 plus x raised to power 4 which is 1 by 2. So, from two different paths values are different. So, this means this limit does not exist and hence this function. So, we can say that the function g x, y, g x, y is discontinuous at 0 comma 0, since, because the limit does not exist and the function is discontinuous at 0 comma 0.

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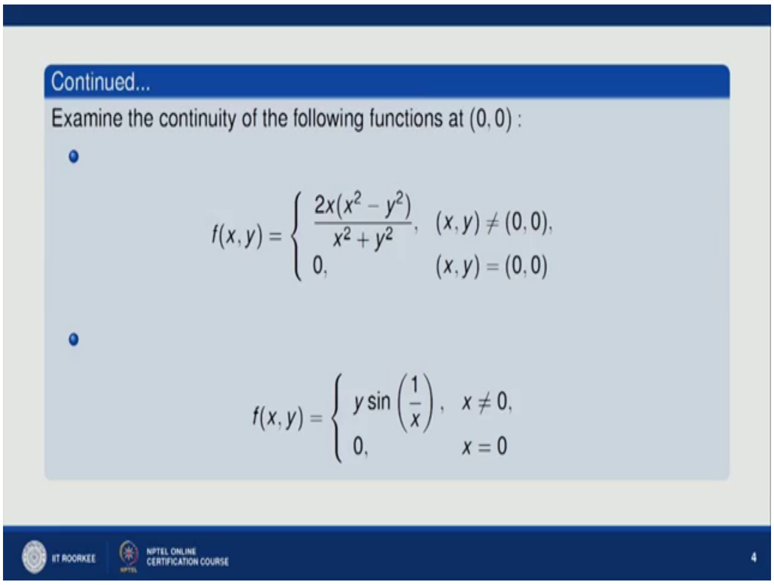

Now we will see two more examples based on the continuity of function of the origin.

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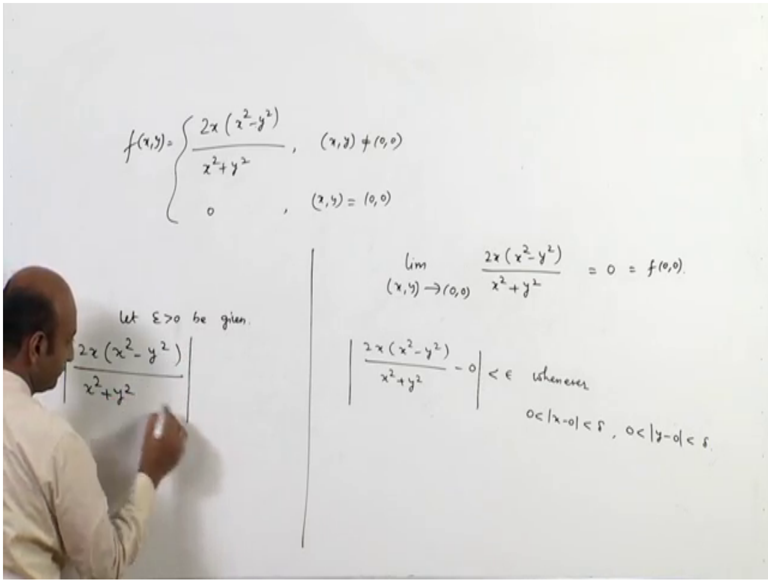

Suppose, you want to see the first problem the first problem is f x, y is equals to it is 2x x square minus y square upon x square plus y square x y not equal to 0, 0 and 0 when x, y is equal to 0 comma 0 .

Now, again to see whether this function is continuity of origin we first see whether the limit double limit at 0 comma 0 this is a double limit. Now, to check whether this is continuous or not we have to show that this limit is equal to 0. Now, because if we see different paths the values of values always come out to the same which is 0. So, we can check by delta epsilon definition whether this limit is equal to 0 or not. So, so this is this is 0 which is equal to f 0, 0 basically. So, we are trying to show that the value of limit of the limit of the function is equal to value of the function at 0, 0 at 0 comma 0.

So, how can we show this? So, we will try to show you delta epsilon definition. So, let epsilon greater than 0 be given ok. So, what we have to show basically now? If this limit exists, so, we have to show that this is this quantity minus 0 is less than epsilon whenever 0 less than mod x minus 0 less than delta and 0 less than mod y minus 0 less than delta ok, this you have to show. So, you take this side this is mod of 2x into x square minus y square upon x square plus y square ok.

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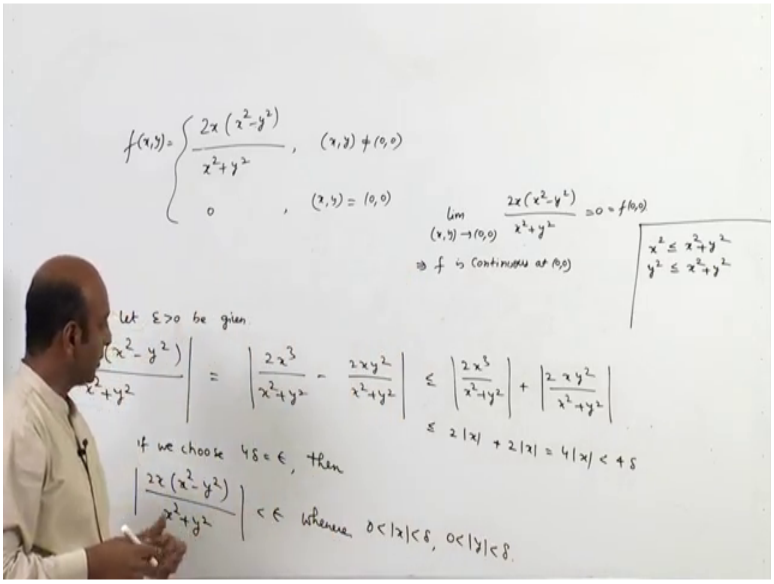

Now, this quantity is equal to it is 2x cube upon x square plus y square minus 2 x, y square upon x square plus y square this modulus which is less than equal to 2x cube upon x square y square mod plus 2x, y square upon x square plus y square is modulus.

Now, this is this is x into x square and x square is always less than equal to x square y square. Similarly, y square is also less than equals to x square y square ok. So, we can say that x square upon x over y square less than equal to 1. So, it is less than you equals to 2 into mod x because it is x into x square and x square upon x square y square is n equal to 1. So, it is less than equal to mod x plus 2 times. Again, y square upon x square y square is less than equals to one. So, it is less than equals 2 mod x. So, it is 4 mod x, and this mod x is less than delta, so, it is 4 delta. So, if we choose 4 delta is equal to epsilon then this quantity then this quantity will be less than epsilon whenever 0 less than mod x less than delta and 0 less than mod y less than delta.

So, we have shown the existence of delta for every epsilon for which the inequality hold hence we can hence we can say that the that this limit that the limit of this function is 0 hence we can say that this limit this limit exists and is equal to 0, which is the value of the function at 0 comma 0 hence we can say that function is continuous at origin. So, this implies f is continuous at origin ok, because this limit exists and is equal to 0 this is the value of the function at 0 comma 0 this means function is continuous at the origin.

So, similarly we can try a more example based on this. Now, this problem we can also show by converting it into polar coordinate system. We can take x equal to r cos theta y equal to r sine theta then also we can show that this limit is equal to 0.

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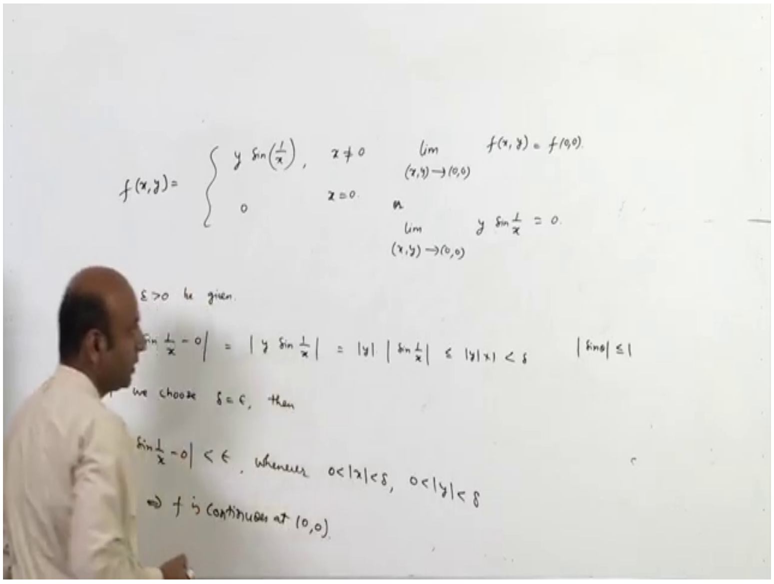

Now, you can take this example it is f x, y is equal to it is y sin 1 by x when x is not equal to 0 it is 0 by x equal to 0. So, again we will see whether this limit x, y tending to 0 comma 0 this f x, y is equals to f 0, 0 or not or limit x, y tending to 0, 0 y sin 1 by x is equals to 0, 0 because when x is 0 value is 0, ok. Now, now again we will try to show this using delta epsilon definition; now how can we use it.

So, let epsilon greater 0 be given. So, I will take this side mod y sin 1 by x minus 0, we will take this side f x, y minus l ok. Now, this quantity is equal to mod y sin 1 by x which is equal to mod y into mod of sin 1 by x. Now, sin theta is always mod of sin theta is always less than equal to 1. So, it is less than equals to mod y into 1 because mod of sin theta is always less than equals to 1 for any theta and this is less than delta. So, if we choose delta equal to epsilon then mod of sin 1 by x minus 0, will be less than epsilon whenever 0 less than mod x less than delta and 0 less than mod y less than delta. So, we have shown the existence of such delta for this inequality hold, hence we can say that this limit exists and is equal to 0, this is the value of the function is 0 comma 0. So, this means this implies f is continuous at 0 comma 0.

So, we have seen that this if we will talk about this function y sin 1 by x. So, this function is continuous at origin, ok. We have shown this by delta epsilon definition. So, this is double limit. So, double limit of this function is 0.

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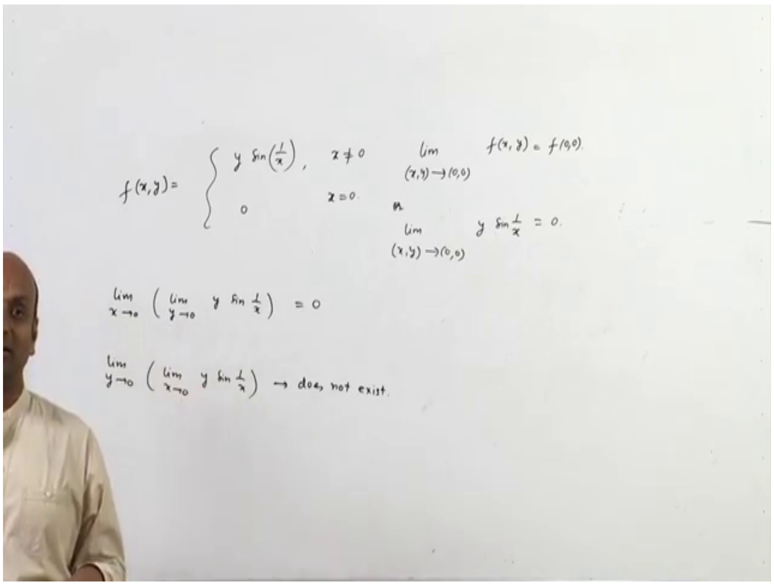

Now, if we talk about iterated limit of the same problem say we want to find out limit x tend to 0, limit y tending to 0, so, y sine 1 by x or limit y tend to 0 limit x tend to 0 y sin 1 by x, let us let us talk about iterated limit of the same problem ok.

Now, if we take y tend to 0 in this for this function now, this is a bounded function and y tend to 0, so, this value is always 0. So, this is 0. Now, this is x tend to 0, you forget about y, this is x tend to 0 when x tend to 0 then this limit does not exist because this is basically lying between minus 1 and plus 1 it is not finite. So, this limit does not exist.

So, this limit does not exist means the entire limit does not exist. So, we can say that this limit does not exist.

Basically, what I want to say from here you see that these iterated limits are not equal, are not same is still double limit exists and equal to 0, why? The reason is simple because that implication that double limit if it double limit is l, then the iterated limit is also are equal at equal to l exist that implication will exist only when this inside limit exists. Now, here this limit exists, but this limit does not exist.

So, that implication does not hold is it clear because we can talk about its existence of iterated limit only when the inside limit exists ok. So, hence here in this example these limits are not equal, it is still the double limit equal to 0. So, that is all about limit and continuity of several variable functions.

Thank you.