Multivariable Calculus

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Lecture – 35
Applications of Line Integrals

Hello friends. So, welcome to the 35th lecture of this course and in this lecture I am going to introduce an Application of Line Integral that is called work done.

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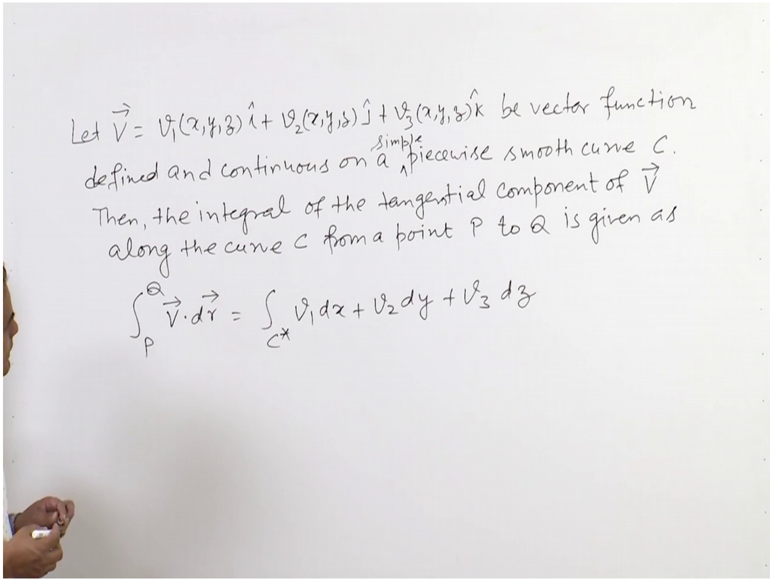

So, let V be a, so, v let the components of V are v 1 x, y, z in i direction, v 2 x, y, z in j direction plus v 3 x y z in k direction. So, let V be a vector function which is defined and continuous on a piecewise a smooth curve C.

So, V is a vector function which is continuous on a piecewise smooth curve C. So, let us put simple piecewise. Then the integral of the tangential component of V along the curve C from a point P to a point Q is given as. So, this is given by a line integral and it will be something like this integral P to Q and then V dot d r. Let us define C star.

So, C star is the arc of curve C from point P to Q. Then it will become v 1 dx plus v 2 dy plus v 3 d z. So, this is the integral of tangential component of v along the curve C from the point P to Q. Now based on this we are going to define the work done.

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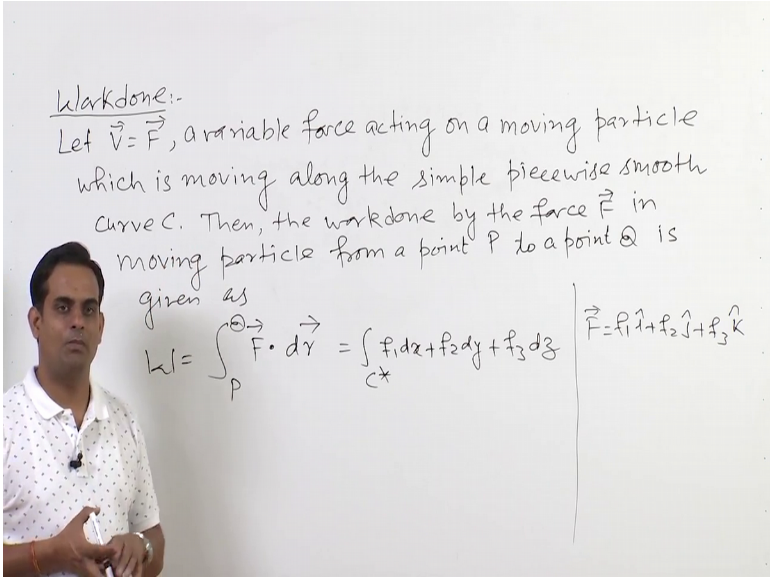

So, let V the vector function equals to F, which is a variable force acting on a moving particle. So, this moving particle is moving along the simple piecewise a smooth curve C. And this force F is acting on this moving particle. Then the work done by the force F in moving particle from a point P, which is a point on the curve C to a point Q is given as. So, work done W is given by the line integral P to Q the force vector f dot d r.

So, this is the same integral or I will say the same line integral which is we have taken for the integral of tangential component of V. And this is if F is having the components f 1, f 2 and f 3 means f is defined as f 1 i plus f 2 j plus f 3 k; where f 1 f 2 and f 3 are functions of x, y and z; then it will become again C star f 1 dx plus f 2 dy plus f 3 d z; where C star is the arc of the curve C from the point P to the point Q.

Now we are going to take few examples of work done and then we will see that what will happen if the force vector f is a conservative force vector. Means there exist a potential function phi; such that that the vector function f equals to gradient of phi, but before that let us take some simple example.

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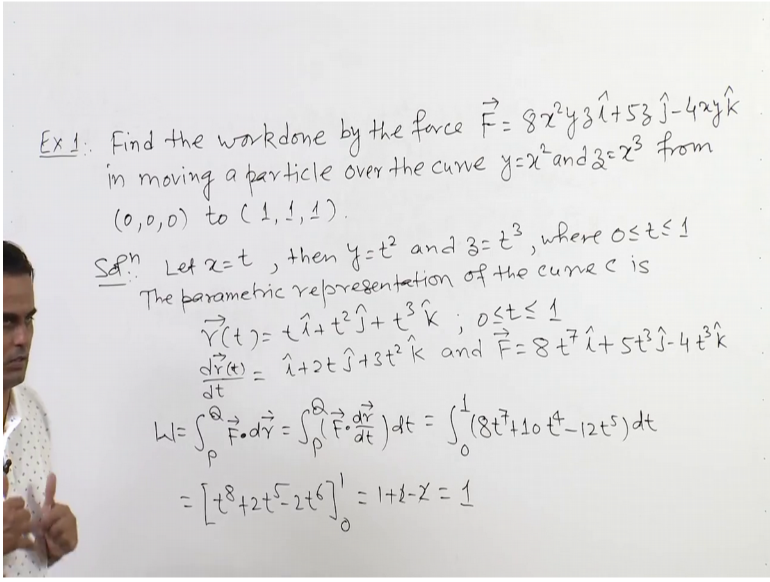

So, example 1; so, find the work done by the force F that is 8 x square y z i plus 5 z j minus 4 x y k; in moving a particle over the curve y equals to x square and z equals to x cube. So, this is the curve in 3 dimensional space from let us define our point P and Q. So, let us take P as 0, 0, 0, 1, 1, 1. So, first of all as we have done in line integral we need to find out the parametric representation of the curve C and then only we can evaluate the line integral. So, let x equal to t, then y will become t square and z which is x cube will become t cube; where t is in close interval 0 to 1.

So, the parametric representation of the curve C is r t equals to t i plus t square j plus t cube k and t is between 0 to 1. Now if I calculate dr over dt it will become i plus 2 t j plus 3 t square k. Now work done and here the force F in terms of t given as 8 times x square will become t square; y will be t square and z will be t cube.

So, total will become t raised to power 7 i plus 5z. So, 5z means 5 t cube j minus 4; x is t y t square; so, 4 t cube k. So, now, work done is given as from the point P to Q line integral that is F dot d r. So, it will become P to Q F dot dr over dt into d t. So, it will be 0 to 1 the limit of the t F dot dr will become 8 into t raised to power 7 plus 5 into 2; 10 and here I am having t cube here I am having t.

So, 10 into t raised to power 4 and then minus 3 t square here; 4 t cube here. So, the total will become 12 t raised to power 5 and then d t. So, this will become t raised to power 8 plus 2 t raised to power 5 minus 2 t raised to power 6 and that is 0 to 1. So, it will become 1 plus 2 minus 2; so, this cancel out 1. So, the work done is 1; for moving the particle by the force F along this curve y equals to x square and z equals to x cube from the point 0, 0, 0 to 1, 1, 1. So, this is simply represented by a line integral.

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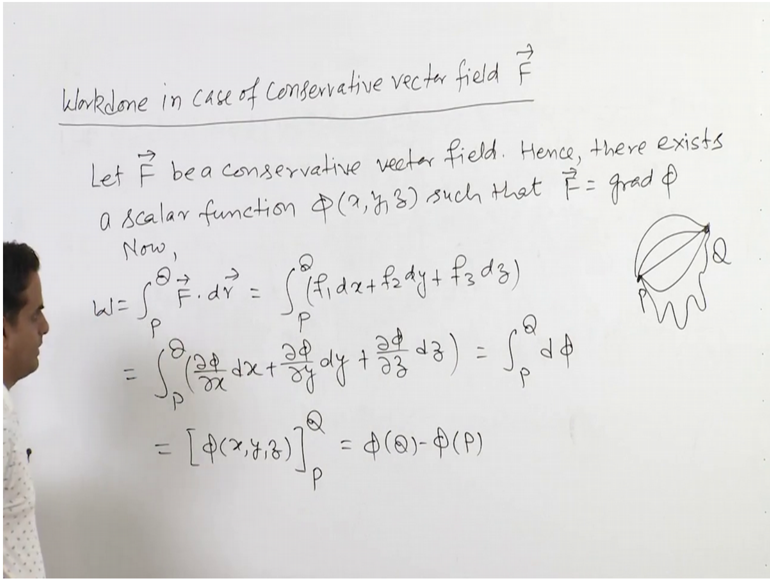

Now let us see the work done in case of conservative vector field F. So, let F be a conservative vector field. This means, there exist a scalar function phi x, y, z such that F equals to gradient of phi; means phi is a potential function of the vector field F. Now work done W is P to Q F dot d r.

So, it will become P to Q f 1 dx plus f 2 dy plus f 3 d z. Now, this F equals to grade of phi; so, hence this f 1 equals to del phi over del x; f 2 equals to del phi over del y; f 3 equals to del phi over del z. So, this is equals to P to Q del phi over del x into dx plus del phi over del y into dy pus del phi over del z into d z. And this will become P to Q d phi and if I integrate it, it is simply phi x, y, z P to Q; that is phi Q minus phi P.

And hence, what we can conclude that the work done depends only on the initial point and the terminal point; means on the point Q and the point P. It is nothing to do with the path moving from P to Q; means the work done or line integral in case of conservative vector field is path independent. You go along any path, so, if this is P; this is point Q, you move like this, you move like this, you move like this, you move along a straight line or you move like this or you move any other path.

Because there will be infinitely many path between P and Q. Along all the curves integral will be same in case of conservative vector field and that is the idea mentioning here that work done in case of conservative vector field. So, if question is like that; that let F be a conservative vector field given by this; then find a work done by this conservative vector function or conservative force by moving a particle from P to Q. So, what you need to do? You need to find out the potential function phi corresponding to conservative vector field F and then phi Q minus phi P that is all.

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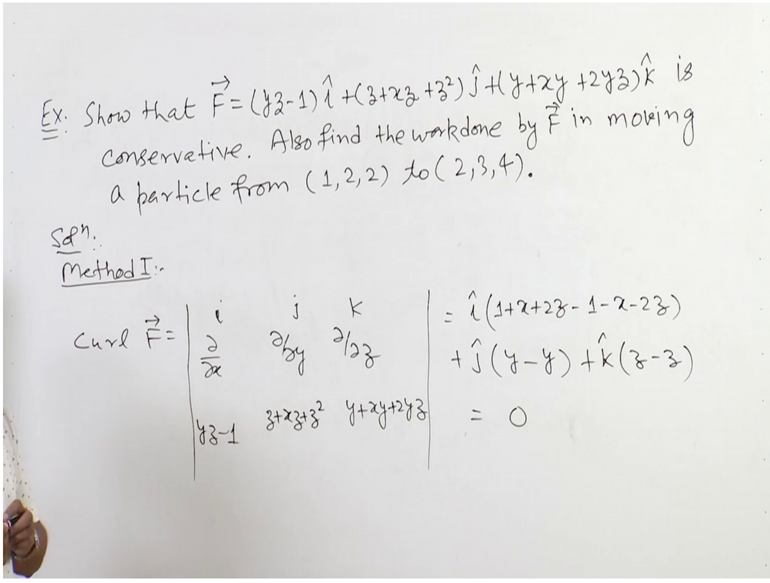

Let us take an example of this case. Show that the force vector F which is given as yz minus 1 i plus z plus xz plus z square j plus y plus xy plus 2yz k is conservative. Also find the work done by F in moving a particle from 1, 2, 2 to 2, 3, 4 ok. So, in the first sentence of the question, I need to show that the force vector is a conservative vector field. This I can do using three methods. Let us take method 1; if F is conservative means; F equals to grade phi and curl of grade of any scalar function is always 0.

So, if curl of F is 0; means F is a conservative vector field. So, calculate curl of F. So, it is given by i, j, k, del by del x, del by del y and then yz minus 1, z plus xz plus z square, y plus xy plus 2yz. So, this equals to i and then del by del y of this function. So, 1 plus x plus 2 z minus del by del z of this function; so, minus 1 minus x minus 1 z plus j component; so, del by del z of this function; so, y minus del by del x of this function. So, y minus y plus k component; so, del by del x of this; so, it will be z minus z. So, it comes out to be 0.

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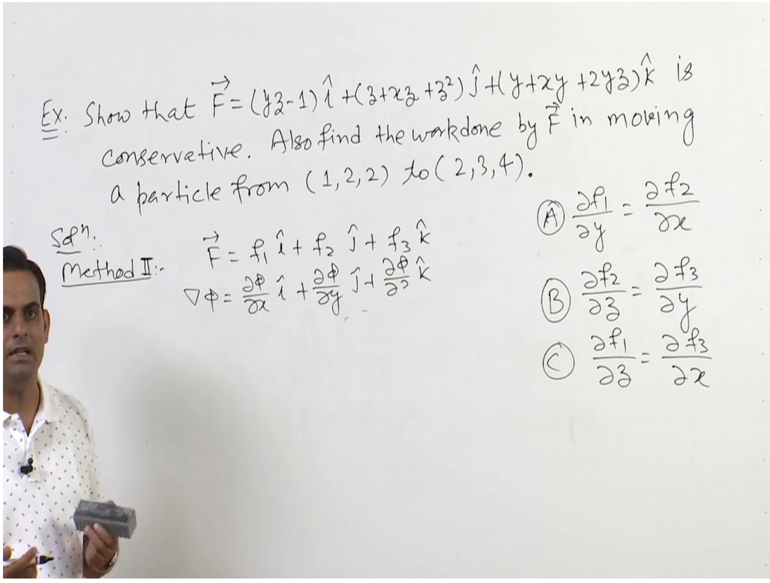

So, curl of F is 0; so, F is a conservative vector field. Method second; you know that F is let us say f 1 i plus f 2 j plus f 3 k; where f 1 is y z minus 1; f 2 is z plus xz plus z square and f 3 is y plus x y plus 2yz. Now it is equals to grade of phi.

So, it means del phi over del x i plus del phi over del y j plus del phi over del z k. If F is conservative these two are equal. It means f 1 equals to del phi over del x and f 2 equals to del phi over del y and f 3 equals to del phi over del z; relation 1, relation 2, relation 3. Let us differentiate partially relation 1 with respect to y and relation 2 with respect to x.

So, what I will get in this case, del f 1 over del x no del x. I am doing first with respect to y. So, del f 1 over del y; this equals to del 2 phi over del x del y. Now differentiate partially relation to with respect to x; so, I will be getting del f 1 over del x; this will become del 2 phi over del x del y. So, right hand side equal.

So, from here I got del f 1 over del y equals to del f 2 over del x. Similarly what you do? You differentiate partially the second relation with respect to z and the third relation with respect to y. So, what you will get? del f 2 over del z equals to del f 3 over del y. And then what you do? You differentiate first relation with respect to z partially and the third relation with respect to x partially. So, what I will get? del f 1 over del z equals to del f 3 over del x.

So, if the components of a vector field satisfy these 3 equalities, then the given vector field is a conservative. And if you see in this example all these three conditions are satisfied and hence you can say that F is conservative. My third way of is to find out the potential function phi such that f equals to grade phi.

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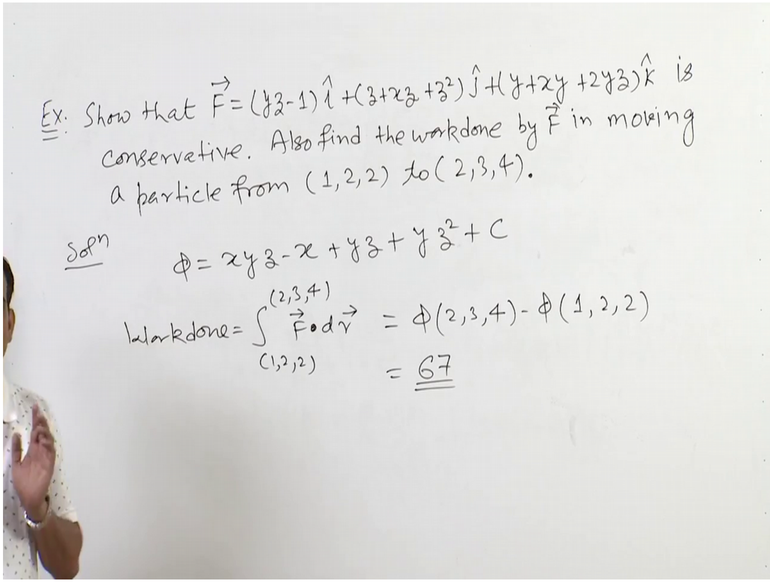

And if I do it phi comes out to be in this case for this vector field is x, y, z minus x plus y z plus y z square plus constant C. And if you see that del phi over del x, del phi over del x will become y z minus 1 f 1 component; del phi over del y will become x z plus z plus z square means f 2 component.

And similarly del phi over del z will become y plus x y plus 2 y z that is the f 3 component. Now work done is 1, 2, 2; 2, 3, 4; F dot dr here curve is not given, but just now we have prove that it will become phi of 2, 3, 4 minus phi of 1, 2, 2; since F is a conservative vector field and this comes out to be 67.

So, in this way we can solve this kind of example, where the force vector is a conservative vector field ok.

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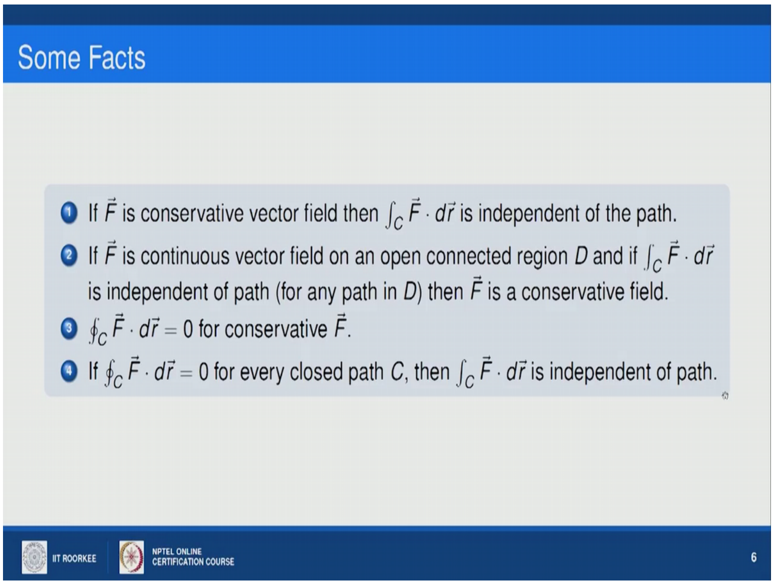

There are some more facts I want to tell you that if F is a conservative vector field then the integral over a curve C F dot dr is independent of the path. We have seen just in this example and we have prove it earlier. If F is continuous vector field on an open connected region D and if the line integral over a curve C F dot dr is independent of path for any path in domain D then F is a conservative vector field.

The third remark I would like to tell you that the line integral over a close curve C; if this line integral that is F dot dr is 0, then F is conservative. Because in case of close curve the initial point and the terminal point will be the same and hence we are subtracting the same amount from the same amount. So, it must be 0.

Finally, if F dot dr equals to 0, over the a close curve C; for every close path C, then F dot dr is independent of path. So, with these remarks I will end this lecture. So, in this lecture we have learn the concept of work done that is basically representation of line integral. So, if someone ask you what is the physical representation of a line integral, you can say that it is wok done.

So, thank you very much.