Multivariable Calculus
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Lecture - 38
Surface Integral

Hello friends. So, welcome to the 38 lecture of this course and in this lecture I will introduce surface integral. In previous unit, we have seen that first I have introduced (Refer Time: 00:34) and then I define the concept of line integral in the same way, I will generalized the concept of line integral to the into the surface integral. So, the surface integral can be seen as a generalization of the line integral.

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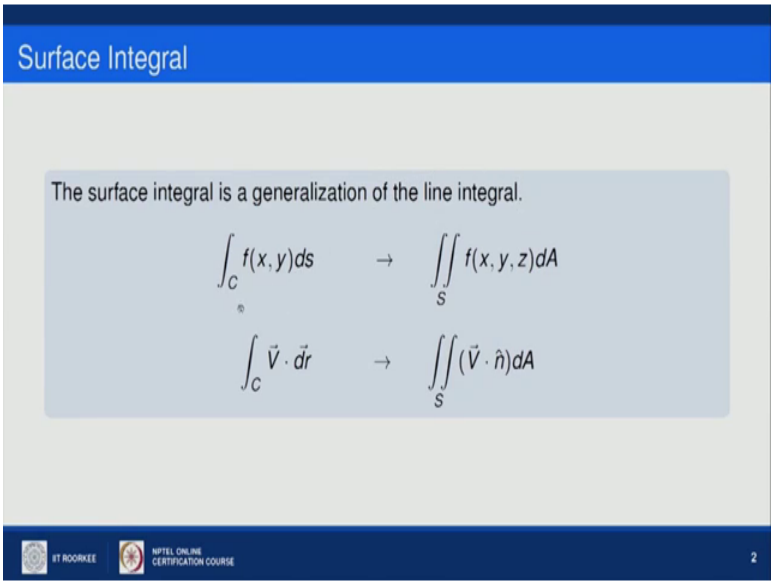

For example the line integral with respect to (Refer Time: 00:56), its opaque curves along a curve C is given by this formula. So, this can be generalized in this way, and this I will say the surface integral of the scalar field or scalar functions over a surface S. Similarly this is the line integral of the vector functions along a curve C having the parametric representation r. So, it is given by integral over C V dot d r.

So, this I am going to generalized as the surface in integral; that is V dot n into dA over the surface s. So, here n is the unit normal vector oriented outward to the surface s. So, now, let us define the surface integral of a scalar function. So, let g of x y z be a given function defining 3 D space, and let S be a surface which is given by the graph either z is equals to f of x y of y equals to h of x z or x equals to h of y z.

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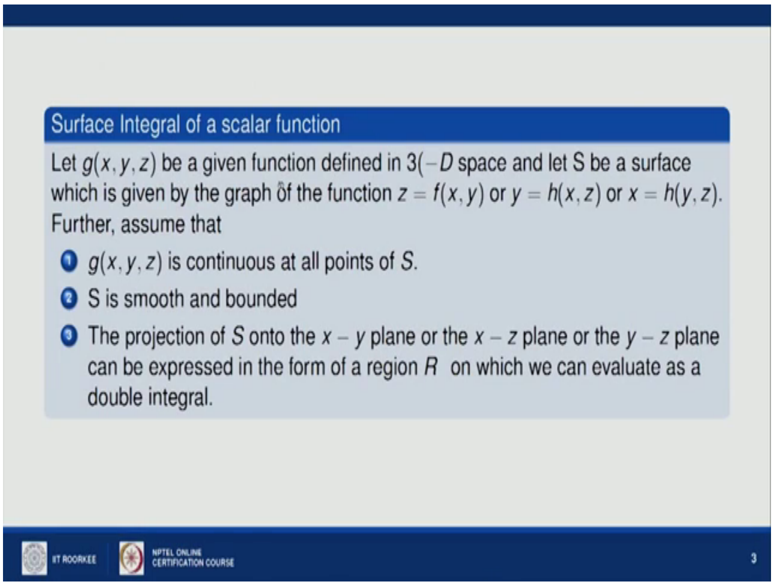

Further assume that g of x y z is continuous at all points of S, S is smooth and bounded the projection of S on to the x y plane or x z plane or y z plane, can be expressed in the form of region R. Region R means on which we can evaluate a double integral, means in 1 variable. The limits would be the variable function of other and for the other variable the limits should be constant if all these conditions are satisfied.

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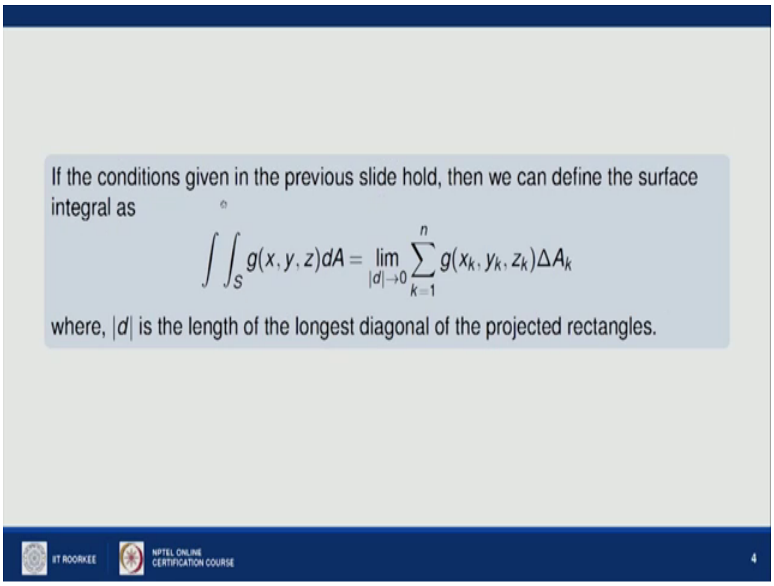

Then the surface integral is defined as the integral over the surface S; that is the double integral g of x y z dA and this equals to limit d tending to 0 summation from k equals to 1 to n g of x k y k z k delta A k here. This d and the length of the is the longest diagonal of the projected rectangles, because we will project the surface either on x y plane y z plane or z x plane or u v plane and surface is divided into small patched or small elementary surface s.

We learn how to do it in the previous lecture. So, those are projected and d is the length of the longest diagonal among the projected rectangles. We will generalize this formula and the first one is. If the surface is given in parametric form; that is that is R of u v, then surface integral of g over S and S is the surface is given as g x y z and then dA.

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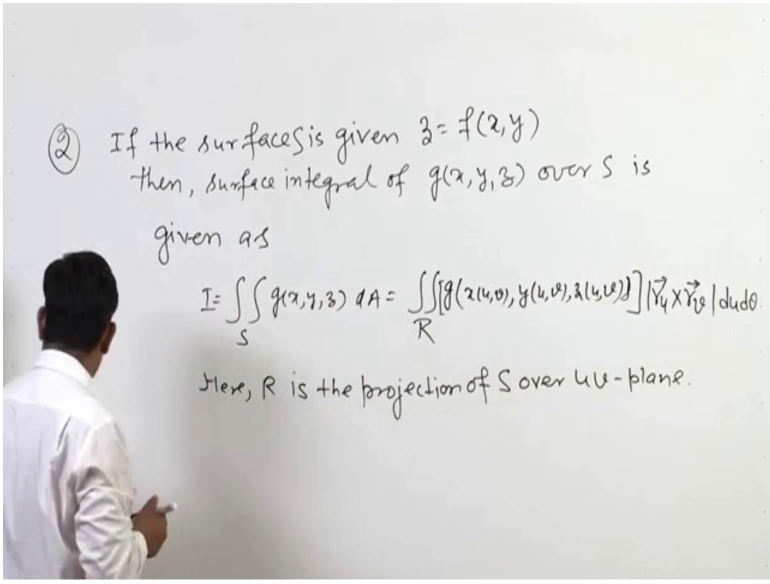

So, here I will project the surface onto u v plane. So, the projection will become r, then g x of u v y of u v z of u v. So, you can write this g in terms of u and v. Now, dA will become that is the surface area. So, r u cross r v and magnitude of this into d u d v. So, when the surface is given in this form; that is in the parametric form.

I will use this formula for calculating the surface integral of g over the surface S and here R is. So, R is the projection of surface S over u v plane. Now, my second formula, I just edit it. So, if the surface S is given as z equals to f of x y ok, then surface integral of the function g of x y z over the surface S is given as.

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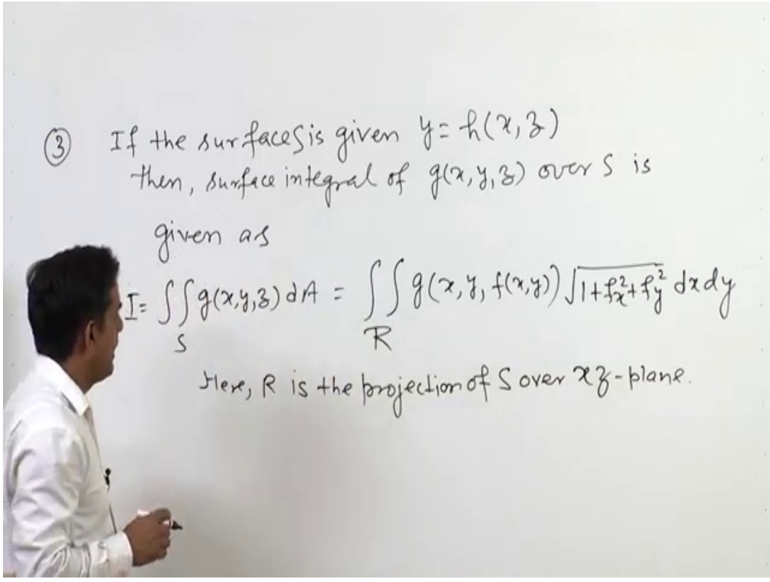

So, I will use the formula like this. So, I need to calculate the g of x y z dA over the surface S. So, if the equation of the surface is given in this form z equals to f of x y, then what I will do? I will take the projection of S over x y plane. So, here I will evaluate this double integral over a region R, where R is a projection of S over x y plane and then I can write g x y and z is again f of x y.

So, the g is completely written as a function of x and y only then dA and from the formula of surface integral oh. Sorry now from the formula of surface area which we have taken in the previous lecture, we know that dA can be written in this case. So, in that way I will use this formula, if the surface S is given in this form

Next formula is, if the surface S is given as y equals to some function h of x and z, then surface integral of the function g of x y z over the surface S is given as g x y z dA. And now from the previous lecture, you know that if the surface is given in this form ok, then I will project the surface S over y z plane. So, here it will become y z plane and then R is the projection of S in this plane, sorry not y z. So, then area R is the projection of S over x z plane.

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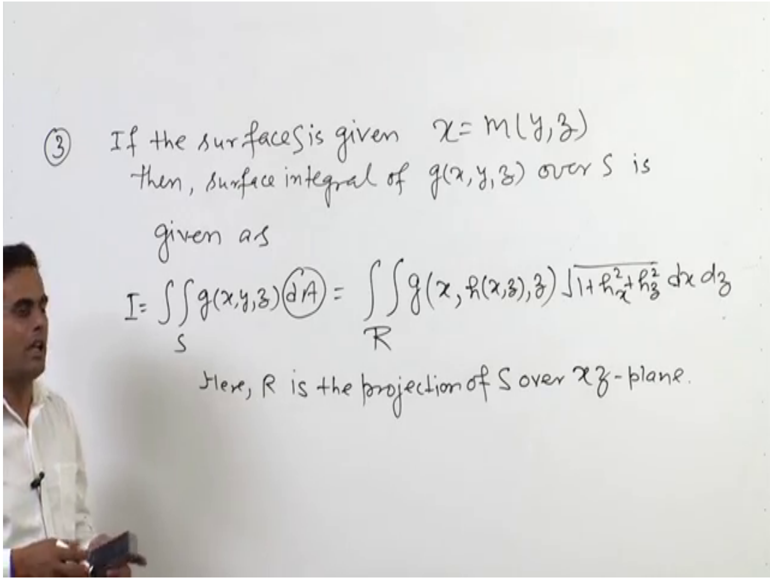

And now the final form of this formula will be g x y can be written in terms of x and z. So, h of x z z is square root 1 plus h x square plus h z square d x d z ok, and the last, if the surface S is given in the form x equals to some m of y z.

So, we are doing the same as we have done in the previous lecture. Only thing we are changing the rule of calculating this dA over the surface S, means calculating the surface area of S.

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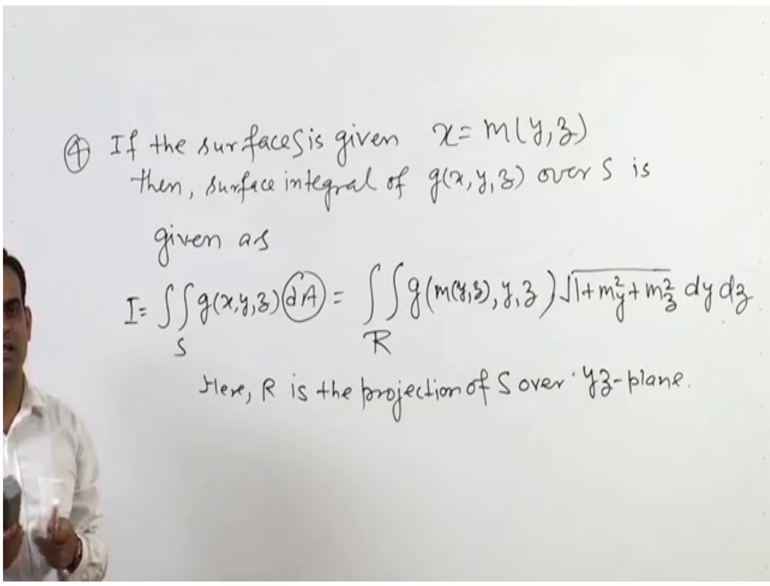

So, in this case, this can be written as. So, g will become a function of x y, sorry y z, because x equals to m of y z y z and then square root of 1 plus m y square plus m z square d y d z, and here the region R is the project of S over y z plane. So, this is formula number 4. So, with the help of these 4 formulas we can evaluate the surface integral of scalar functions.

Moreover, here in a definition we have taken S as the smooth surface, it as it piece wise is smooth surface, then like the case of plane integral we can define. So, if S is piecewise smooth having is smooth surfaces S 1 S 2 S k,

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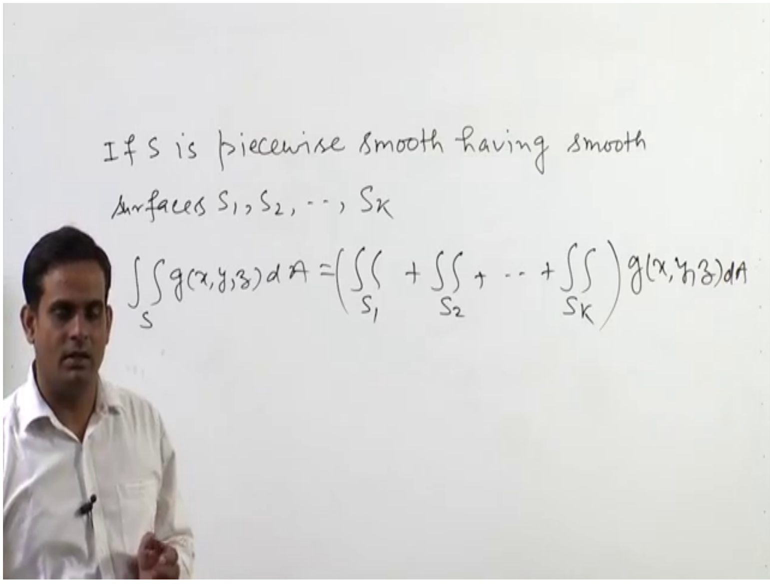

Then the surface integral of a function g over the bigger surface S which is piecewise smooth equals to. So, integral over the each is smooth portion of the surface separately. For example, if you are having a closed cylinder, then that closed cylinder will be having three surfaces; top, bottom and the side surfaces.

So, for these three surfaces we will evaluate the surface integral separately. Now, let us take an example.

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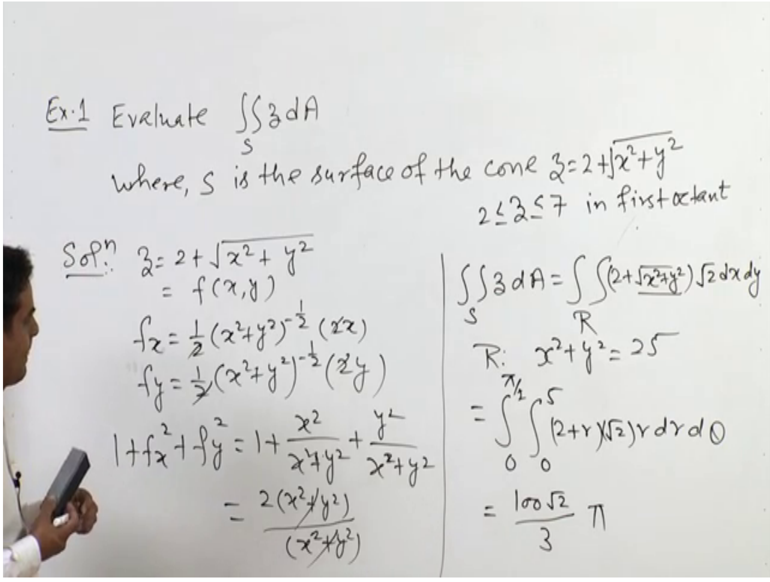

So, example 1 evaluate, let us say z of dA. So, g of x y z is only z, where S is the surface of the cone z equals to 2 plus x square plus y square and z is from 2 to 7 in first octant. So, let us try to solve it. So, here surface is, surface will be a cone; that is z equals to 2 plus square root x square plus y square.

So, it will be like this, it is z x is x and y. So, this is 0 0 2 and this will be this point 0 0 7 and this length means length of the circle. This particular circle, the radius of the circle will be how much it will be root x square plus y square equals to z minus 2. So, z is having maximum 7; So, 7 minus 2 5. So, x square plus y square equals to 25.

So, if I project this particular surface onto x y plane what I will get? I will get a circle of radius 5 ok. So, now, my region R is given by this circle, circle of radius 5, having centre at the origin in x y plane. So, the surface is given as z equals to 2 plus square root x square plus y square. So, this is my f of x y, let us say ok. Then what I will do? I need to calculate z and then dA.

So, first of all I need to calculate f x ok. So, f x is to 0. So, 1 by 2 x square plus y square is to power minus half into 2 x. Similarly I will get f y. So, 1 by 2 x square plus y square raise to power minus half into 2 y ok. So, this is 2 will be cancel out. So, 1 plus f x square plus f y square will become 1 plus x square upon x square plus y square, and then plus y square upon x square plus y square. So, this comes out to be twice of x square plus y square upon x square plus y square. So, this cancel out. So, 1 plus f x square plus f y square equals to 2.

Now, the surface integral over S z a becomes R, R is a circle and z is 2 plus square root x square plus y square, and then what I will be having square root of 1 plus f x square plus f y square. So, square root of 2 will become root 2 d x d y ok, and the area R is a circle x square plus y square equals to 25, a circle of radius 5. So, changing into polar coordinates R is 0 to 5 theta is 0 and one more thing, it is here in first octant.

So, first octant means theta will move only from 0 to pi by 2; so, 2 plus r. So, this is my R root 2 r d r d theta and this integral we can solve quite easily, it becomes 100 root 2 upon 3 into pi ok. So, this way we can calculate this particular integral. So, for we have defined the surface integral for scalar function; now, we will do it for vector function.

So, the surface integral of vector function can be defined in this way, let us be a smooth orientable surface. Also let the vector V, which is having component v 1 v 2 v 3.

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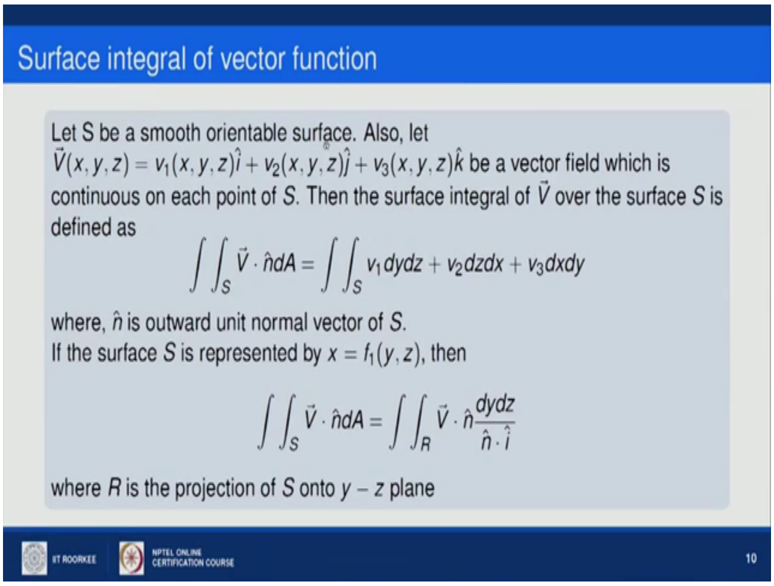

Those are functions of x y and z be a vector function which is continuous on each point of S, then the surface integral of V over the surface is, it defined as. So, this double integral over S V dot n dA and this can be written as double integral over S v 1 d y d z, v 2 d z d x plus v 3 d x d y, here in this definition, by unit normal vector n is outward unit normal vector of S.

So, now if the surface S is represented by x equals to f 1 y z, then this will become this surface integral of V will become in this way. So, double integral over the region R V dot n and dA can be written as d y d z over n dot i, where R is the projection of S onto y z plane ok. Similarly if the surface S is represented by z equals to f 2 of x y, then the surface integral of vector V over the surface S is given by this formula.

So, double integral over the region R, where R is the projection of S in x y plane and then dA can be written as d x d y upon n dot k. Finally, if the surface is given in this form y equals to f 3 of x z, then what we will do? We will project the surface S on to x z plane and projection of on x z plane.

Let us be R, region R, then the surface integral equals to double integral over the region R V dot n d x d z over n dot j and n j are unit vectors. So, let us take an example of the surface integral of vector function.

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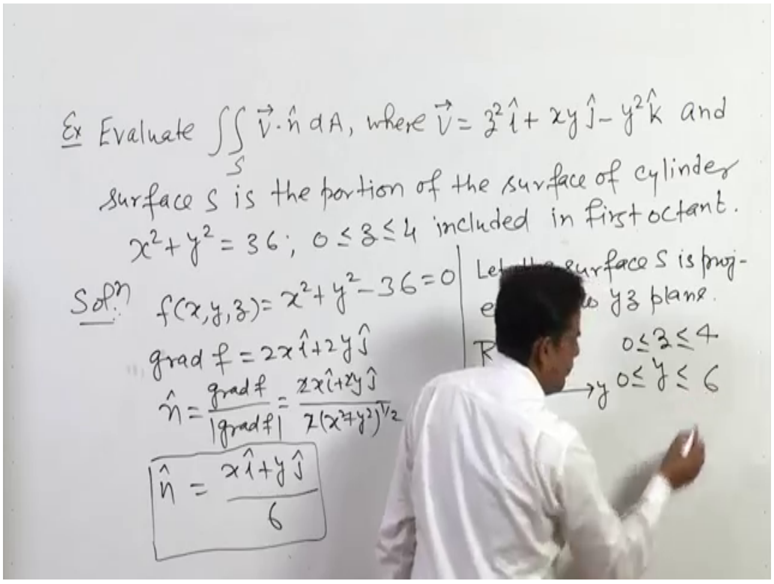

So, example is evaluate V dot n dA, where V is given as z square i plus x y j minus y square k and surface S is the portion of the surface of cylinder x square plus y square equals to 36 0 to z to 4 included in first octant ok. Here surface is given by this. So, I can write it z equals to, i can take the surface as f of x y z equals to x square plus y square minus 36 and this equals to 0.

So, now I need to find out a normal to the surface. So, grade f will be 2 x i plus 2 y j and then the unit normal vector n to the surface f is given by grade f upon magnitude of grade f. So, this will become 2 x i plus 2 y j, and magnitude of this will become 4 x square plus 4 y square. So, for i can take 2 out 2 into x square plus y square raise to power 1 by 2. So, 2 will be cancel out, it is x i plus y j upon square root of x square plus y square, you know that x square plus y square is 36. So, square of will become 6. So, this is the unit normal vector to the surface.

Now, I need to evaluate V dot n dA. Now, the question is, how to write dA, because there are three ways for writing dA and that depend, those three ways depend where you are going to project your surface S on x y plane y z plane or z x plane. Now if I project here on x y plane what will happen. So, my dA will be d x d y upon n dot k, and when I will evaluate n dot k n is given by this. So, n dot they will become 0. So, I will get a 0 in the denominator and hence I cannot project the surface onto x y plane.

So, what are the options either project it on y z plane or x z plane. So, let us project it on y z plane. Let the surface S is projected onto y z plane. So, then what will be my region R in y z plane, it is a cylinder x square plus y square minus 36 z is going from 0 to 4 ok. So, y will go from 0 to 6. So, when you will project a cylinder onto y z plane, where the cylinder is along z axis, it will become a rectangle ok.

And so in x y y z plane; So, y z it will become a rectangle and rectangle will be given as z is 0 to 4 and y is 0 to 6, because I am talking about first octane. So, that is why I am taking y from 0 to 6. Now,. So, I am having the region R I am having normal n. So, what I need? I need to evaluate the surface integral.

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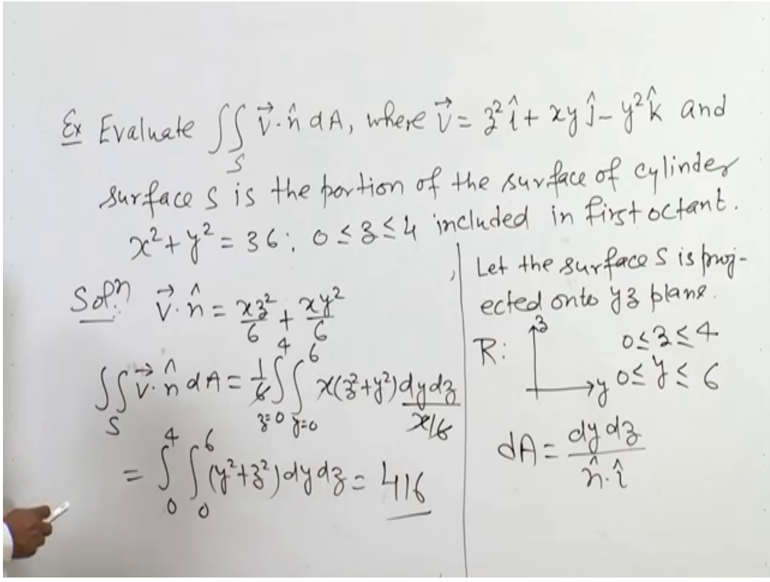

So, V dot n will become x z square upon 6 i, because it is dot product. So, I will not come square upon 6 plus x y square upon 6 and z component is 0 open. So, now, the integral V dot n dA over the surface S will be 1 upon 6 region is R. Now, so, 0 to 4 is z, y is 0 to 6 and then x z square plus upon 6. So, this I have taken out x z square plus y square d y d z.

So, only thing I need to and then d y d z upon n dot i, because you are projecting on y z plane. So, dA will become d y d z upon n dot i and what is n? dot i n dot i is x upon 6 i component of the normal vector that n. So, this x are cancel, the 6 is cancel. So, I simply got 0 to 4, y is 0 to 6 y square plus z square d y d z and this comes out to be 416 ok.

So, in this way we can evaluate the surface integral of vector functions. So, in this lecture what we have seen? We have seen the definition of surface integral over a surface S. Then we have seen some formula for finding the surface integral of the scalar functions. Then we have seen formula for finding the surface integral of vector functions, we have taken one more example for each case ok.

In the next lecture we will do the divergence theorem of the gauss; that is related to the surface integral of a vector function and the triple integral from the multiple integral section. So, with this I will end this lecture.

Thank you very much.