Multivariable Calculus
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Lecture – 34
Line Integral - I

Hello friends. So, welcome to the 34th lecture of this course and in this lecture I will introduce line integral.

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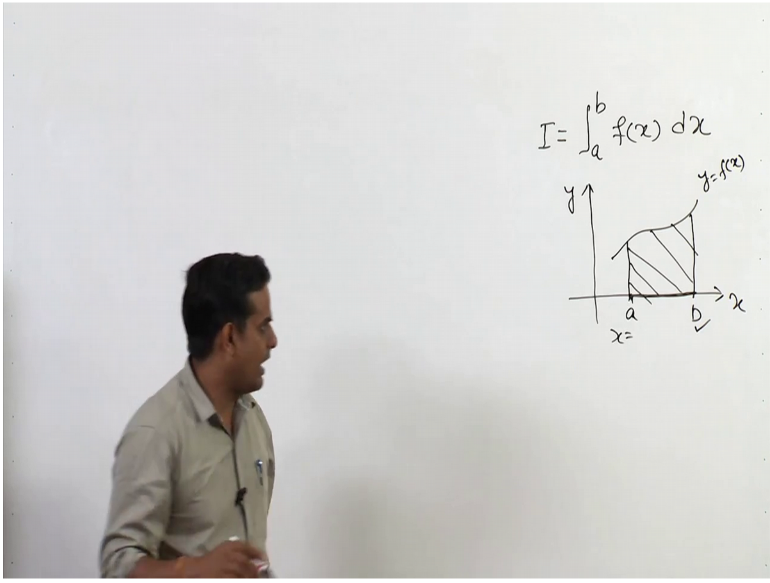

So, we have knew about the definite integral; definite integral is something like this I equals to a to b; f x d x. So, basically what is this? You are having your x axis, y axis, you are having a curve y equals to f x, you are having a point x equals to a; another point x equals to b.

And now this I gives me this particular area; area under the curve and above the x axis and here we are calculating this integral on the horizontal axis; from x equals to a to x equals to b. So, on a straight line, but all the curves are not straight lines. I can have a circle, I can have any other conic, I can have helix which is a space curve; we have seen the parametric representation of that curve just few lecture ago. Now if I ask you, you are having a function f which is a function of x, y, z.

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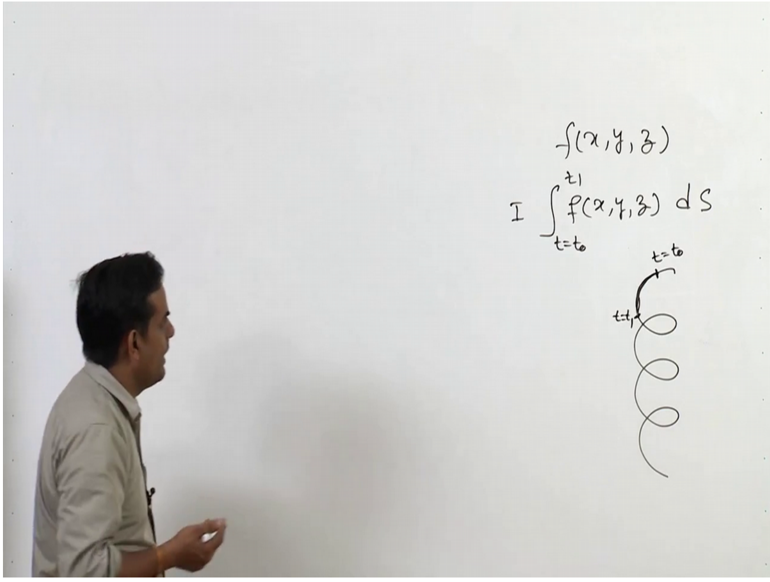

So, f is a function of x, y and z. Now what you do? Calculate f of x, y, z that is the integration of f with respect to arc length of a given smooth curve. For example, if I am having a helix; so, let us say this is having some t equals to t naught and here I am having t equals to t 1. So, find the integral t equals to t naught to t 1 along this arc length.

So, now, please see the difference. In the definite integral we were having a straight line on the horizontal axis; an interval from a to b. Here my t is changing or curve is changing not in a straight line; it is along a curve. So, such type of integral are called line integral. Here it can be a scalar function or it can be a vector function also, it can be with respect to arc length or we can define it in other way also. So, let us define it formally. This is just a motivation for you that what we mean by line integral.

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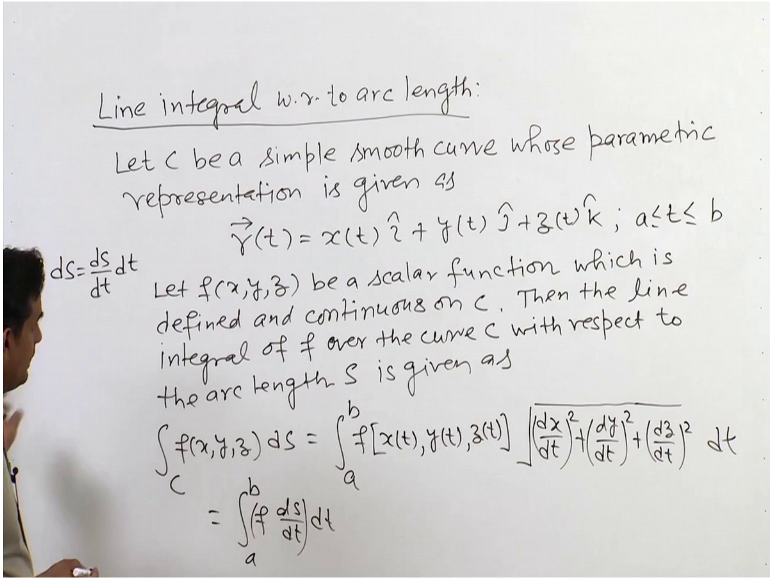

So, we have to integrate a function over a smooth curve for a given arc of the curve ok. So, let me write the formal definition of line integral with respect to arc length. So, let C be a simple a smooth curve whose parametric representation is given as r t equals to x t i y t j cap plus z t k cap; here t is let us say between a to b. Let f of x, y, z be a scalar function which is defined and continuous on C. Then the line integral of function f over the curve C with respect to arc length; let us say arc length is S. So, with respect to the arc length S is given as integral over C f of x, y, z dS.

So, this I can write a to b. Let me write this f in terms of parametric representation means in terms of t. So, x will be x t, y will be y t and z I can write z t. So, now, it will be a function of t. Now let us what we do with this dS? So, you know that I can write dS equals to dS by d t into d t and what is dS by dt? dx by dt is magnitude of r dash t.

So, I can write it a square root of dx by dt whole square plus dy over dt whole square plus dz over dt whole square into dt and the limits of integral is from a to b. So, basically what I am doing? I am writing f dS over dt into dt. So, here everything will be in terms of t and limits are also in of t from a to b. So, we can evaluate this particular integral. So, let us take one or two example of this particular thing and then we will move to next definition.

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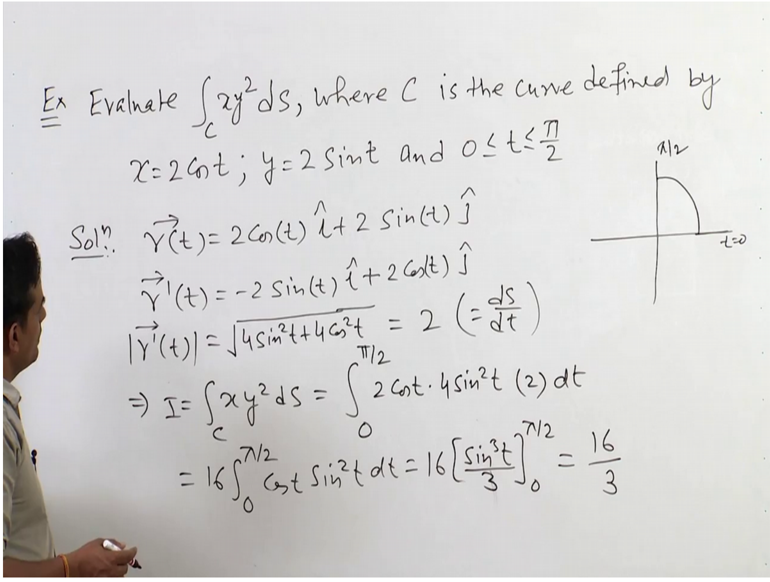

So, let us take example. Evaluate integral over a curve C function is given by x into y square with respect to arc length S over the curve C, where C is the curve defined by x equals to 2cost; y equals to 2sint and 0 to t to pi by 2. So, if we take, so if we see the curve C, so, it is nothing, but just a circle of radius 2 in the first quadrant because x is 2cost, y is 2sint. So, x square plus y square equals 2 4 and since t is moving from 0 to pi by 2.

So, from t equals to 0 to pi by 2; So, here r t is given as 2cost i plus 2sint j. So, now, r dash t will become minus 2sint i plus 2cost j. Magnitude of r dash t will be square root; for sin square t plus for cos square t, so, for sin square t plus 4 cos square t. So, basically it is dx by dt whole square, it is dy by dt whole square. So, it comes out to be 2. So, basically I can write dS as 2 times d t because it is my dS over dt. So, now, line integral is I equals to x y square dS over the curve C can be written as t is moving from 0 to pi by 2, x is twice cost y is 2sint. So, it will become 4 sin square t and then dS over d t is 2 and finally, dt. So, this becomes 16 0 to pi by 2 cost sin square t dt.

This will become 16 0 to pi by 2 or I can write directly the integration sin cube t over 3 0 to pi by 2 and this comes out to be 16 by 3; that is the final answer of this problem. So, hence the value of this integral over this curve, 20 is moving from 0 to pi by 2 is 16 by 3.

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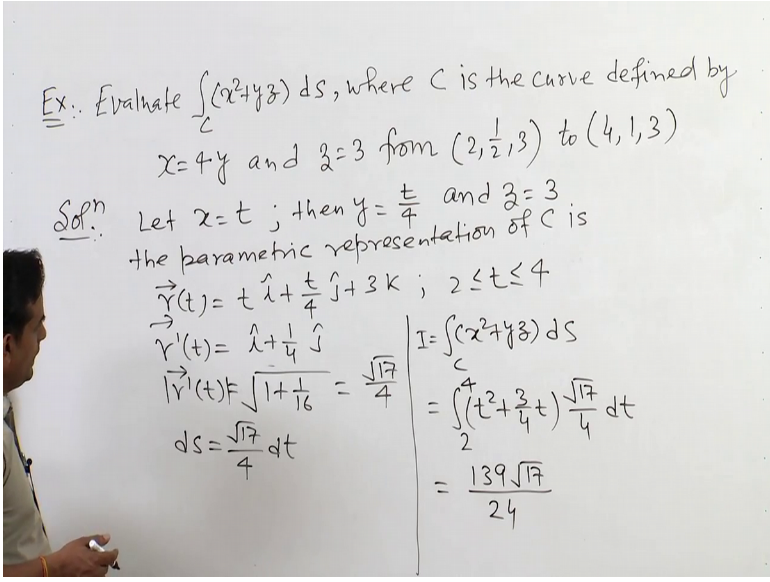

Now let us take one more example in which we will change the definition of curve in another way. So, evaluate a line integral over the curve C of a function x square plus y z with respect to arc length S; where C is the curve defined by x equals to 4y and z equals to 3; from 2, 1 by 2, 3 to 4, 1, 3 ok. So, here first we need to write the parametric representation of the curve C; then we need to calculate magnitude of r dash t and finally, we will solve the integral in terms of t.

So, here let x equals to t. So, I am writing the parametric representation of the curve C. If x equals to t then y will become t by 4 because x equals to 4y and z is given as 3. So, the parametric representation of C is r t equals to t i plus t by 4 j plus 3 k; where t is moving from 2 to 4. Why I am writing this 2 to 4 because t equals x and x is going from 2 to 4. So, from here like the earlier example I will calculate r dash t which will become i cap plus 1 by 4 j cap and so on because this component will become 0. So, here r dash t, the magnitude of this vector will be square root 17 by 4.

So, this will become basically square root 1 plus 1 by 16. So, it will become square root 17 by 4. So, from here I can write dS equals to square root 17 by 4 dt. Now integral over the curve C x square plus yz dS can be written as t is moving 2 to 4; x square is t square y is t by 4; z is 3. So, 3 by 4 t it is square root 17 by 4 dt; so, this comes out to be 139 square root 17 by 24, after solving this particular integral. So, in this way we can solve such this kind of problem.

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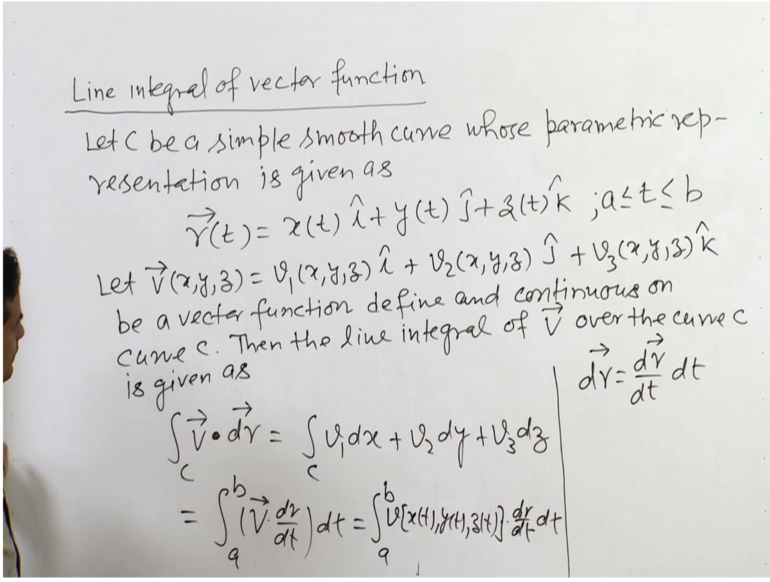

So, my next definition is line integral of vector function. So, again let C be a simple smooth curve whose parametric representation is given as r t equals to x t i cap plus y t j cap plus z t k cap; where t is let us say between a to b.

Let the vector function V which is let us say V x 1 x, y, z is the component in i direction plus v 2 x, y, z is the component in j direction plus V 3 is the component in k direction. So, let vector function V be a vector function defined and continuous on curve C. Here continuous means all the 3 components V 1, V 2, V 3 are continuous on the curve C. Then the line integral of the function V, over the curve C is given as the integral over C V dot dr.

So, basically if I write it in another way it can be written as V 1 dx plus V 2 dy because V as well as dr are the vector functions. So, their dot product will be a scalar function and this is scalar function will be V 1 dx plus V 2 dy plus V 3 dz or I can write it in terms of t. So, t is moving a to b; this dr I can write in another way. And what will be the another way of writing this dr? It will be because we know that.

So, another way will be dr I can write as d r over d t into d t. So, this will become V dot d r over d t into d t. So, this will become as you know integral over a to b V which is now a function of x t; y t; z t and then dot product with d r over d t into d t. So, in this way we can define the line integral of vector functions.

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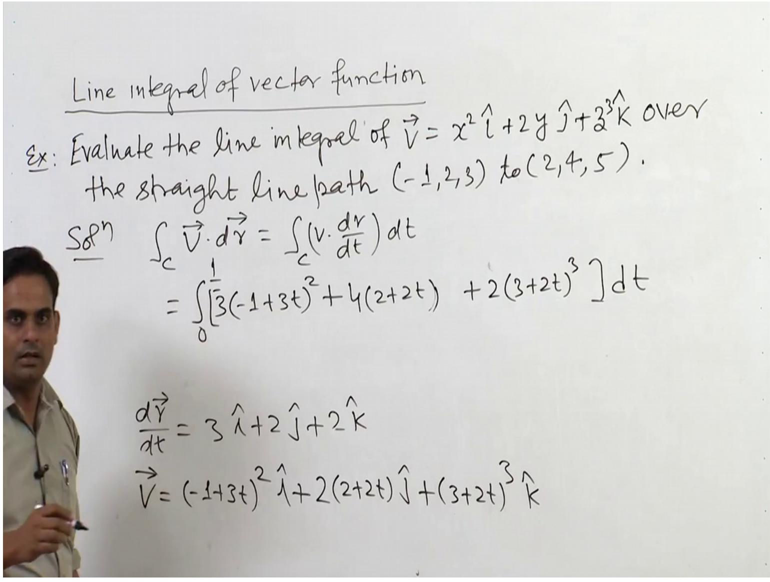

Let us take one example of it that how to calculate this line integral for a given vector function and a given curve. So, evaluate the line integral of the vector function V equals to x square i plus 2 y j plus z cube k over the straight line path; let us take minus 1, 2, 3 to something 2, 4, 5. So, first of all I have to write the parametric representation of this straight line.

So, the parametric representation of the straight line is given as so, r t equals to I will write from here; so, minus i plus 2 j plus 3 k plus t times 2 minus minus 1, so, 2 plus 1; 3 i plus 4 minus 2 2; 2 j plus 5 minus 3; 2 k and t is between 0 to 1. So, when t is 0, I will get minus 1, 2, 3; when t is 1, I will get 3 minus 1; 2; 2 plus 2 4; 2 plus 3 5; the another point end point of the line.

For this I can write minus 1 plus 3 t i plus 2 plus 2 t j plus 3 plus 2 t k and t is between 0 to 1. Now d r over d t; so, it is given as 3 i cap from here plus 2 j plus 2 k. My vector V is given as x square i; so, my x is now minus 1 plus 3 t. So, it will become minus 1 plus 3 t whole square i plus 2 y.

So, 2 y now from the curve y t equals to 2 plus 2 t. So, 2 times 2 plus 2 t into j plus z cube. So, from the parametric representation of curve j t is 3 plus 2 t cube k. So, I am having d r over d t; I am having the vector V in terms of t. So, now, line integral of V over this is straight line can be given as C V dot d r ok.

So, this I am writing C V dot d r over d t into d t and this will become the dot product of these two vectors. So, 3 minus 1 plus 3t whole square plus 2 into 2 4 2 plus 2 t j plus 2 times 3 plus 2 t cube not j here ok. So, this one into d t and t is moving from 0 to 1. So, ultimately what we are having? We are having everything in terms of t and now we can calculate this particular definite integral and by calculating it we can get the value of this.

So, we can further simplify it; we can write in terms of t and then we will get a polynomial of degree 3; we can integrate it over t and we can simply substitute the limits. Now I have taken a constraint that the curve should be a smooth curve.

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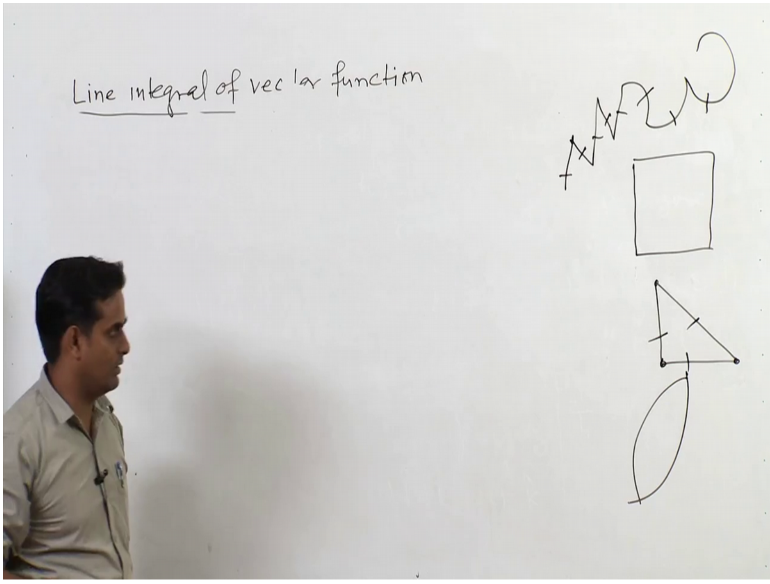

Now let us mild this particular constraint. So, if my curve is not is smooth, but it is piece wisely smooth. For example, it is a square or it is a triangle or it is something intersection of two conics. So, here it is having four piecewise; four sub curves, those are smooth. So, I will say that it is a piece wisely a smooth curve; it is having these three straight lines those are smooth, but if I put these corner points are not a smooth. So, I will say it is a piece wisely smooth curve; similarly these two curves. So, if the curve C is like that or it is something like this.

So, this is piece; this curve is piecewise smooth. These are the pieces of this curve; sub pieces those are is smooth. So, how we will define my line integral of a vector function on such a curve?

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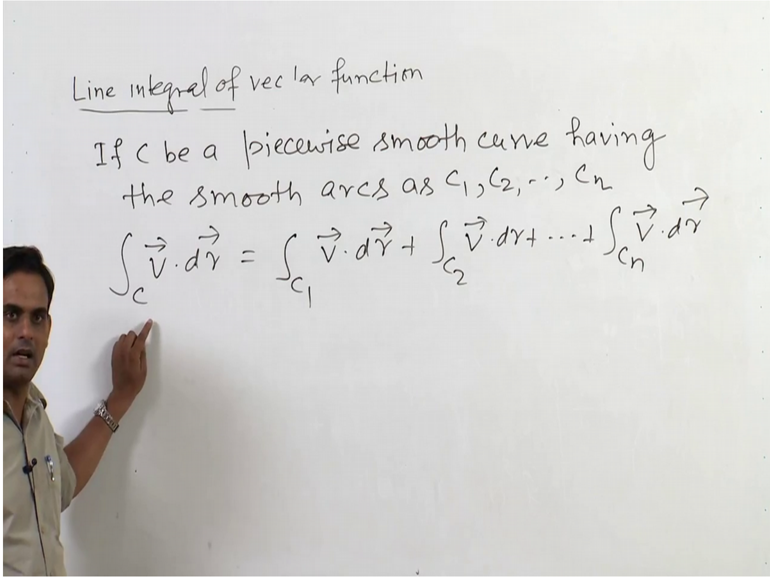

So, if C be a piecewise a smooth curve having the smooth arcs as C 1, C 2, C n; then the integral V dot d r over the curve C can be calculated or can be given as V dot d r plus integral over C 2 V dot d r plus; meaning is you calculate the line integral over each smooth arc and then sum all of them. So, the total will be the integral over the curve C.

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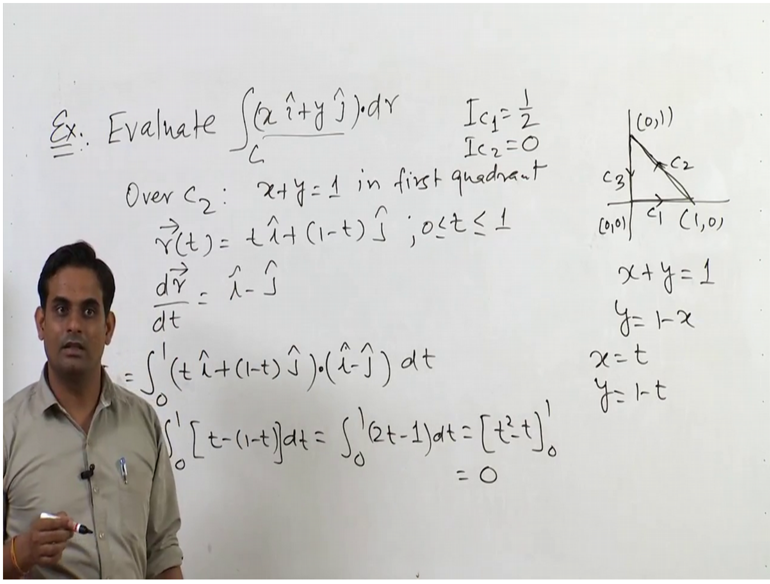

So, let us take a simple example of this. So, example is evaluate let us say x i plus y j and this is d r dot d r over a curve C; where C is a curve given by this triangle.

So, let us say 0, 0; 1, 0. So, here I am having three smooth arc; one is C 1 let us say; orientation is like this C 2 and C 3; so, having the positive orientation. So, first I will calculate is over C 1. So, C 1 is something x is between 0 to 1 and y is 0. So, for C 1 parametric representation will become r t is t i; where t is between 0 to 1. So, d r over d t will be i only and here integral over C 1 will be 0 to 1; x is t here, y is 0.

So, t i dot product with i d t because this is d r over d t and d t. So, it will be 0 to 1 t d t and it comes out to be 1 by 2. Similarly I will calculate on I C 2. So, I C 2 is this line; this line is x plus y equals to 1 or I can write y equals to 1 minus x. So, if I take x equals to t, y will become one minus t; so, for on the curve C 2.

So, let me write it here I of C 1 equals to 1 by 2. So, on curve C 2; so, C 2 is x plus y equals to 1 in first quadrant. So, parametric representation will become t i plus 1 minus t j; where t is between again 0 to 1. So, d r over d t will be i minus j ok and then the integral will become 0 to 1, x is t i.

So, x is t i plus y is 1 minus t j; so, this is my this function over this curve and then dot product with i minus j. And finally, d t and limit is 0 to 1. So, this I will get 0 to 1 t minus 1 minus t d t; 0 to 1 t plus t 2 t minus 1 d t. This will become t square minus t 0 to 1 and then 1 minus 1 is 0.

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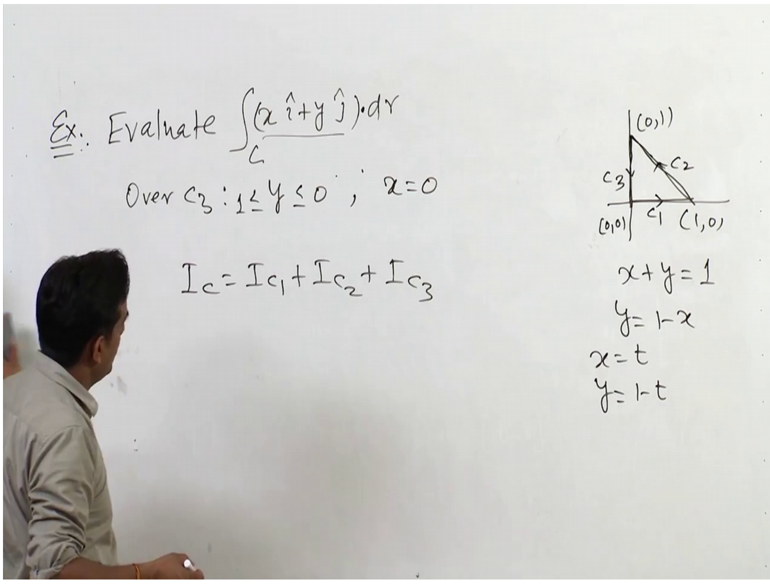

So, I C 2 is 0. Next I will calculate this integral over C 3. So, for C 3 I am having this particular arc. So, y is from 1 to 0 and x is 0. So, similarly you can write the parametric representation, you can calculate the C 3 and then I C will become I C 1 plus I C 2 plus I C 3. So, in this way we can solve this kind of problem where curve is not smooth, but piecewise is smooth.

So, with this I will close this lecture. So, in this lecture I have introduced the concept of line integral. Then we have seen line integral with respect to arc length, we have seen line integral of vector functions. We have taken few examples. In the next lecture we will continue from here and we will see some applications of line integral.

Thank you very much.