Multivariable Calculus
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Lecture – 39
Divergence Theorem of Gauss

Hello, friends. So, welcome to the penultimate lecture of this course. In this lecture I will teach you divergence theorem of Gauss. Basically, this theorem this relation between the surface integral computed over a closed surface and the volume integral of the divergence of the vector on which I need to calculate the integral. So, let us start with the statement of this theorem.

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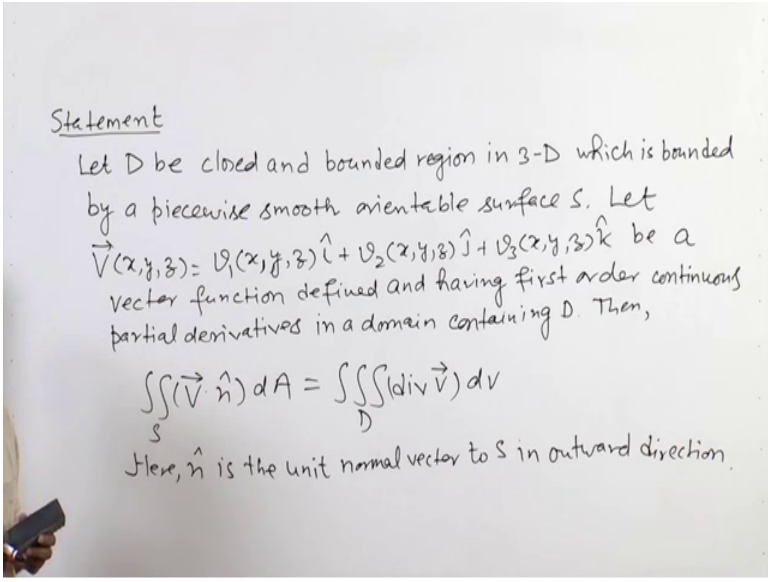

So, let D be a closed and bounded region in 3-D which is bounded by a piecewise smooth orientable surface S. So, here S is the boundary of the region D, and as you cannot down the region is closed as well as bounded. For example, like a sphere if the surface S is a sphere then the region inside the sphere along with the boundary will be the D.

Let V be a vector function of x, y, z, having component as v 1 in i direction, v 2 in j direction and v 3 in k direction. So, this v is a vector function defined and having first order continuous partial derivatives in a domain containing D, then we have the surface integral of V over the surface S this equals to divergence of vector V into dV and this triple integral we are calculating over the region D. Here n is the unit normal vector to surface S in outward direction means in the other side of the region D. So, this statement is the divergence theorem of the Gauss.

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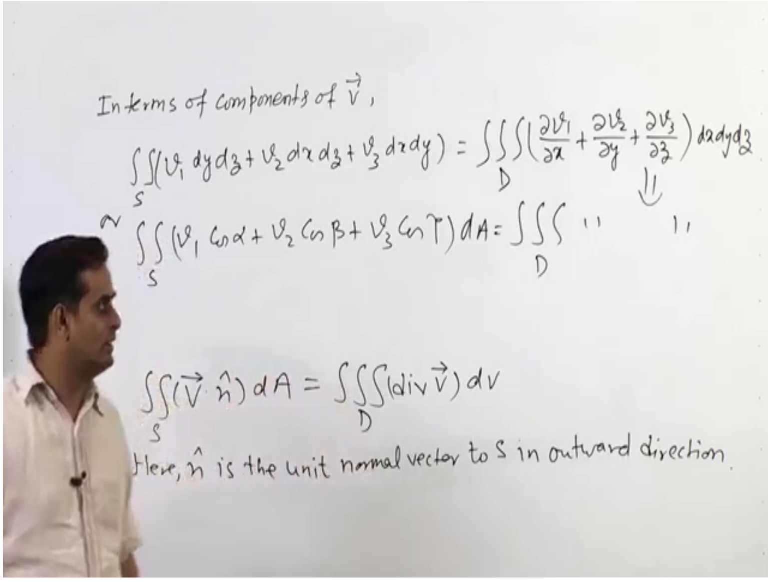

Now, in terms of components of V we can write this theorem as. So, v 1 is the component of vector v in i direction, so, I can write v 1 dy dz plus v 2 dx dz plus v 3 dx dy. So, this is replacing this V dot n dA and we have seen it in surface integral of the vector function. This equals to triple integral over the region D and now, divergence we will become del v 1 over del x plus del v 2 over del y plus del v 3 over del z and this into dx dy dz.

So, this is the alternative formula of the divergence theorem of Gauss in terms of components of a vector or I can write this as v 1 cos alpha plus v 2 cos beta plus v 3 cos gamma; So, v 3 cos gamma and then dA equals to the right hand side given by the same thing.

Here alpha beta and gamma are the angles which the unit normal vector n max with x axis y axis and z axis in positive directions respectively. So, alpha is the angle between x axis and unit normal vector n, beta is the angle between y axis and unit normal vector n and gamma is the angle between z axis and unit normal vector n.

So, we can consider any of the three forms as I told you and all of these are called the divergence theorem of the Gauss. So, now I will talk few examples where I can apply this theorem for calculating either the surface integral or the volume integral.

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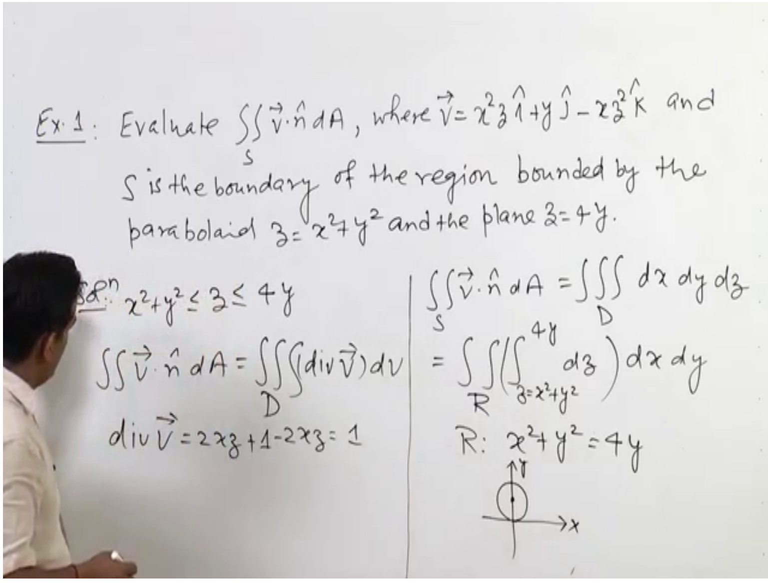

So, my first example is evaluate the surface integral V dot n d A ok, where V is given by a vector function x square z i plus y j minus x z square k and S is the boundary of the region bounded by the parabolaid z equals to x square plus y square and the plane z equals to 4y.

So, let us solve it. So, here the range of z is clearly given, it is from x square plus y square means the surface of the parabolaid and upper one is the plane z equals to 4y. Now, I need to calculate V dot n dA. So, by the divergence theorem this will be equals to the triple integral over the region D divergence V dV. Now, divergence V will become 2xz plus 1 minus 2xz. So, it comes out to be 1.

So, hence the surface integral of V over the surface S equals to the region D dx dy dz because here divergence of V is 1, this equals to z equals to x square plus y square and then 4 y dz into dx dy and now dx dy. So, now, this double integral is over the region in x y plane which is the projection of the region D on x y plane.

So, now, projection will become from here it is z equals to x square plus y square it is a parabolaid going in this way and then we are having a plane z equals to 4y. So, hence the projection on x y plane will become this over R and here R will become x square plus y square equals to sorry 4y which is the upper boundary of the region and if you note down what is this it is a circle in x y plane of radius 2 and having centre at 0, 2. So, centre is 0, 2 radius is 2.

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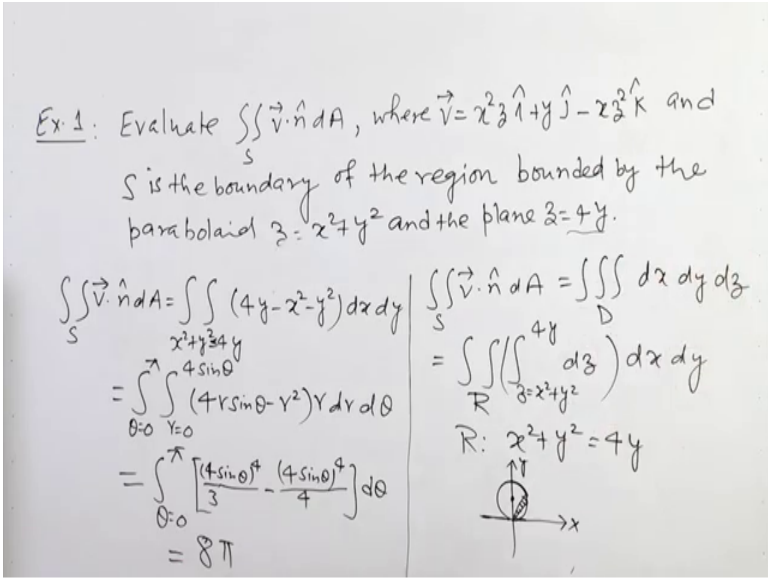

So, now my integral or surface integral of V over the surface S becomes integral over the region x square plus 5 y square equals to 4x this region and this will become 4y minus x square minus y square dx dy. So, here it is 4y 4y. So, if I change it into polar coordinate R will start from 0 it will go up to 4 sin theta and theta will be from 0 to pi because still will go like this.

So, r equals to 0 and ended over this curve which is 4r equals to 4 sin theta and 0 to pi because it will start from here and ended over this line and then it will become 4 r sin theta minus r square because x square plus y square is r square r dr d theta, this comes out to be theta equals to 0 to pi So, this will become r square r cube upon 3.

So, it will become one by 3 or 4 sin theta raise to power 4 upon 3 minus 4 sin theta raise to power 4 upon 4 and then d theta once you solve it comes out to be 8 pi. So, in this way we need to evaluate surface integral, but by using the divergence theorem of Gauss what we have done we have evaluated the surface integral in terms of a triple integral.

However, please be careful we can apply this theorem only when my surface is a closed surface, you cannot apply it on the open surface. For example, if we have to apply it on a cylinder let us say x square equals to y square plus z square then what you need to do you should have the upper circle which is closing the cylinder as well as a circle circular disc in the bottom to make the cylinder as a close surface.

If you want to evaluate it on hemisphere where z is positive you have to take a circular disc to close the hemisphere. So, always this theorem is applicable on close surfaces if the surface is open somehow make it close calculate the integral and then finally, subtract the integral over that particular portion of the surface which you have used for closing the surface ok.

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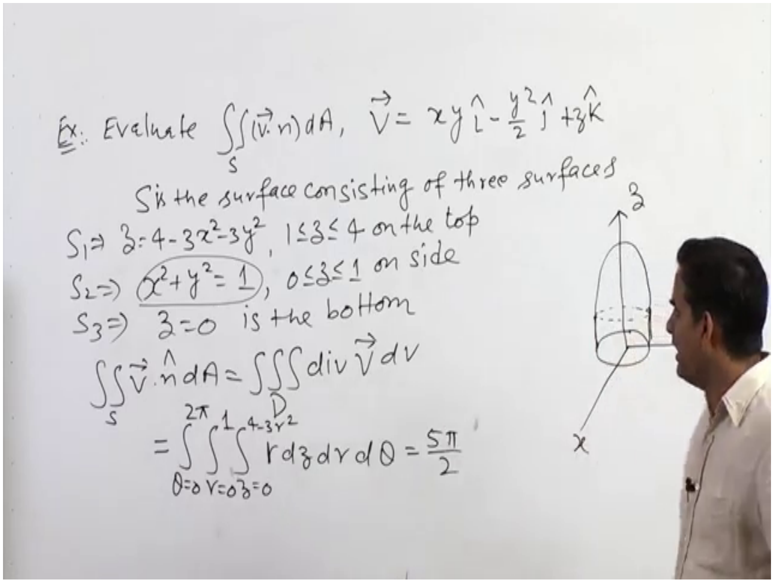

Let us take another example and this example is used the divergence theorem to evaluate again I need to calculate. The surface integral of a vector function where vector function v or let me write in where the vector function V is given as xy j minus y square upon 2. So, this is i in the i direction, this is in the j direction and plus k plot k. In fact, I will take z k ok.

So, this is my vector function and now where surface S is also need to be defined. So, S is the surface consisting of three surfaces that is z equals to 4 minus 3x square minus 3y square and here z is 1 to z to 4 on the top, then I am having a cylinder x square plus y square equals to 1 for 0 to z to 1 on side and finally, it should be closed from the bottom then only we can apply the divergence theorem. So, z equals to 0 is the bottom.

So, I can say this is my S 1 S 2 S 3. So, this is like this if I am having; So, x, y and z. So, what I am having? I am having a z equals to 0 in the bottom that is the xy plane I am having a cylinder along z axis which is going from z equals to 0 to 1. So, let us say up to here and then I am having a this kind of thing it is a close surface since it is consisting three surfaces and if I need to evaluate the surface integral what I need to do I need to evaluate this surface integral over the whole surfaces by solving three surface integrals one is on S 1 another one is on S 2 and the last one on S 3 and by adding all of them I will get the final answer.

But, what is the easy way instead of doing this since it is a close surface I can apply the divergence theorem here. So, let us try to apply divergence theorem. So, here by the divergence theorem the surface integral of V over the surface S will be the triple integral over the region D divergence V dV ok. So, here region D is the closed region. So, let me take it what will be the limit here divergence V if I will calculate it will become y minus y plus 1. So, it becomes 1 ok.

Now, the limits will become z equals to 0 is the lower limit of z because this is the disc in x y plane what is the upper surface of the region. So, this will be if I use the cylindrical coordinate it will I can write 4 minus 3 r square where r is r square is x square plus y square x is r cos theta, y is r sin theta then the projection of this particular surface on the x y plane. So, what will be projection it will be a circle of radius. So, it will be a circle of radius 1, see the cylinder. So, this is the side surface. So, this will project as a circle of radius 1.

So, r will be 0 to 1 and theta will be 0 to 2 pi and then divergence V is 1. So, dV will become r dz dr d theta, and when I will evaluate it comes out to be 5 pi upon 2. So, in this way we can solve this integral this surface integral by the divergence theorem just by solving this triple integral over this region D. Let us take one more example where we need to verify the divergence theorem.

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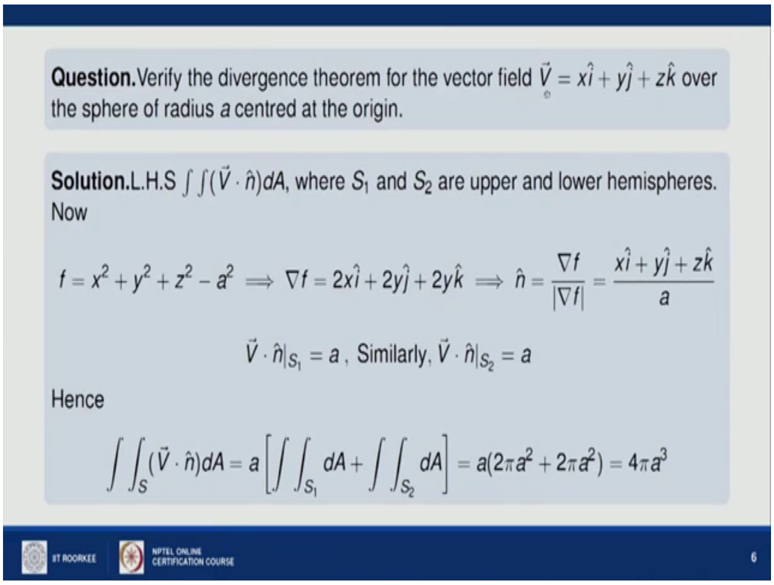

So, verify the divergence theorem for the vector field V equals to x i plus y j plus z k over the sphere radius a centre at the origin ok.

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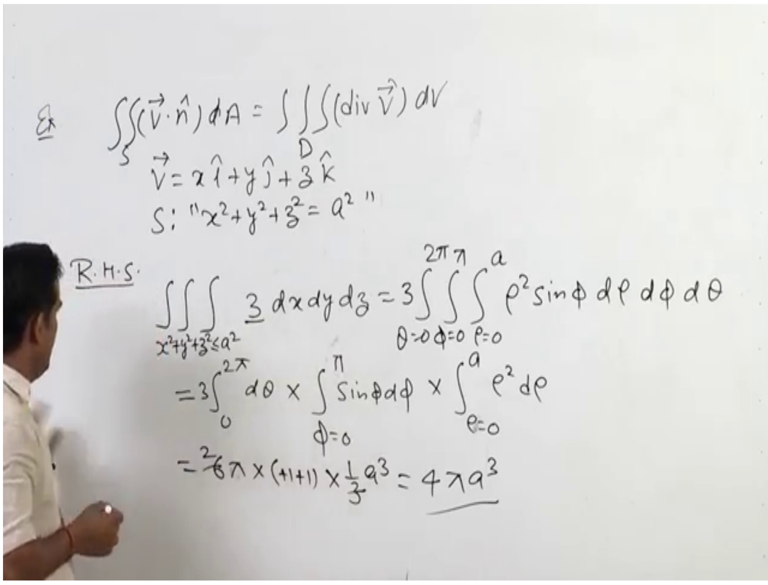

So, my divergence theorem is V dot n dA and this equals to over the surface S equals to triple integral D divergence of V dV. Here V is given as x i plus y j plus z k and surface S is the boundary of the sphere x square plus y square plus z square equals to radius x centre at the origin so, this one. So, S is a represented by this surface.

Now, let us take right hand side first and the right hand side I need to calculate triple integral or a spherical region that is I need to calculate this x square plus y square plus z square less then equals to a square and then divergence of V. So, divergence of V will become del x over del x plus del y over del y plus del y del z of z. So, it will become become 3 and then dx dy dz. So, better to convert from the multiple integral we know whenever we are having we have to evaluate some multiple integral over a spherical region better to change it in spherical coordinates. So, let me do it.

So, for a given is sphere rho will go from 0 to a that is the radius of the sphere then phi will go 0 to pi and theta will go 0 to 2 pi and dx dy dz will be. So, 3 I have taken out that I got from the divergence of V and then dx dy dz will become rho square sin phi d rho d phi d theta all the limits are constant.

So, I can write 3 0 to 2 pi d theta into phi 0 to pi sin phi d phi into rho 0 to a rho square d rho. This equals to 6 pi into sin phi will become cos phi and here when I will put pi cos pi is minus 1 cos 0 is 1; So, minus 1 minus 1 at. In fact, sin phi will become minus cos phi; So, minus minus plus; So, 1 plus 1. So, I will get 2 form here and this is 1 by 3 into rho cube rho cube means a cube. So, 3 2, so, 2 into 2 to 4. So, I got 4 pi a cube.

So, this is the value of right hand side now, I will calculate left hand side and if left hand side equals to right hand side then the theorem is verified.

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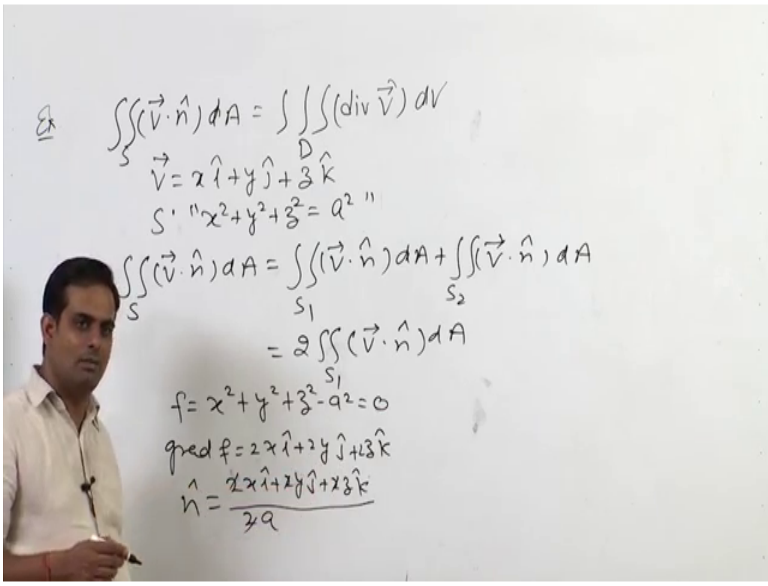

So, I need to calculate the surface integral over S that is V dot n D a here I am having a complete sphere. So, I can divide this complete is sphere into two surfaces one is hemisphere in the upper a half means above xy plane another one the hemisphere below xy plane; So, two hemispheres. So, let us say first is S 1 V dot n dA plus S 2 V dot n dA since both the surface is are same. So, I can write two times S 1 V dot n dA. So, now, I need to calculate this vector surface integral of this vector over S 1, where S 1 is the hemisphere in the over above the x y plane.

So, now the equation of hemisphere is given by x square plus y square plus z square minus a square equals to 0. So, from here my gradient f will come 2 xi plus 2 yj plus 2 zk normal vector to this hemisphere will be grade f over magnitude of grade f. So, it will become 2 xi plus 2 yj plus 2 zk upon square root of 4 x square plus 4 y square plus 4 z square. So, 4 I will take common.

So, it will become 2 and n square root of x square plus y square plus z square; x square plus y square plus z square is a square. So, it will become twice of a. So, 2 will be cancelled out. So, basically normal vector is x upon a in i direction, y upon a z direction and z upon a in k direction.

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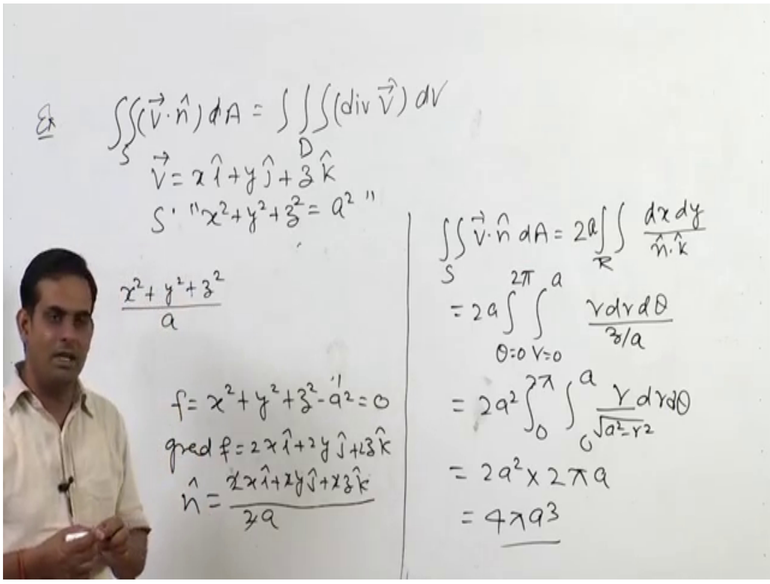

Now, we need to calculate S V dot n dA. So, this is twice of now we will calculate V dot n. So, my n is given by this, V is given by this. So, V dot n will become x square plus y square plus z square upon a this is a square. So, a square upon a this comes out to be a.

So, I will take this a out, and then I need to write this dA. So, let me project the surface on x y plane. So, when I will project it in x y plane it will become dx dy upon n dot k this equals to twice a in x y plane I am having the region r. So, r will be a circle of radius a. So, r will go from 0 to a theta will go from 0 at 2 pi dx dy will become r dr d theta and n dot k will become z upon a; So, z upon a.

So, twice a square this a I will take here theta 0 to pi 2 pi r 0 to a r d r d theta upon z please for the upper hemisphere z will become square root a square minus x square minus y square. So, basically this is a square minus r square. So, twice a square and then this will comes out to be 2 pi a. So, this comes 4 pi a cube which is the same as we are having the right hand side. Hence, left hand side equals to right hand side and theorem is verified.

So, in this lecture we learn about divergence theorem of the Gauss and then we have taken few examples based on this theorem. With this I will close this lecture. In the next lecture I will discuss another important theorem that is the Stokes' theorem.

Thank you very much.