Multivariable Calculus

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Lecture – 33
Some Identities on Divergence and Curl

Hello friends welcome to the 33 lecture of this course. So, this lecture is again in the continuous, is in the continuation of the previous lecture. So, in the previous lecture we have learnt about divergence and curl. In the last lecture we have taken an example like a function phi is given to you which is a scalar function and then we need to find out curl of grad phi and divergence of grad phi.

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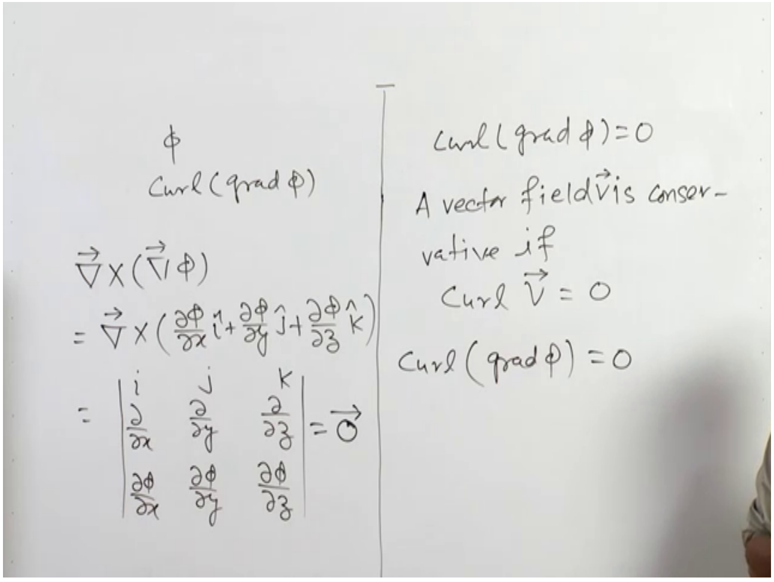

So, we have taken an example, a particular example. Now, let us see in general what these things are. So, curl of grad phi. So, basically curl of grad phi is given as del cross del phi. So, this will be del cross del phi over del x i plus del v over del y j plus del v over del j k. So, this will become i j k del by del x del by del y del by del z, and then del phi over del x j component is del phi over del y del phi over del z.

Now if I compute this thing the i component will become del 2 phi over del y del z minus del 2 phi over del y del z. So, 0 j component will become again 0 and k component will be 0. So, it comes out to be 0. So, hence in the previous lecture I have directly written it 0. So, what I want to say that curl of gradient of a scalar function will be always 0. So, if curl of grad phi equals to 0, then can we relate it with some other concept. Yes we can relate and what is that concept. That particular concept is about potential function or conservative vector field.

So, a vector field is conservative if or let us say vector field v curl of v equals to 0. Why I am writing it, because if v is conservative then I can write v as the gradient of a potential function. So, that is basically gradient of phi and curl of gradient of phi. We have just shown that it is always 0. So, this is another way of testing whether a given vector function is conservative or not. Instead of finding the potential function phi what you can do?

You can just calculate the curl of that vector field and if it is 0, then the vector function is conservative. Another thing which we have calculated in the previous lecture was the divergence of grade of phi and this I have calculated for a particular example in the previous class. Now, let us see what is this

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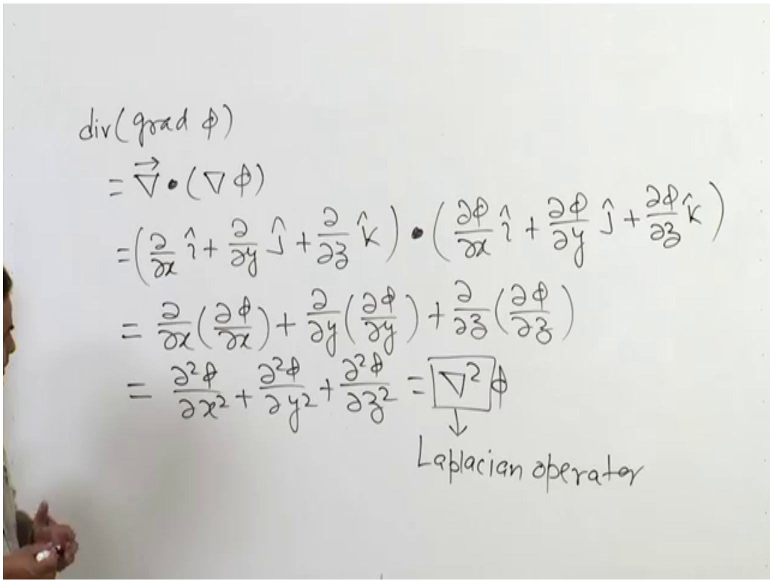

This is basically del of vector of dot product with del phi. So, this become del by del x in the direction of y del by del y in the direction of j, del by del z in the direction of k and dot product with del phi over del x in the direction of i plus del phi over z. Sorry del y in the direction of j plus del phi over del z in the direction of k. So, this basically in the second down bracket we are having del phi; that is gradient of phi.

Now, if I do the dot product of these two vectors it will become del by del x over del phi over del x plus del over del y del phi over del y plus del by del z del phi over del z and this comes out to be del square phi over del x square plus del square phi over del y square plus del square phi over del z square.

And this I can basically I can denote with this notation this particular operator is called Laplacian operator. So, basically Laplacian operator is del square over del x square plus del square over del y square plus del square over del z square and denoted by this. Now the equation del square phi equal to 0 is called Laplace equation, and it is a famous partial differential equation.

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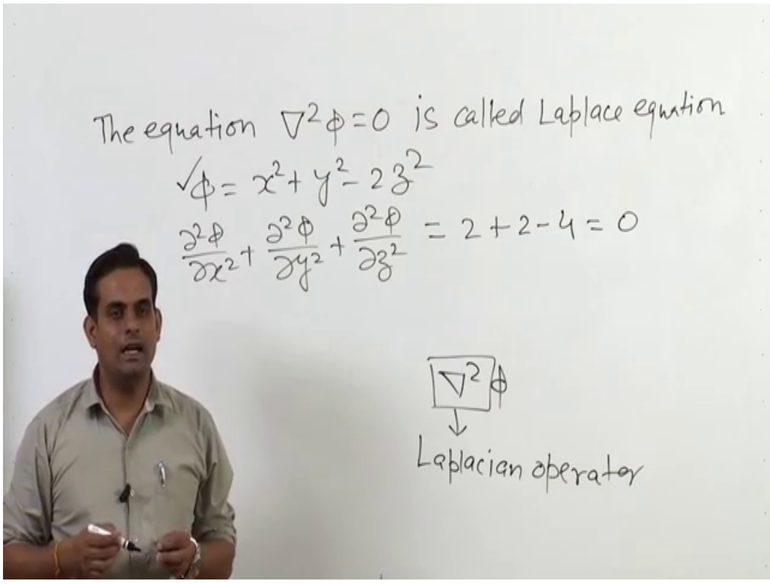

Can you tell me a function which is satisfied this equation. So, so here we can easily construct such a function. For example, if I take phi equals to x square plus y square minus twice of z square, then del 2 phi over del x square plus del 2 phi over del y square plus del 2 phi over del z square which is basically Laplacian operator operating on phi.

So, this term is 2 plus second order partial derivative of this phi with respect to y is 2 minus 4 and this comes out to be 0. Hence, this particular scalar function phi satisfy the Laplace equation. After this we will take few more identities on divergence and curl. So, with the these two previous properties the next identity is, let me write as i 1.

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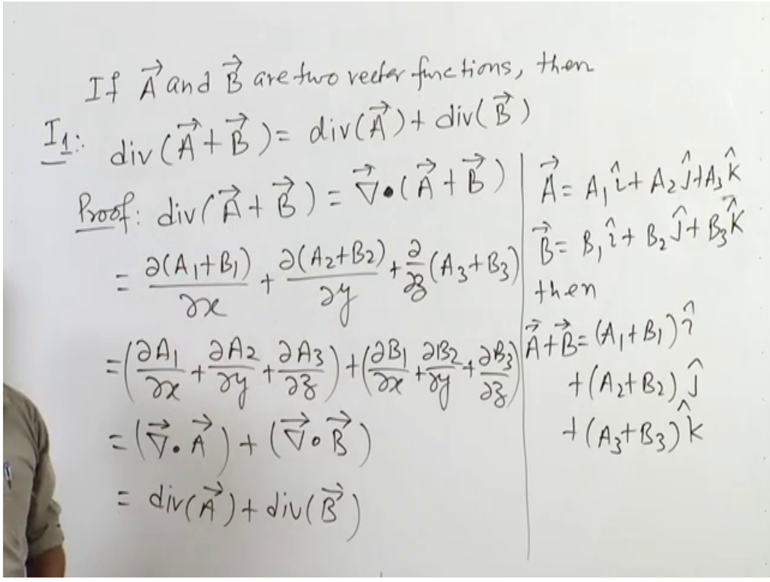

So, it is saying if let me write A and B. So, if A and B are two vector functions then, or let me write like divergence of A plus B equals to divergence of A plus divergence of B. Let us see the proof of it. So, divergence of A plus B will become del operator dot product with the sum of these two vector functions. So, we need to prove that divergence of vector A plus B equals to del dot A plus B.

Now if vector a is having component like A 1 i plus A 2 j plus A 3 k and vector B is having component as noise] B 1 i plus B 2 j plus B 3 k. Then as you know that the sum of these two vectors will be having A 1 plus B 1 is the i component plus A 2 plus B 2 as the j component plus A 3 plus B 3 as the k component. So, now, divergence of A plus B will become del by del x of phi component plus of phi component, it mean j component and del by del z of k component.

So, this since the partial derivative operator is a linear operator. So, I can write del A 1 over del x plus del A 2 over del y plus del A 3 over del z plus del B 1 over del x del B 2 over del y plus del B 3 over del z and this is nothing just dot product of del operator with vector function A plus the second square bracket is round bracket, is the dot product of the second round bracket is dot product of del operator with vector B A vector function B and this is divergence A plus divergence B.

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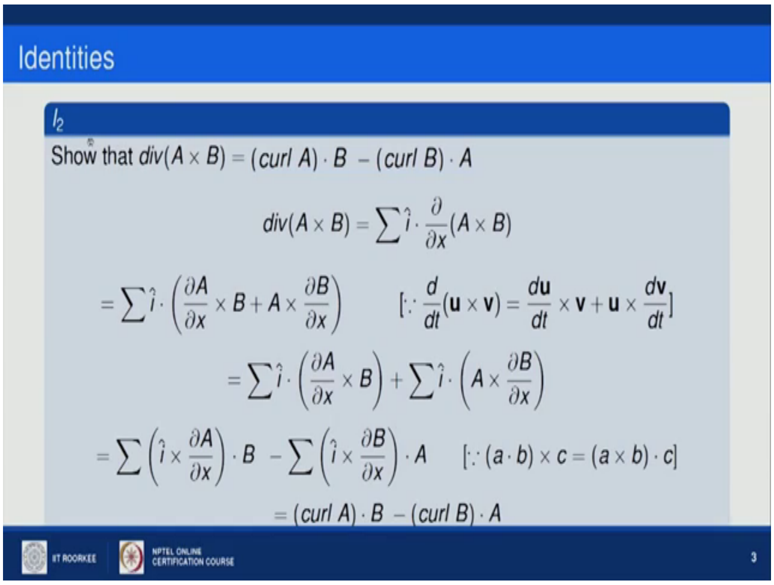

This is what we need to prove the next identity which are going to prove, is show that divergence of vector A cross vector B, the cross product of A and B equals to curl A dot B minus curl B dot A. So, here divergence of A cross B written as in the short form and what is this short form. The short form is saying that the i component of this divergence can be written A in this way. So, if I take the dot product with i component, I will get del by del x of A cross B and since this is here. So, similarly we can extend it; the components of j and component of k.

Means it will become plus j dot del over del y over A cross B plus k dot product with del by del z over A cross p. So, in this way, so this is the short notation. Now, del by del x of A cross B, if I am having B by d t of u cross v, I can write it d u or d t cross v plus u cross d v over d t. So, applying this property here I will get disturbed del by del x of A cross B as del over del x cross B plus A cross del v over del x. Now, I can write it in the term, in the sum of two different terms. So, first term will become i cap dot del A over del x cross B.

So, up to here plus I kept A cross del B over del x. Now we are having this particular property of vectors that a dot b cross c can be written as a cross b dot c. So, in this way, this particular term I can write as i cross del A over del x dot B, and this I have taken the minus, because I am taking inside i cross del v over del x. So, I am changing the order of this A cross del B over del x.

So, I have written i cross del v over del x dot product with A. Now what is this? This is curl of A dot B minus curl of B dot A and this way we have done the proof of this particular identity. The next identity again a bit tricky proof and it is saying that. So, that curl of cross p like in the earlier one, we are having divergence here.

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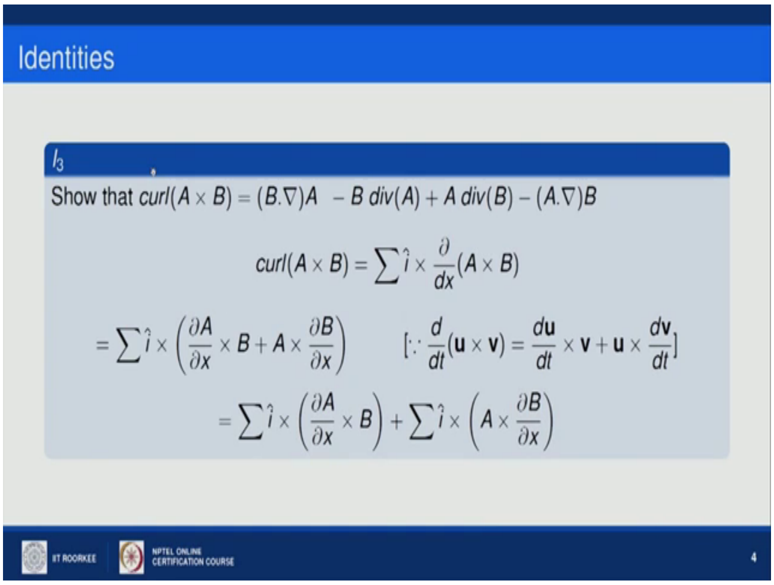

Now, we are taking curl here equals to B dot del A minus B, divergence A A divergence B minus A dot del B. Please note that these two square round bracket term B dot del and A dot del, these are basically operators and how I can define these operators. So, if my B is, let us say B 1 i plus B 2 j plus B 3 k del operator is define as del by del x del by del y del by del z k.

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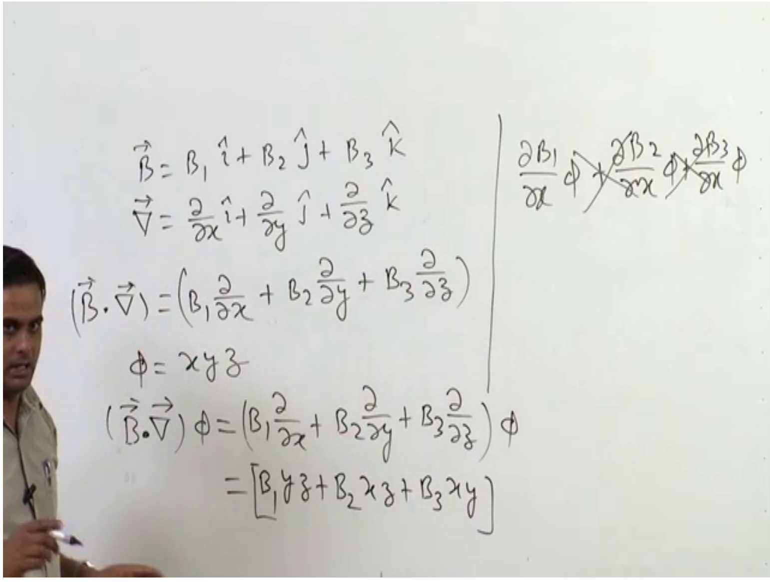

So, now, B dot del is in operator and it is something B 1 del by del x. Please do not write it del B 1 over del x no plus B 2 del over del y plus B 3 del by del z. For example, if one my some a scalar function is phi. So, let us say phi x y z, then B dot del phi will become B 1 del by del x plus B 2 del by del y plus B 3 del by del z phi ok. So, it will become B 1 del phi over del x. So, now, this B 1 del by del x.

Operating on phi; So, it will become B 1 y z plus B 2 x z plus B 3 x y. So, it will be a scalar quantity; this operator operating on a scalar function. As I told you do not write it del B 1 over del x. If you write it del B 1 over del x, then del B 1 over del x into phi plus del B 1 B 2 over del x into phi plus del B 3 over del x into phi. These two teams are quite different here, you are operating partial derivatives on phi. Here you have made a mistake, because you have done the partial differentiation of B 1 B 2 and B 3.

So, this is wrong. So, basically you cannot write it del B 1 over del x plus del B 1 over del y del B 3 over del z. please be careful about this operator. So, now, I need to prove that curl of A cross B v dot del operator operating on A minus B divergence A plus A divergence B minus. the operator A dot del operating on B. Now if I take A curl of cross B in short notation, I can write it some i cross del by del x of A cross B.

Now again using the same property which we have used in the earlier example that d by d t of u cross v equals to the d over d t cross v plus u cross d v over d t, this can be written as i cross del over del x cross B plus A cross del B over del x. Now again like in the earlier identity I will break it in two terms; one is i cross del A over del x cross B plus i cross A cross del B over del x.

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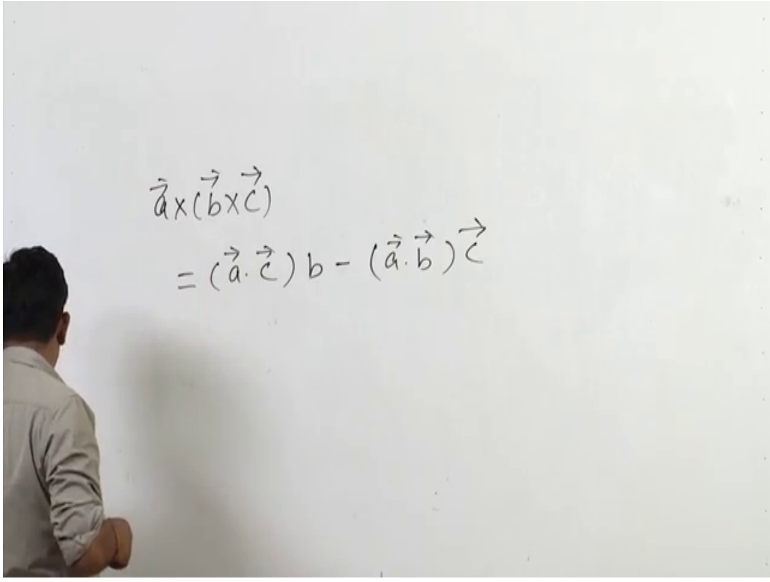

And now I will use the property of vectors which is saving say mediate a cross b cross c equals to a dot c into b minus a dot b into c, because here in the identity which I am doing, I am having this kind of two terms; So, using this property. So, using the property just I mentioned I can write these two terms.

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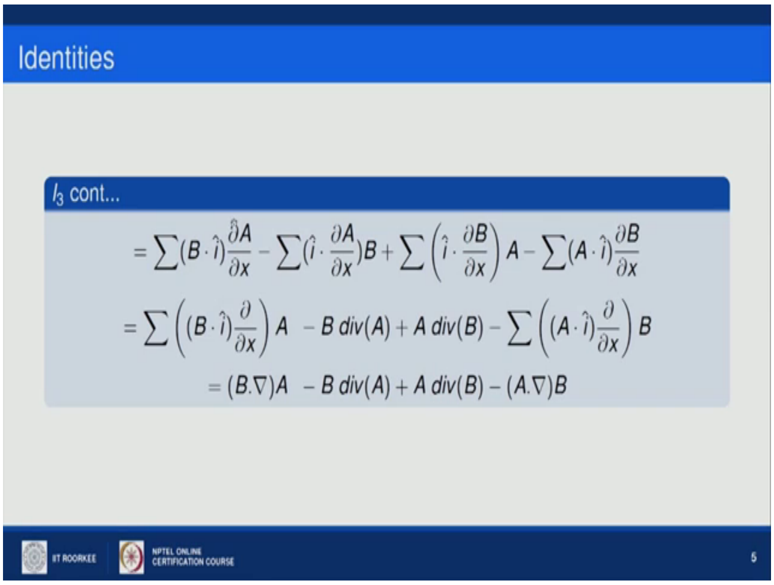

In this way and finally, B dot i del del A over del x, I can write B dot i del by del x at a, because v dot i will be the component of B in the direction of i. So, this i dot del A over del x into B I can write, because it is nothing just divergence of a. So, I can write B times divergence of a. This i dot del B over del x is A times divergence of B.

And finally, this A dot i del B over del x, i can take del y del x inside the bracket and B i can take out. So, this is B dot del operating on A minus B divergence A plus A divergence B minus A dot del operating on B and this is what we need to do my next identity is show that gradient of A dot B equals to B dot del plus A dot del B plus B cross curl A plus A cross curl B.

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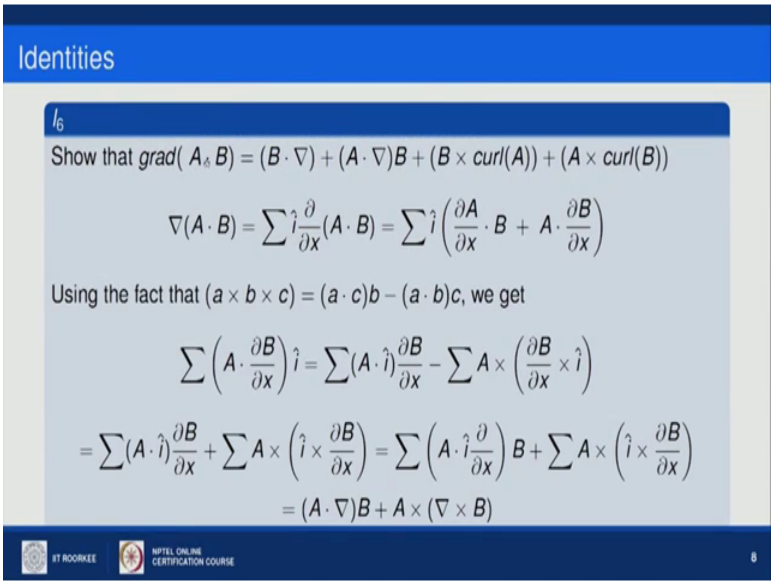

So, sorry A is missing here. So, it is B dot del operating on a, so gradient of A dot B. In the short I can write it i del by del x operating on A dot B. So, it will be summation i del A by del x dot B plus A dot del B by del x. Now using the fact that a cross b cross c can be written in this way we get, sorry this bracket should be here a cross b cross c. So, I can write it in this way A dot del v over del x i can be written in this way; that is this term can be, this term can be written as this minus this one.

So, this term I am taking this side and this term I have taken in the right hand side. So, A dot del v over del x i can be written as A dot i del B over del x minus A cross del v over del x cross i. So, this way I can write this A dot i del B over del x minus A cross i cross del B over del x equals to A dot i del by del x. So, del by del x I have taken inside, like we have done in the earlier identity plus A cross i cross del B over del x. And this I can write as A dot del operating on B plus A cross del cross B. So, this I am getting from this term which is this one the second term. Similarly, I will get from the first term which is given as B dot del A over del x i, I can write in this way.

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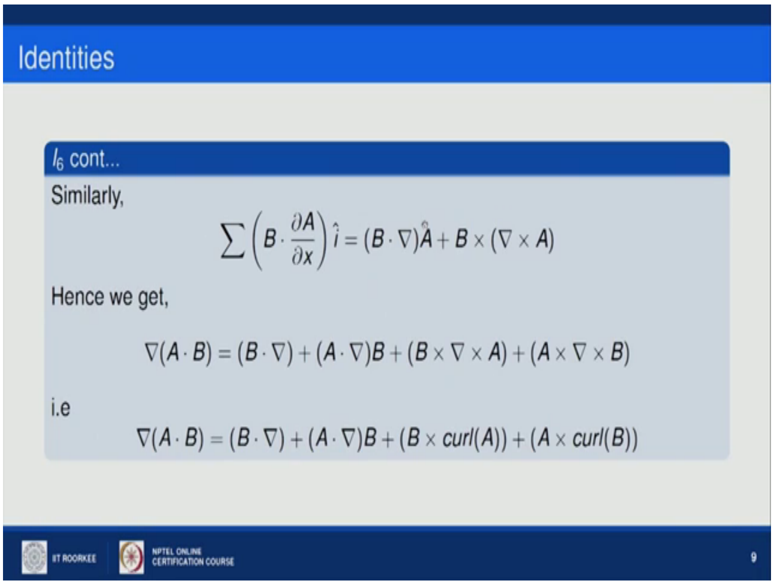

Hence, adding these two I will get the required identity. So, in this particular lecture we have seen some identities on divergence and curl, and we have seen about the Laplacian operator and which is nothing just divergence of gradient of a scalar function. We have also seen that a conservative vector field will be having curl 0.

So, with this I will end this lecture and we will learn a new concept; that is a new type of integral that is called line integral in the next lecture.

Thank you very much.