Multivariable Calculus

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Lecture – 36
Green's Theorem

Hello friends. So, welcome to the 36th lecture of this course and in this lecture I will discuss about Greens theorem. As you know that in the previous lecture, we have learn about the some applications of line integral, where we have seen work done by a force in moving a particle from a point to another point.

And then we have seen conservative vector fields; means how can we calculate work done in case when the force is conservative. Today we will see another thing for calculating the line integral over a closed curve and we will see that we can calculate the line integral over a close curve just in form of a double integral and vice versa. So, let us start with the statement of Green's theorem.

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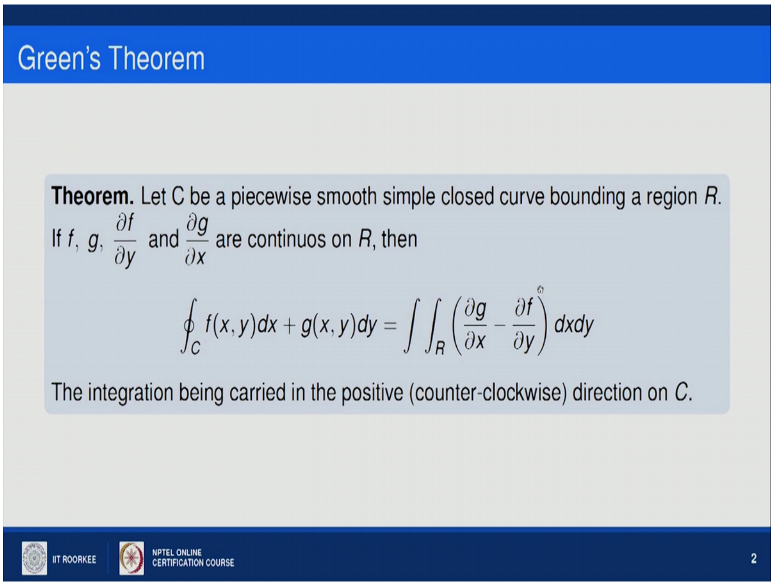

So, let C be a piecewise smooth simple closed curve bounding a region R. So, here C is a smooth simple and closed curve and it is bounding the region R in a plane; let us say in x-y plane. If f, g, del f over del y and del g over del x; all these functions are the functions of x and y; all these are continuous on the region R. Then the line integral over the closed curve C f dx plus g dy will be equal to the double integral over the region R and where integrate each del g over del x minus del f over del y.

Here the integration being carried in the positive that is the counter clockwise direction on C. So, let us see the proof of this theorem.

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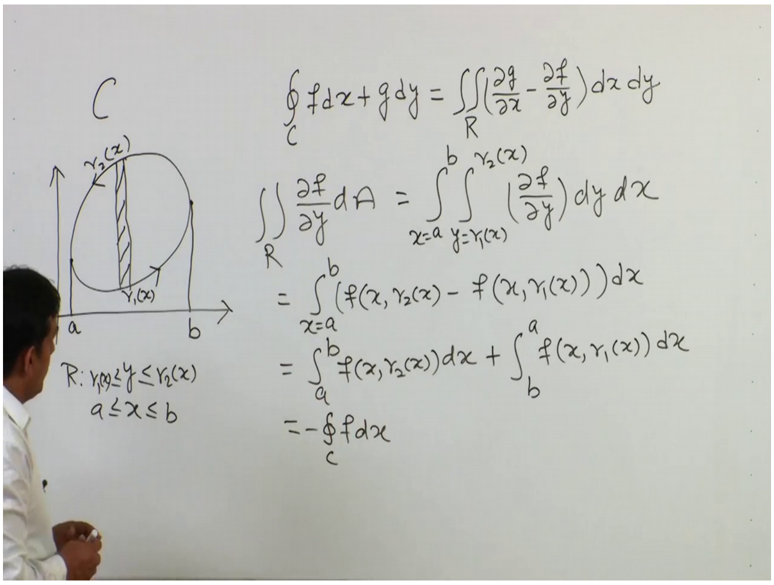

So, here we are assuming that we are having a simple closed curve C, which is bounding a region R. So, let us assume this is our region R. So, this is curve C which is having the orientation in counter clockwise direction, this is the region R. So, let us denote this region in this way. So, my x is going a to b and let us say this r 1 x and this is the curve r 2 x. So, this region R is given as y is between r 1 x to r 2 x and here x is between a to b.

Now the line integral over the whole curve C that is line integral of f dx plus g dy that is basically capital F dot d R; where capital F is defined as f component in I direction and g component in y j direction. This equals to double integral over the region R del g over del x minus del f over del y and then dx dy. So, let us try to integrate del f over del y over the region R. So, over this region I can write y is going from r 1 x to r 2 x means I am taking a strip in this way.

So, lower limit of y is r 1 x; upper limit of y is r 2 x and x is from a to b; del f over del y into dy dx. This equals to x equals to a to b and then after integrating I can have f of x r 2 x minus f of x r 1 x dx. This I can write limit is from a to b f of x r 2 x dx and then here plus I am signing interchanging the limit. So, I am taking from b to a f of x r 1 x dx.

So, what we are doing? We are moving R to x from a to b. So, in the clockwise direction and then from b to a, I am going through b to a through R 1. So, I am completing this closed curve, but in the clockwise direction. So, it means this is equals to integral over C minus f dx.

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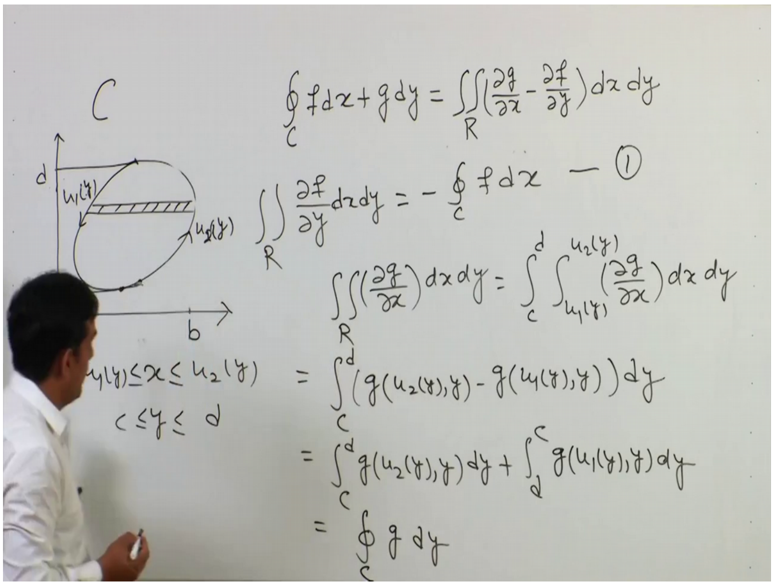

So, here I obtained at del f over del y over dx dy; this equals to now c f dx. So, let us say this is relation number 1. Let us see this region in another way where we will be having the constant limit for y and variable limits for x.

So, let us take the same closed curve and the same region. So, here; so, let us say c to d and again counter clockwise direction. Let us say this is u 1 x or I will say u 2 x and u 1 x. So, I am having the same region R the same closed curve which is bounding this region R that is C; only I have change now the representation of this particular region in this way. Here my y is between sorry; x is between u 1 y to u 2 y. And then I will be having y between c to d. Now what I will be having? Again I will calculate the double integral over this region R.

And now I will take del g by del x; that is the this one this term dx dy. So, I will take a strip like this that is the strip in horizontal direction. So, this equals to u 1 y to u 2 y;

these are the limits of x and then the limit of y c to d; del g over del x dx dy. This equals to integral from c to d.

And when I it will become g of u 2 y y minus g of u 1 y y and then dy. This I can write c to d g of u 2 y y dy plus d to c. So, I have taken the, I have interchange the limit; so, this minus become plus and then g of u 1 y y dy. So, what we are doing? We are going from c to d along u 2 y; means in this direction and then from d to c, I am coming allow u 1 y. So, basically it becomes integral over c g dy. So, here this thing becomes, so, let us say relation number 2.

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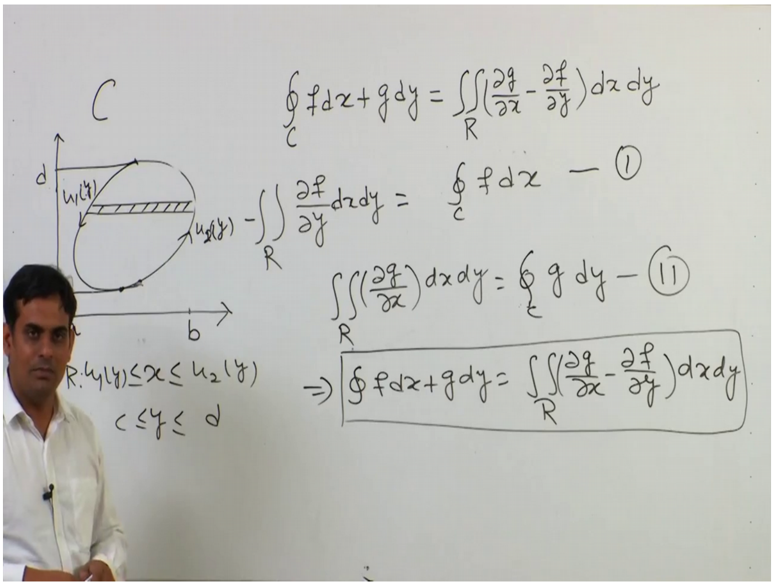

Now, what we are having from first and second? I can write it by adding first and second. So, I can write I can take this minus here. So, minus f dx plus g dy over the curve c; this equals to double integral over the region R del g over del x minus del f over del y dx dy and this is the relation of Green’s theorem.

So, in this way we can prove this theorem which is quite simple while considering a region and a closed curve bounding that region; taking the two different representation and just following this process.

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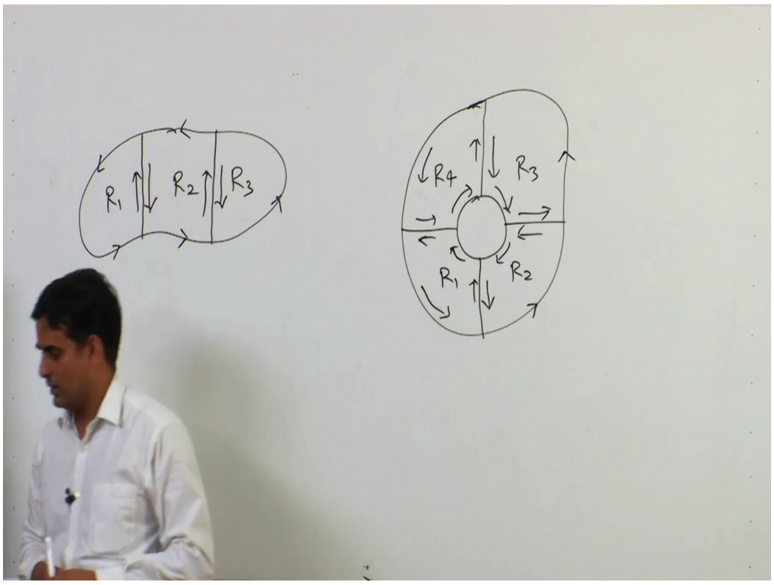

So, we can generalize the result of Green’s theorem for more general regions. For example, if you are having a region like this. So, here you can divide this region in two parts; let us say R 1, R 2, R 3. So, here you can have this representation like this in the counter clockwise direction and for the second one, I can have like this and for the third one it will become like this.

So, in this way I can prove Green’s theorem for such kind of region also following the same process. Another region may be like this; so, it is this smooth here and then this kind of region. So, again divide this region into different parts. So, let us say R 1, R 2, R 3, R 4 ok and here again I will take, so, here again I will take the representation like this in this direction. So in the counter clockwise orientation of the closed curve bounding these regions and finally in this way. So, again I can prove the Green’s theorem for each of the different region and finally I can add the results; to get the proof of Greens theorem for the complete region. So, hence I show you the two examples of different region.

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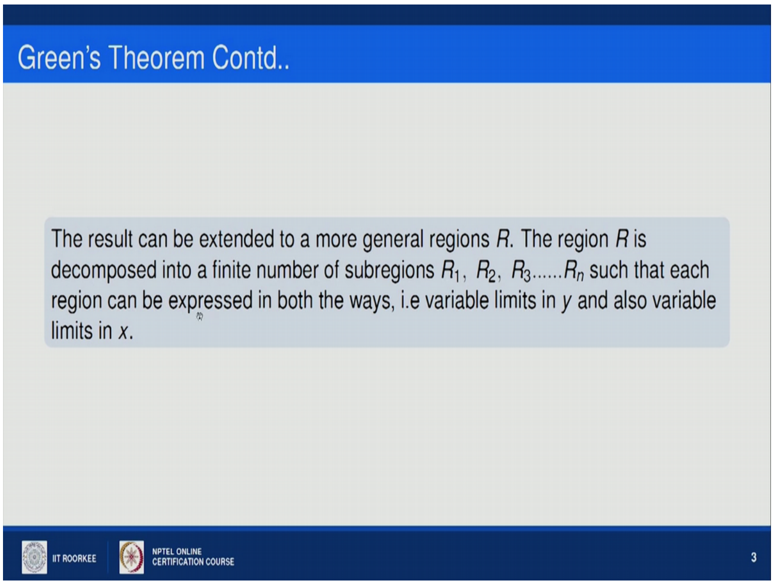

So, the result of Green's theorem can be extended to a more general regions R. The region R is a decomposed into finite number of subregions like R 1, R 2 up to R n such that each region can be expressed in both the ways. Both the ways means the constant limit in x and variable limits in y and vice versa. I have given two examples just now. Now we will take some examples on Green's theorem.

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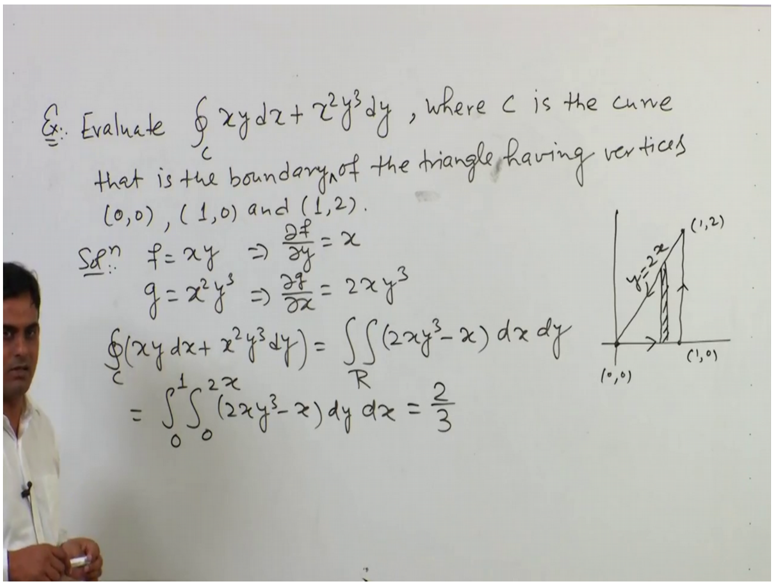

So, my first example is evaluate the line integral over a closed curve C x y dx plus x square y 3 dy; where C is the curve that is the boundary of the triangle having vertices 0, 0; 1, 0 and let us say zone 1, 2. So, here and is the boundary in counter clockwise direction. So, that we are assuming here; so this is 0, 0; 1, 0 and 1, 2. So, I am having this triangle. This is the orientation of closed curve C and this is curve. So, here C is not a smooth curve, but it is piece wisely smooth. We are having three arc of C 1 is from 0, 0 to 1, 0 that is a smooth 1, 0 to 1, 2 that is. So, let us say C 1, C 2 and C 3.

So, if I need to calculate this particular line integral, I need to calculate this integral first for C 1, then for C 2 and finally, for C 3 and then I need to add the these results. Another way of doing it let us apply the Green's theorem. So, here if I compare this with the statement of Green's theorem; I can write f equals to xy and g equals to x square into y cube.

So, from here I can have del f over del y as x and del g over del x as 2xy cube. So, now, according to Green's theorem, the line integral of xy dx plus x square y cube dy over this curve C can be written as the double integral over the region R; where R is this region bounded by these three lines in the region of the triangle. So, this is and then del g over del x that is 2xy 3 minus del f over del y that is x dx, dy. This is quite simple region, you can take either the vertical strip or the horizontal strip. So, let us take variable limits in y.

So, I am taking the vertical strip. So, this line is y equals to 2x; so, this equals to 0 to 2 x and x is going from 0 to 1 twice x y cube minus x dy and an dx. After simplifying this I will get this integral as 2 by 3; so, in this way we can solve this line integral in a very simple manner just by applying the Green's theorem ok.

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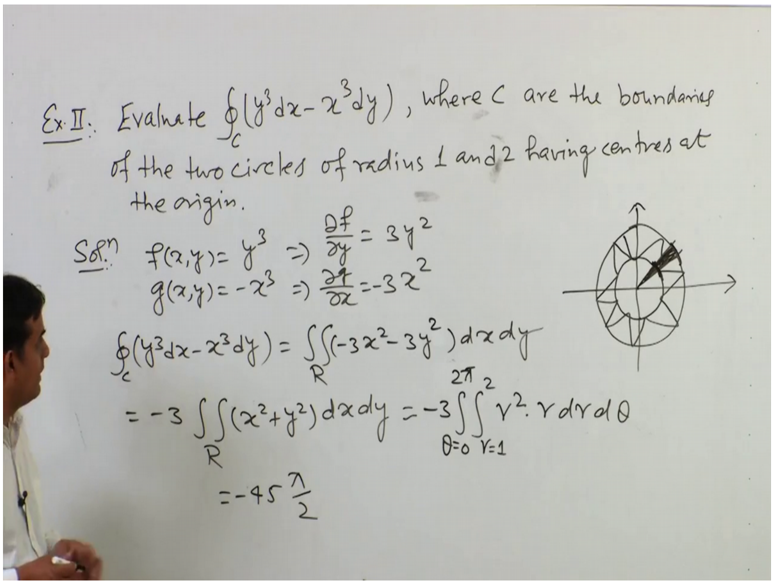

Let us take one more example with different kind of boundary. So, again evaluate the line integral over the closed curve C and it is given by y 3 dx minus x cube dy; where C are the boundary of the two circles of radius 1 and 2 having centers at the origin ok. So, let us sketch our region; so, if this is my x and y coordinates. So, I am having a circle of radius 1; center at origin, so, this circle, another one this one. So, basically C is given by these two boundaries.

So, this is the region bounded by C and it is a bit difficult if I apply Green's theorem directly not Green's theorem; if I calculate the this line integral directly using the usual process of calculating line integral. Here I will apply Green's theorem again; so, here if I compare this particular line integral with the statement of Green's theorem.

So, here f of x y is y 3; g of x y is minus x cube. So, from here del f over del y comes out to be 3y square and del g over del x comes out to be 3 minus 3x square. So, from the statement of Green's theorem, y 3 dx minus x 3 dy equals to the region R bounded by the curve c, del g over del x that is minus 3x square and then I will be having minus del f over del y, so, 3 y square dx dy.

So, I am having this one; this can be I can take minus 3 out. So, R x square plus y square dx dy. And now since I am having the circular region, so, I can convert my variables into polar coordinate. So, x equals to r cos theta and y equals to r sin theta. So, this becomes x square plus y square as r square; dx dy as r d r d theta and then now limits for r is from 1 to 2; that is the radius boundary of the inner circle to the boundary of outer circle. I will take a radial strip like this; so, which is starts from here and move up to here. So, r equals to 1 to 2 and theta will move from 0 to 2 pi. After simplifying this I will get the value of this double integral as minus 45 times pi upon 2. So, in this way we can solve this particular a very difficult problem if I see it in terms of line integral only, but using the Greens theorem, I can solve it in a very simple manner.

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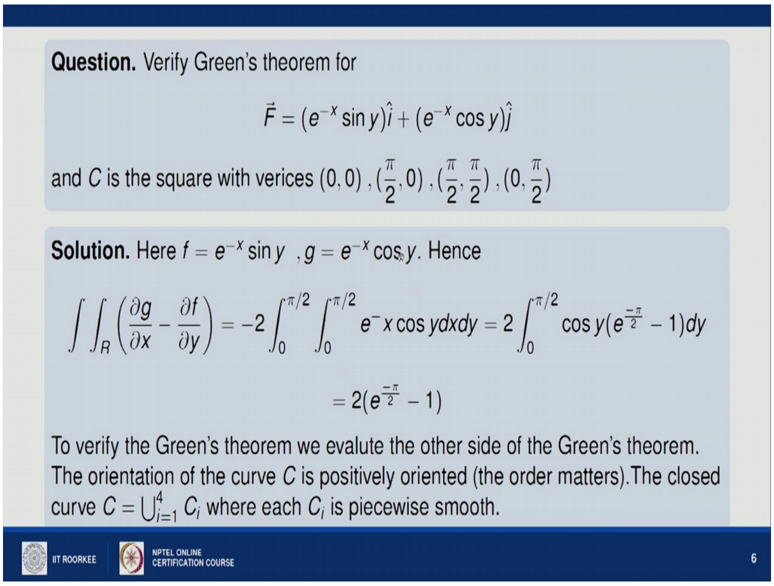

Let us take one more example, this is another type of question we can have on Green's theorem. So, verify Green's theorem for this particular vector function where the i component is e raised to power minus x time sin y and the component in j direction is e raised to power minus x cos y. Here the closed curve is give is a square boundary of a square having vertices 0, 0; pi by 2, 0; pi by 2, pi by 2 and 0, pi by 2.

So, here if we compare with the Green's theorem statement, my small f is given by e raised to power minus x sin y and small g is e raised to power minus x cos y. Hence, if I calculate the right hand side of the Green's theorem that is the double integral over the region R which is the square given by these vertices; del g over del x minus del f over del y dx dy. This comes out to be minus 2 times 0 to pi by 2, 0 to pi by 2 e raised to power minus x cos y dx dy and finally, after simplifying it; it is 2 times e raised to power minus pi by 2 minus 1.

So, this is the result of right hand side. Now what for the verification of Green's theorem, I need to calculate left hand side also and it should be equal to the result of right hand side. Now if we see the closed curve c; so, it is not smooth, entirely smooth, but it is piece wisely smooth. It is having four arcs; one is from 0, 0 to pi by 2, 0 another one from pi by 2, 0 to pi by 2, pi by 2. So, these are the straight lines; horizontal and vertical in the horizontal and vertical directions and they are smooth. So, it means I need to calculate for line integrals one for each line and then I need to add them.

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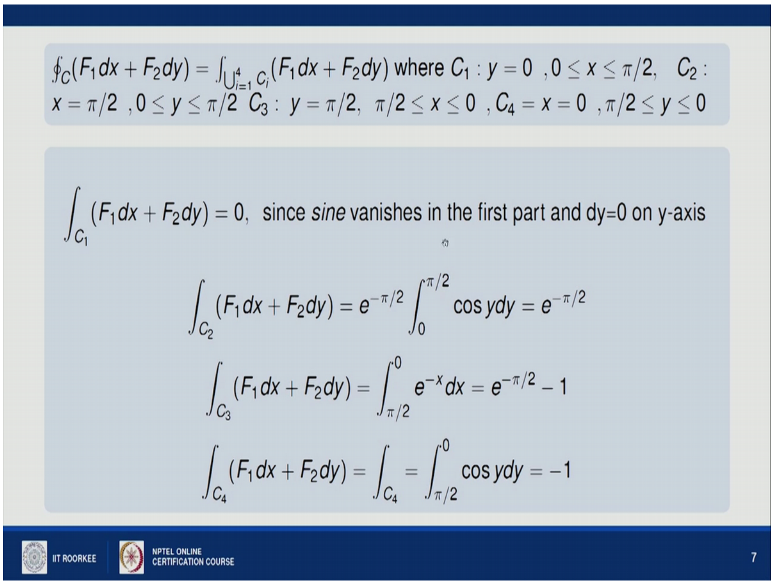

So, let us say I am having these the first arc C 1; where y 0 and x is between 0 to pi by 2. The second curve or second arc is x is pi by 2. So, the vertical line at x equals to pi by 2 and y is moving from 0 to pi by 2; third one is y pi by 2 and x is going from pi by 2 to 0. So, please note that here I am having counter clockwise orientation of the closed curve C and then finally, C 4 is given by this one. So, here F 1 dx plus F 2 dy along the curve C 1 is 0; since sine vanishes in the first part and dy equals to 0 on the y axis; since y is constant, 0 is here. Along the C 2 this comes out to be e raised to power minus pi by 2; along C 3 it is e raised to power minus pi by 2 minus 1 and along C 4 it is minus 1.

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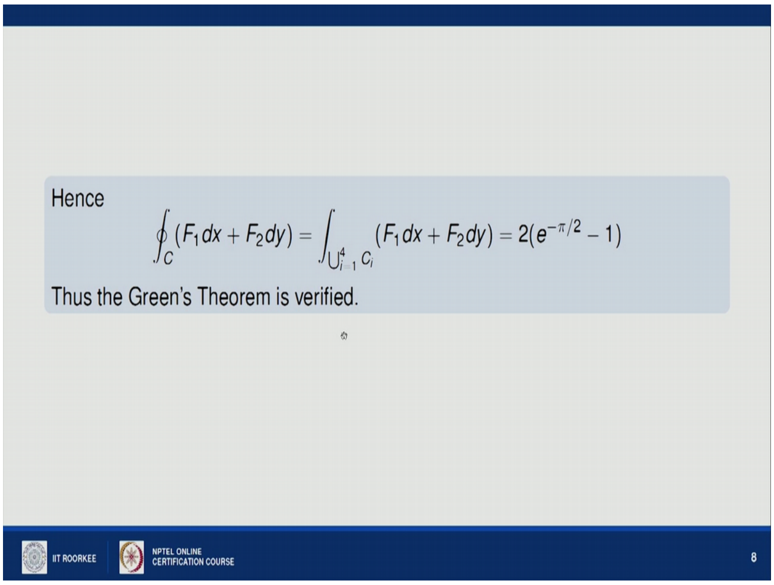

So, now, by adding all of these, I got the value of F 1 dx plus F 2 dy that is the left hand side of the Green's theorem and the line integral on this closed curve C as 2 times C raised to power minus pi by 2 minus 1 and which is the same as the result of right hand side; hence the Greens theorem is verified.

So, with this I will end this particular lecture. In the next lecture we will extend the idea of line integral in terms of surface integral ok. So, like line integral is the integral along a given curve. In the same way surface integral will become an integral over a given surface. So, with this I will stop myself.

Thank you very much.