Multivariable Calculus
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Lecture – 37
Surface Area

Hello friends. So, welcome to the 36th lecture of this course. And in this lecture I will talk about Surface Area. Means, how to find out area of the surfaces?

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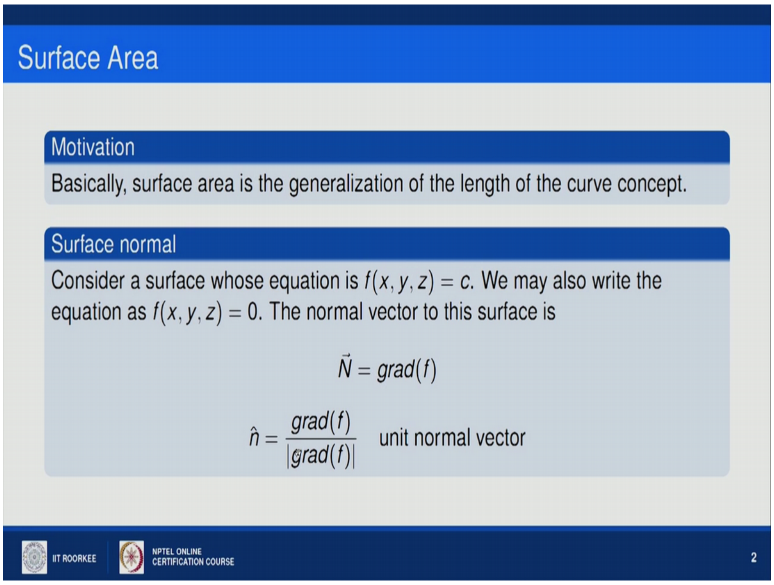

So, basically surface area is the generalization of the length of the curve concept. What we have done for finding the length of the curve? We have divided that curve into small -small arcs in such a way that the length of the arc can be approximated by the length of the tangent. And then we add all those lengths, sum of all those lengths give the length of the curve. In the similar manner, what we will do? The when we need to find out the surface area, we will divide the whole surface into elementary surfaces; in such a way that, the area of each elementary surface can be approximated by the area of its tangent.

So, we will take a point, in each elementary surface we will find out the tangent plane at that point. And then the area of each elementary surface will be approximated by the area of the respective tangent plane. So, before that let us see surface normal. So, we know that if f of x y z equals to c be the equation of a surface. Then the normal vector to this surface is given by gradient of f or unit normal vector will be given by gradient of f upon magnitude of gradient of f.

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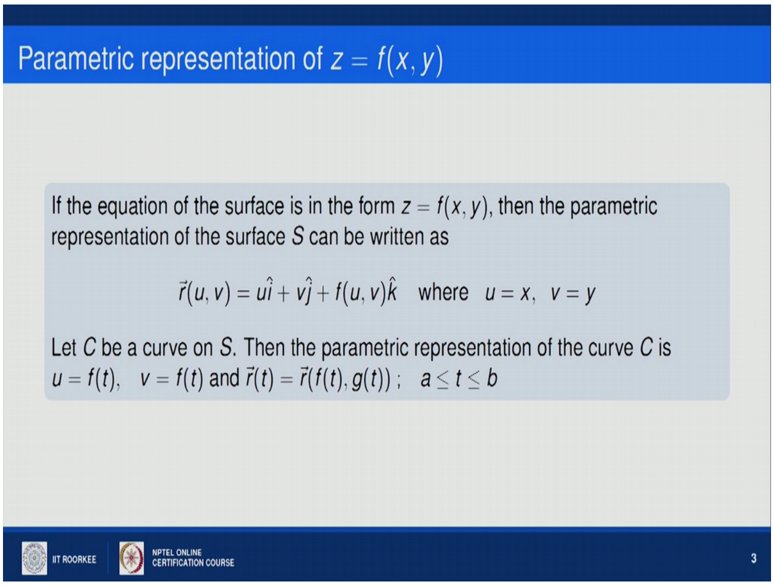

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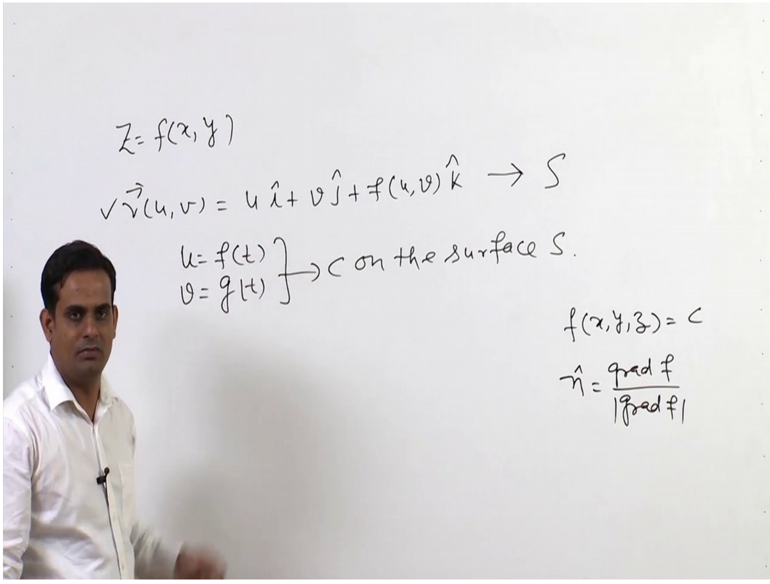

If the equation of the surface is in the form z equals to f of x, y. So, it is something z equals to f of x, y. Then in parametric form I can write it; r u, v equals to u i plus v j plus f of u, v k. Means, what I have done? I have taken u equals to x and v equals to y.

Now, this is the parametric representation of the surface S. Now, consider a curve c on the surface S. Then, let us say the parametric representation of curve is given by u equals to f t and some v equals to g t and the curve is c on the surface S.

Then, what will happen? We will be having parametric representation of the curve is r t equals to r of f t and g t; means f t i plus g t j; and where t between a to b.

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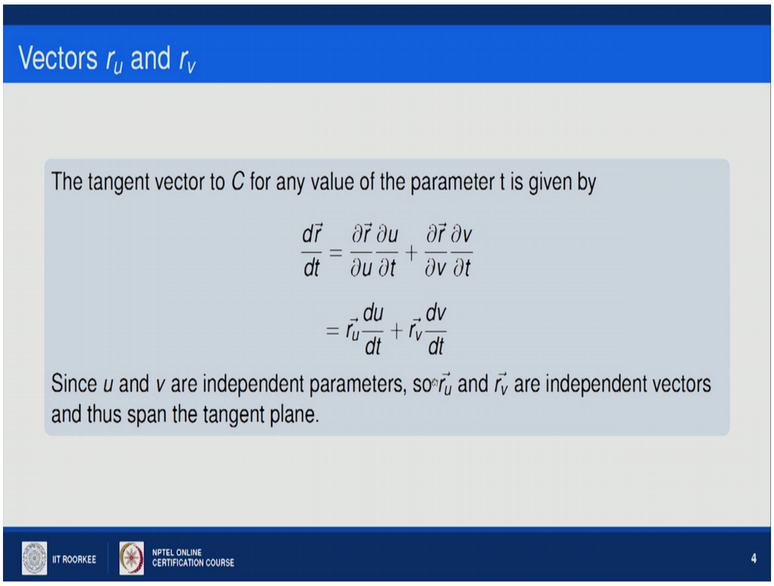

Now, the tangent vector to curve C is given by d r by d t. And by the chain rule, it is it can be calculated since r is a function of u and v and u and v are the functions of t. So, I can calculate, d r over d t as del r over del u into del u over del t plus del r over del v into del v over del t. So, these I can take as r u this I can take as r v. So, I can write this r u d u over d t plus r v d v over d t. Here, u and v are independent parameters. So, r u and r v are independent vector and any vector in the tangent plane, to the surface S at a given point p, can we written as the linear combination of r u and r v. Hence these are the two vectors in the tangent plane.

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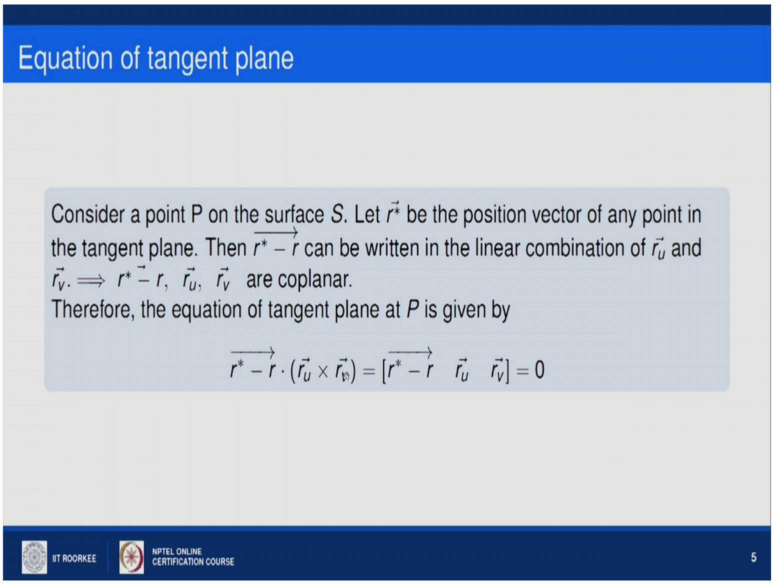

The equation of the tangent plane can be given by this particular equation. So, consider a point P on the surface S. Let r star be the position vector of any point in the tangent plane. Then r star minus r can be written in the linear combination of r u and r v. It means r star minus r is also a vector in the tangent plane and r star minus r, r, u and r, v, are coplanar. Therefore, the equation of tangent plane at P is given by r star minus r dot product with r u cross r v equals to 0.

Now, we know that this is the tangent plane and any vector which is normal to this tangent plane is called normal vector. And so far, we know this representation of normal vector that if the surface is given by f of x, y, z equals to C; then union normal vector is gradient of f upon its magnitude. Now, if the surface is given in this form; means in the parametric representation, then how to find out the normal vector in parametric form.

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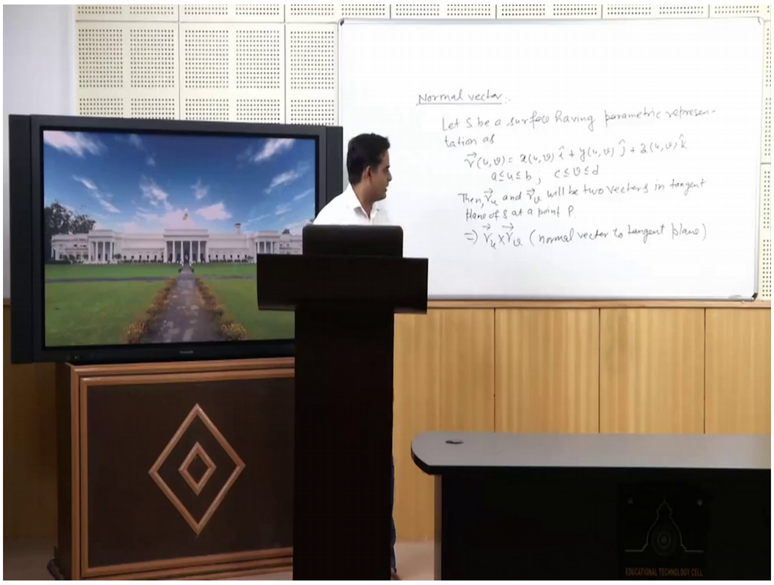

So, this is another definition of normal vector to surface ok. So, let S be a surface having parametric representation as r u, v equals to x of u, v in i direction; y of u v in j direction and z of u, v in k direction. And we are having some bound on u and some bound on v. Then, we just now we have seen that r u and r v will be two vectors in tangent plane. So, if these are the two vectors in the tangent plane, then the vector; the cross product of r u cross r v will be the vector perpendicular to the tangent plane. And hence this vector will be the normal vector to tangent plane.

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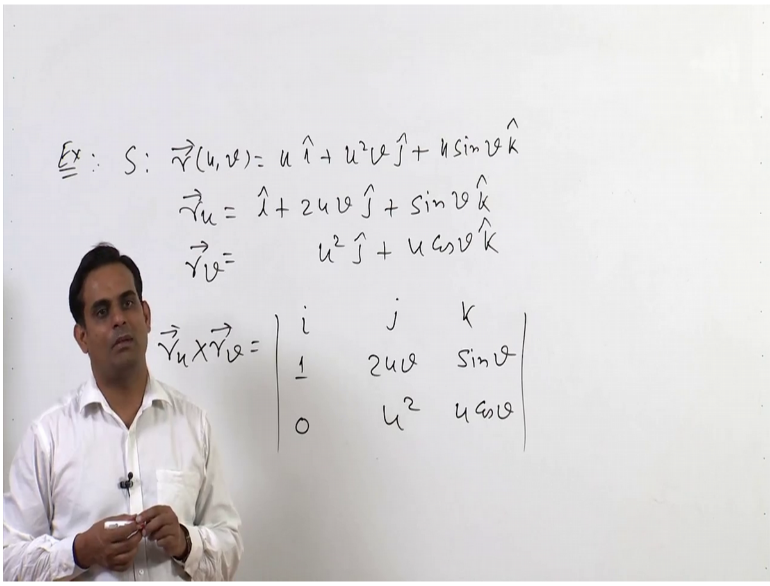

So, for example, if I am having a surface let us say, r of u, v equals to, so, let us say S is given by r of u, v equals to some u i plus u square v j plus u sin v k ok and I need to find out the normal vector to this at some given u and v. So, what I will do? I will calculate r u; that is a vector in a tangent plane. So, r u will become i plus 2 u v j plus sin v k.

Similarly, I will find out r v. So, r v will become; u square j plus u cos v k. Now, the normal vector r u cross r v given as; i, j, k that is the cross product of these vectors; so; 1, 2uv, sinv, 0, u square, ucosv. And by calculating this, I can find out the normal vector in parametric form ok. So, now we are having the parametric representation of surface, the equation of tangent plane and the equation of normal vector in parametric form.

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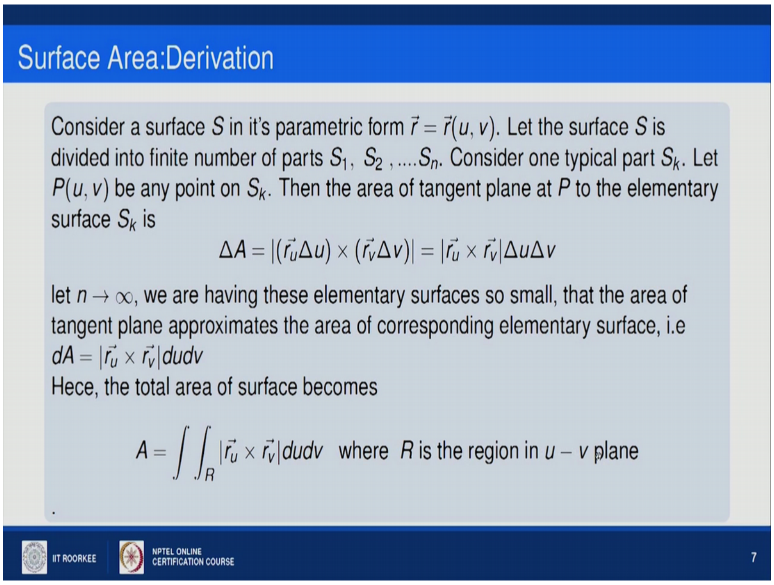

Now, let us define the surface area with all these. So, consider a surface S, in its parametric form r equals to r u, v. Let the surface S is divided into finite number of parts S 1, S 2, S n. Consider one typical part S k. Let P u, v be any point on this elementary surface, elementary part S k. Then the area of tangent plane at P to the elementary surface S k is given as del A. And del A will become r u del u cross r v del v. That is magnitude of r u cross r v into del u del v. Because, each side of the parallelogram that is the tangent plane to the elementary surface S k will be r u del u; other side will be r v del v. Because any vector on that particular plane, we can write in the linear combination of r u and r v. And hence side will become r u del u and r v del v. So, del A will become here product and it will become the cross product of vector r u and r v and magnitude of this new vector into del u, del v.

Let n tends to infinity; means we are having a very large number of elementary surface. And why we are having it? Because I want to approximate the area of each elementary surface by the area of its tangent plane. For that, the surface would be very small ok. Then, I can write when in limiting case, when n is tending to infinity, dA this del A will become d A and it will be the magnitude of r u cross r v; that is the normal vector to the tangent plane dudv. And now, the total area of the surface will become the sum of the areas of far tangent planes. That will be the total area of the surface. And I will say it is surface area.

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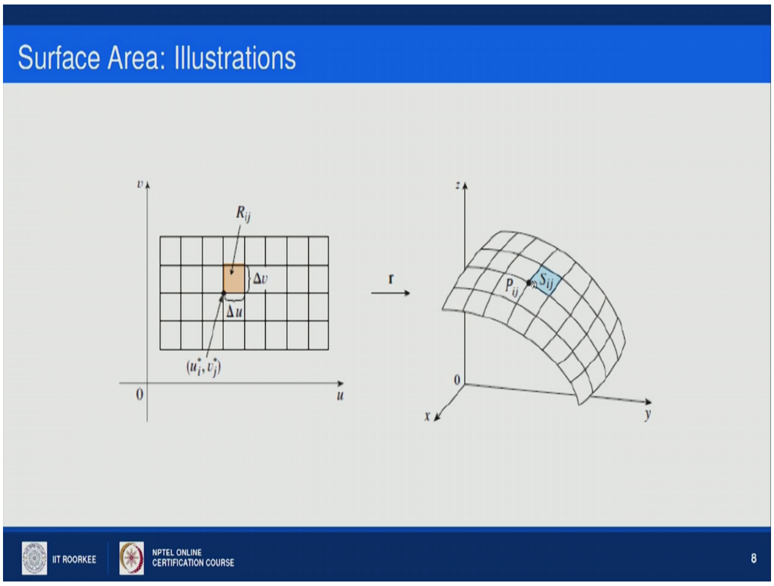

So, surface area is given as integral over r, r u cross r v; magnitude of it dudv; where r is the region in u v plane. So, basically it is like that, suppose I am having this surface. And this S i j is an elementary surface as I told you S k. I am taking a point P i j, at this particular point what I am having? My surface will be like this ok. And this will be the tangent plane at this particular point.

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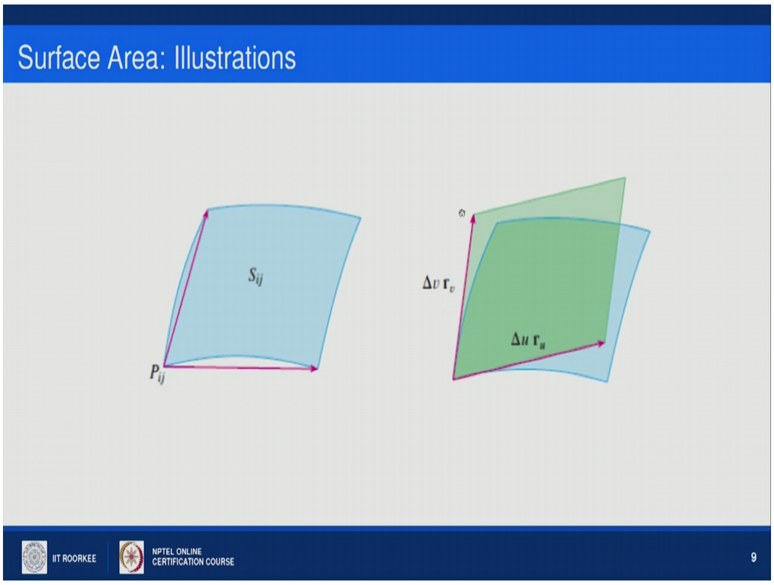

So, here I will be having del u r u and del v r v as the side of the parallelogram; that is the tangent plane and then its area. So, if surface is very small then the area of the surface can be approximated by the area of tangent plane. So, geometrically we can see here.

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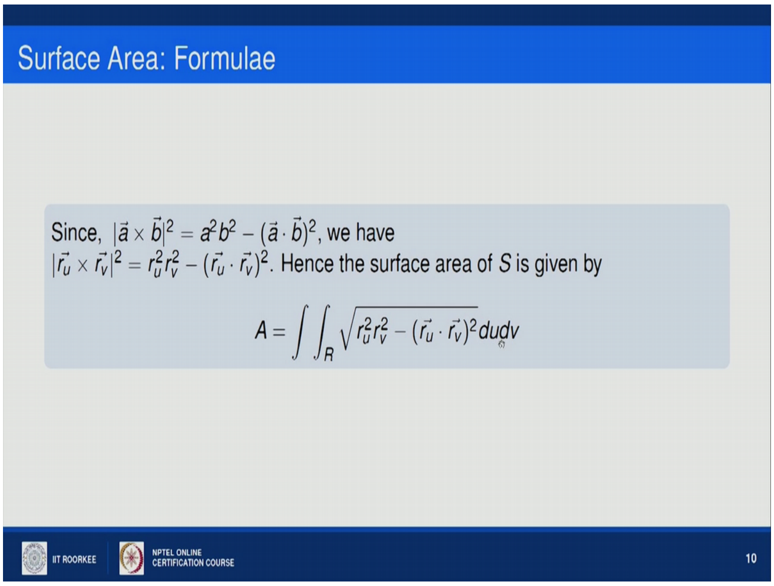

Also we know that, a cross b magnitude whole square equals to a square b square minus a dot b whole square; where a and b are the magnitude of vector a and vector b. So, in this way, we can have this as in this way. So, I can write another formula for surface area as integral over R; that is the region in u v plane; its square root r u square; r v square minus the dot product of vector r u and r v and square of this du dv.

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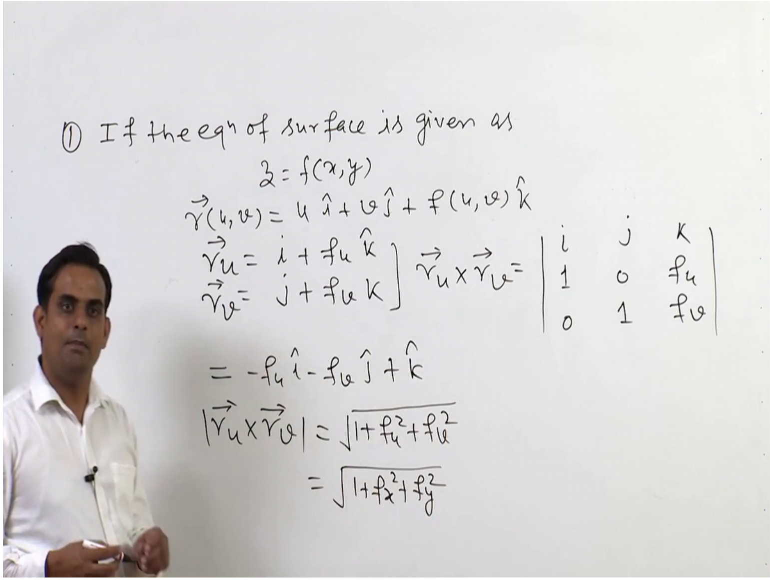

We have seen the area, if we are going with parametric representation. But if we are going in other form let us say if the equation of the surface is z equals to f of x, y. Then the parametric representation will become r of u, v; r of u, v will be u i plus v j plus f of u, v k ok.

Now, r u again we can be calculated as i plus f of u k and r v will become j plus f of v k. So, from here r u cross r v; the cross product of r u and r v can be written as i, j, k, 1, 0, f u, 0, 1, f v. So, this comes out the, i component will become minus f u; the j component will become minus f v and the k component will be 1 ok.

So, now, magnitude of this normal vector to the surface is given as a square root of 1 plus f u square plus f v square. And hence and here please note that what I am taking? I am taking x equals to u and y equals to v because it is the representation f u, v i plus y u, v j. So, x is u y is v. So, this is also equals to 1 plus f x square plus f y square.

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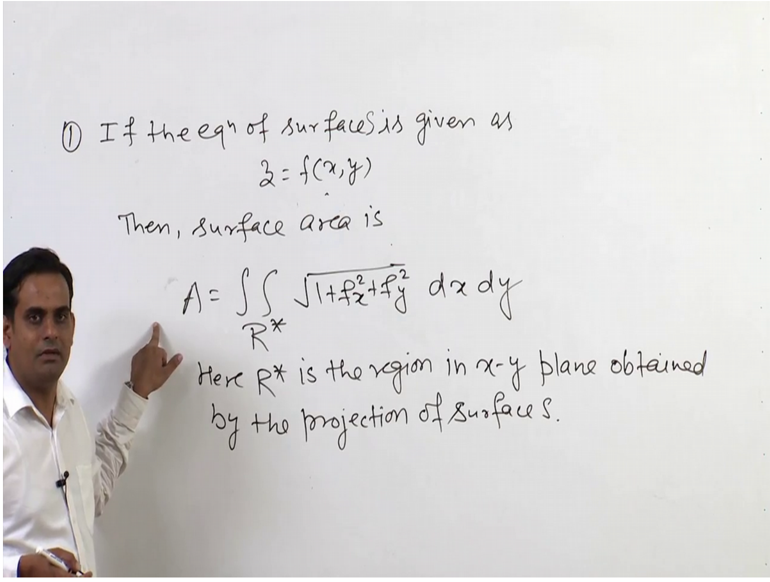

So, if the surface is given in this way, then surface area is A equals to integral over some region R star. I will tell you what is R star? Is square root 1 plus f x square plus f y square d x d y. Because du dv will become dx dy; u equals to x; y equals to v. Here, R star is the region in x-y plane obtained by the projection of the surface S. So, if the equation in this way, you just take the projection on x-y plane, calculate this quantity and solve this double integral. It will give the surface area of the surface S.

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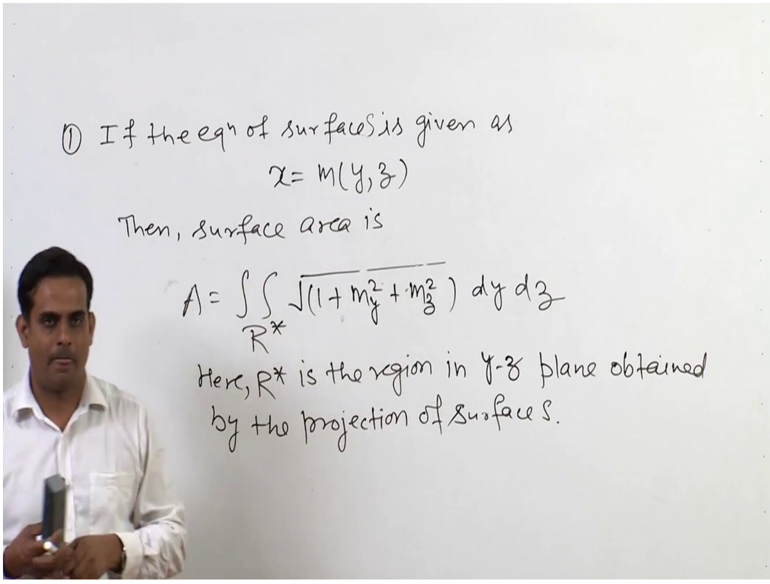

If I am having the equation as y equals to let us say some h of x, z ok. Then surface area is given as in the same way I can write 1 plus h x square plus h z square and then d x d z.

Here, R star is the region in x-z plane now, obtained by the projection of surface S on x, z ok.

If the equation of the surface is given let us say, x equals to some m of y, z. Then surface area A is given as; this square root 1 plus m x sorry m y square plus m z square and then dy dz. Here R star is the region in y-z plane; obtained by the projection of surface S. So, in that way, I can calculate the surface area of a given surface S based on it based on the equation of the surface.

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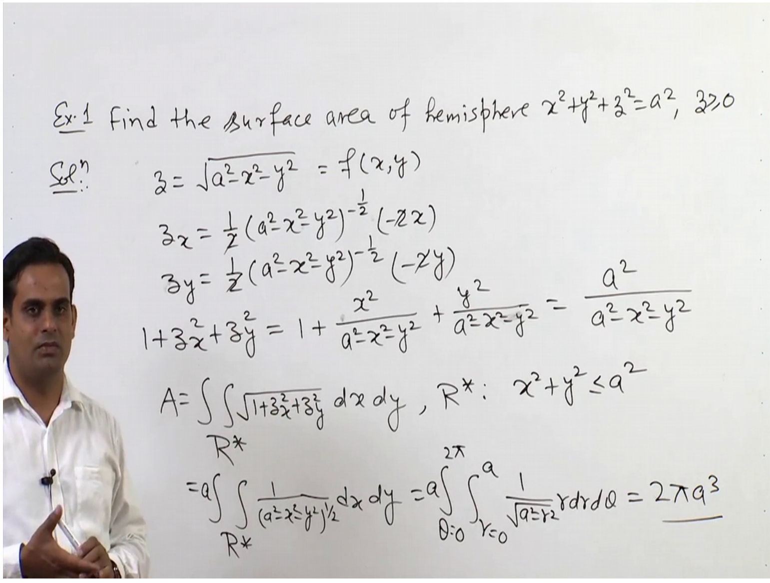

Let us take some example. So, example 1 is find the surface area of hemisphere x square plus y square plus z square equals to a square; z is greater than equals to 0. So, I have to find out the surface area of the upper hemisphere. Solution: So, here I can write this equation of the surface S which is an hemisphere this is square root a square minus x square minus y square from here. And why I am taking positive square root? Because, my z is greater than equals to 0.

So, for the upper hemisphere I need, I can take this equation of the surface. So, this is something z equals to f of x, y. So, now calculate del z over del x. So, it will become 1 by 2 a square minus x square minus y square raise to power minus half and then minus twice of x. I will be having z y. So, z y will be half a square minus x square minus y square raise to power minus half and then minus 2y. So, 1 plus z x square plus z y square please note that z x z y are f x and f y only. This will become 1 plus x square upon a square minus x square minus y square plus y square upon a square minus x square minus y square. And it will be a square upon a square minus x square minus y square ok.

So, now surface area A is given as integral over a region R star is square root of 1 plus z x square plus z y square dx dy. And please note that here, I am projecting the surface on x-y plane. So, when I will project this is sphere hemisphere on x-y plane; it will become a circle of radius a center at 00.

So, here R star is given by x square plus y square less than equals to a square. It means I will be having a square; so, when I will take a square root a will come out and then 1 upon a square minus x square minus y square raise to power half ok and then dx dy over this region R star. If I convert it into polar coordinates, it will become a is outside theta is going from 0 to 2 pi and R is going from 0 to a; 1 upon square root a square minus r square rdr d theta. And when I will solve it it comes out to be 2 pi a cube. Hence, the surface area of the given hemisphere is twice pi a cube. Because 2 pi, you will get from here and it will give you r upon a square minus r square 0 to it will give you a; a square as the.

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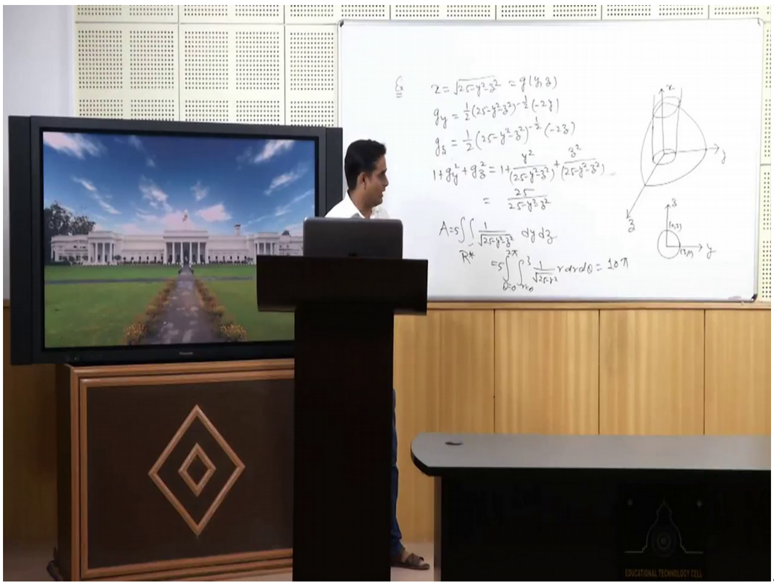

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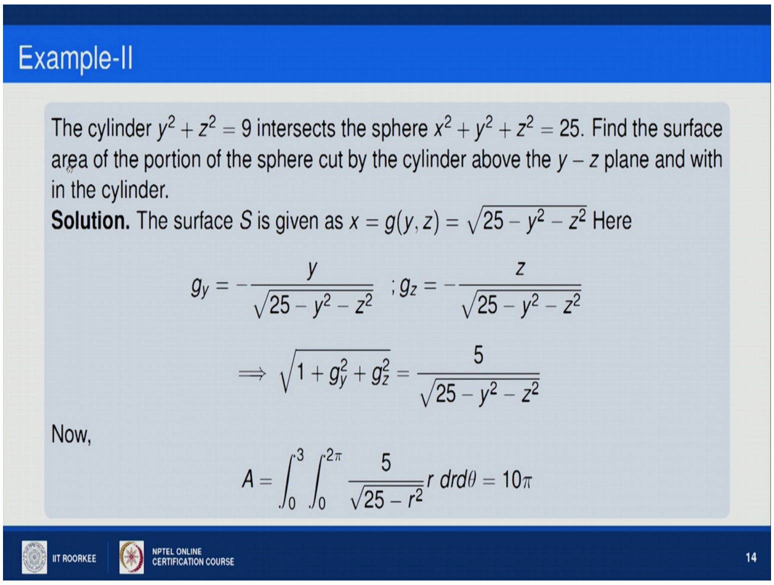

Let us take another example. So, the example is like this. The cylinder y square plus z square equals to 9 intersects the sphere x square plus y square plus z square equals to 25. Find the surface area of the portion of the sphere cut by the cylinder above the y-z plane and within the cylinder. So, basically what I am having? So, let us say y, z. Now, I am having a sphere like this. So, coordinates are 500 sorry 050; 005; 500; so, xxx and 005.

Now, I am having a cylinder which intersect this sphere and cylinder is given by y square plus z square equals to 9 and above. Now, we need to find out the area of this particular portion which is cut by the cylinder of the sphere ok or another way which is common in both and above the y-z plane.

So, here I can write the equation of the surface as x equals to square root 25 minus y square minus z square. Because, ultimately my surface is sphere ok and then what I will I will calculate? And let us say this is g of y, z. So, I will calculate g of y; I will calculate g of z. So, g of y will become 1 by 2 25 minus y square minus z square minus half into minus 2y. Similarly g of z will become 25 minus y square minus z square raise to power minus half into minus 2z ok. And then 1 plus g y square plus g z square will become 1 plus y square upon 25 minus y square minus z square plus z square upon 25 minus y square minus z square. So, it will become 25 upon 25 minus y square minus z square.

Now, surface area is. So, now, I am writing the equation of the surface as x equals to g of y z. So, I will project it on y-z plane. So, now this my R star is the projection of this region; which is common in cylinder and it is sphere on the y-z plane. And then square root of 1 plus g y square plus g z square dy dz. So, when we will solve it so, it will become 5 I can take out, 1 upon square root 25 minus y square minus z square.

And now R star will be the circle of radius 3. Because the cylinder equation is y square plus z square equals to 9. So, projection of this cylinder on y-z plane will give us the circle of radius 3 ok. So, projection will be like this; y, z axis. So, projection will become this circle. So, this is 3 g; here I will say 0, 3 in y-z plane. So, it means this equals to r is 0 to 3; theta is 0 to 2 pi; 1 upon and 5 is out is square root 25 minus r square rdr d theta ok. And when we will solve it; it comes out to be 10 pi. So, hence the surface area of this particular common region is 10 pi. So, with this I will close this lecture.

Thank you very much.