Multivariable Calculus
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Lecture – 32
Gradient (Identities), Divergence and Curl (Definitions)

Hello friends. So, welcome to the 32nd lecture of this course and in this lecture I will talk about few identities related to gradient, and then I will introduce the concepts like divergence and curl for the given vector functions. So, let us take a look of few identities on the gradient of a scalar function and let us assume that.

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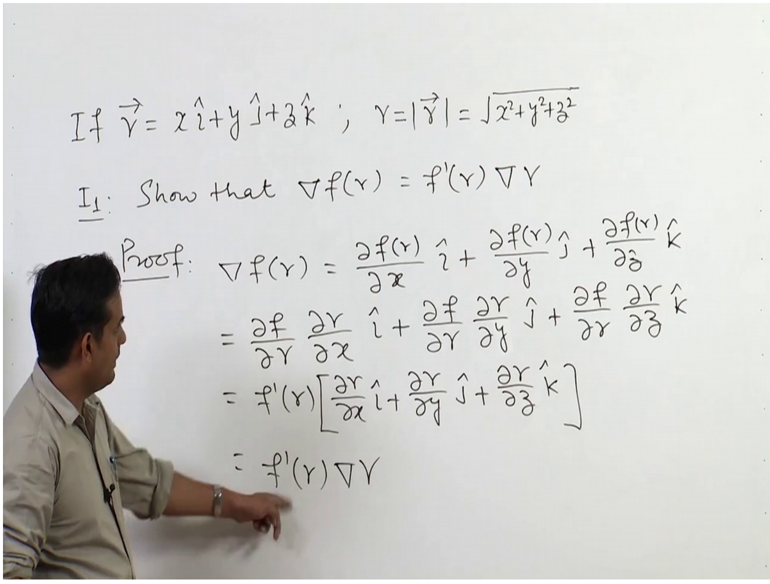

We are having a vector r which is given by x i plus y j plus z k. So, here let us say r is giving the magnitude of this vector. So, this r cap r arrow is a vector function, but this r is the magnitude. So, it will be a scalar function and it is given by magnitude of vector r. Now, my first identity which I am going to prove is show that, gradient of f of r. So, here f is another function which is a function of r equals to f dash r multiplied with gradient of r so, let us try to obtain it.

So, del of f r will become del f r over del x in the direction of i cap, del f r over del y in the direction of j cap plus del f r over del z in the direction of k cap. Now, f is a function of r, and r is a function of x y and z. So, del by del x of f r can be written as del f over del r into del r over del x. Plus del f over del r into del r over del y plus del f over del r into del r over del z. .

Basically, f is a function of r so, del f over del r is nothing just f dash r. So, if I take this common, what I will be having del r over del x i, plus del r over del y j plus del r over del z k. So, this will be f dash r and what is the term in square bracket? Bracket that is nothing just gradient of r and this is what we need to prove in this identity.

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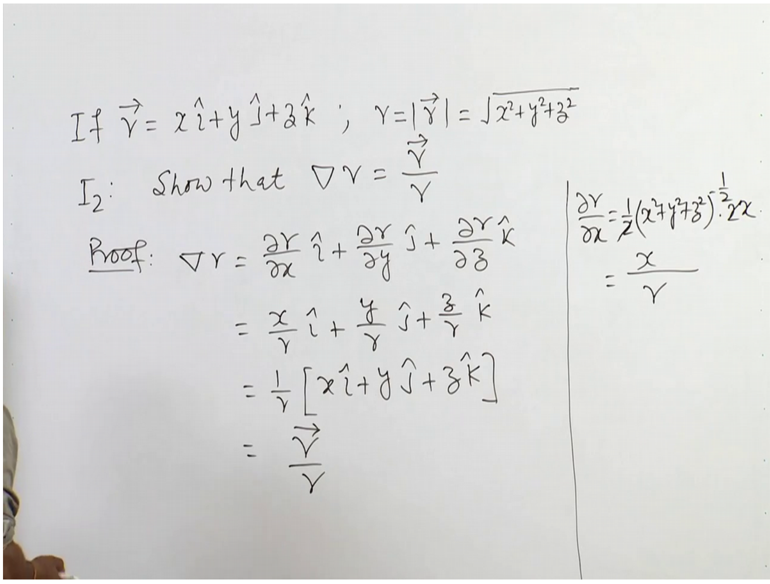

Let us take the second identity on this. So, here I need to show that, del of 1 upon r or let us take this first del of r equals to the vector r upon r. Let us try to obtain it so; I need to find out del r. So, del r will become del r over del x, del r over del y, del r over del z k. So, here if you see r is square root of x square plus y square plus z square. So, if I find out del r over del x here it will be 1 by 2, x square plus y square plus z square, raised to power minus half into twice of x. So, this two will be cancel out so, I will get x upon x square plus y square plus z square raised to power half, which is nothing my r. So, basically del r over del x is x upon r. Similarly, del r over del y will become y upon r and del r over del z will become z upon r.

So, put these value so, x upon r i cap, y upon r j cap plus z upon r k cap or if I take 1 upon r i out I will get x i, plus y j. plus z k and that is the vector r upon r, this is what we need to prove ok.

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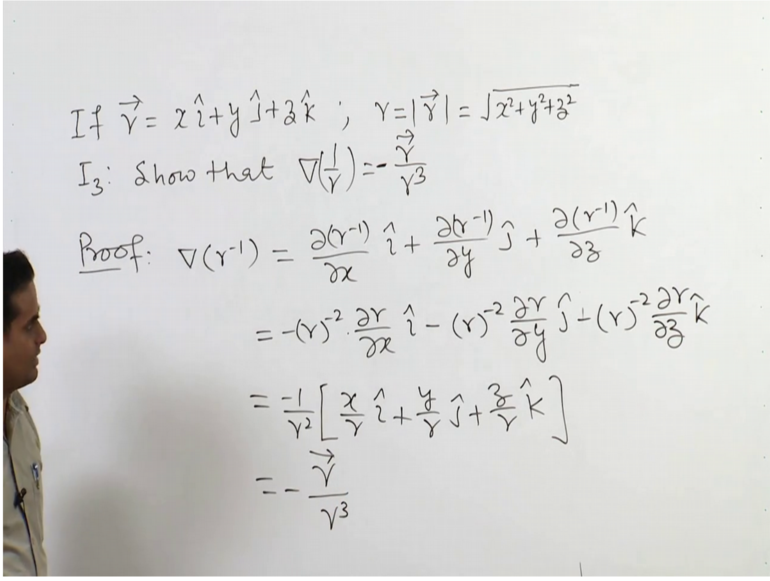

The third identity is given as the gradient of 1 upon r equals to vector r upon r cube. So, let us try to obtain the proof of this identity. So, here del of 1 upon r. So, basically del of r inverse. So, del of r inverse will be. So, I can write in this way. Now, it will become minus r raised to power minus 2 into del r over del x i plus and then it will become minus. So, again r raised to power minus 2 del r over del y in the direction of j. And, similarly minus r raised to power minus 2 del r over del z in the direction of k. Sorry, it is minus here this equals to minus of this. So, now if I take minus or let me write in this way minus 1 upon r square I take out.

So, this I will be having del r over del x, which is x upon r we have seen in the previous identity, plus this I have taken out. So, y upon r j plus z upon r k and that is minus if I take 1 upon r out. In the denominator I will be having r cube and then it will become x i plus y j plus z k which is your vector r that so let us take one more identity for the same vector r.

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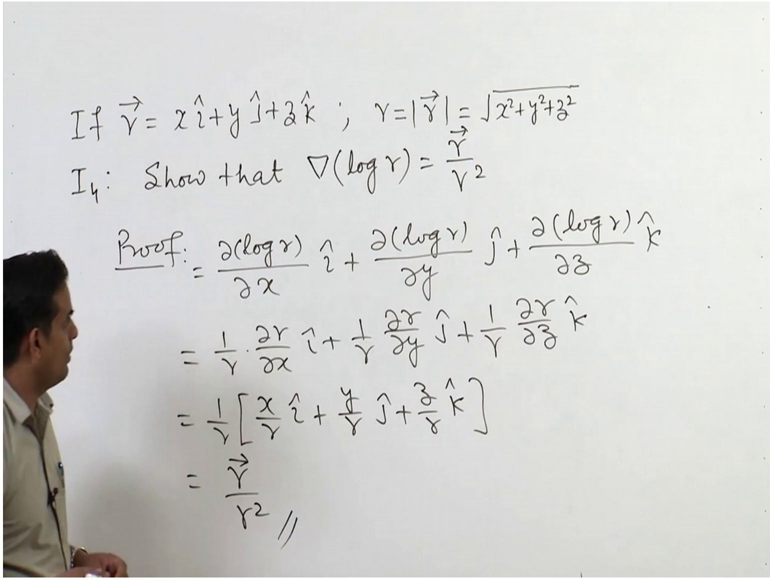

So, here the identity is show that, gradient of log r equals to it should be vector r upon r square. So, again we will follow the same process. So, del of log r so, it will become del ok. So, this is del of log r left hand side, this equals to 1 upon r into del r over del x, 1 upon r del r over del y j cap plus 1 upon r del r over del z k cap. I can take 1 upon r out, so it will become del r over del x, x upon r i plus y upon r j, plus z upon r k. And, then it is nothing, but vector r upon r square because this 1 upon r you can take out and, in the bracket you will be having x i plus y j plus z k. So, this is the end of the proof and let us take the final identity, on this track.

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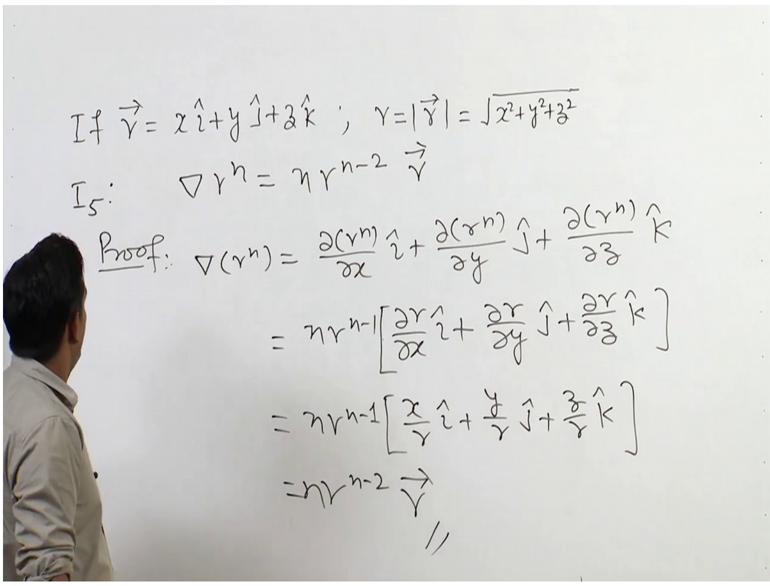

So, it is saying that the gradient of r raised to n can be written as n r n minus 2 times vector r.

So, again we will go with the same process. So, del r raised to power n this will become ok. So, now let us do it, so it will become n r n minus 1 into del over r over del x i plus because this term will coming each so I have taken out this will become del r over del y j cap plus del r over del z k cap. .

Again, the same process which you have done in the previous three cases del r over del x will become x upon r, y upon r j, plus z upon r k. And, this is nothing n times r raised to power n minus 2 because n minus 1 here, 1 upon r we have taken out into vector r and that is the end of this proof. So, we have taken these five identities on the gradient, for a given vector r which is x i plus y j plus z k and we have done proof of all these ok.

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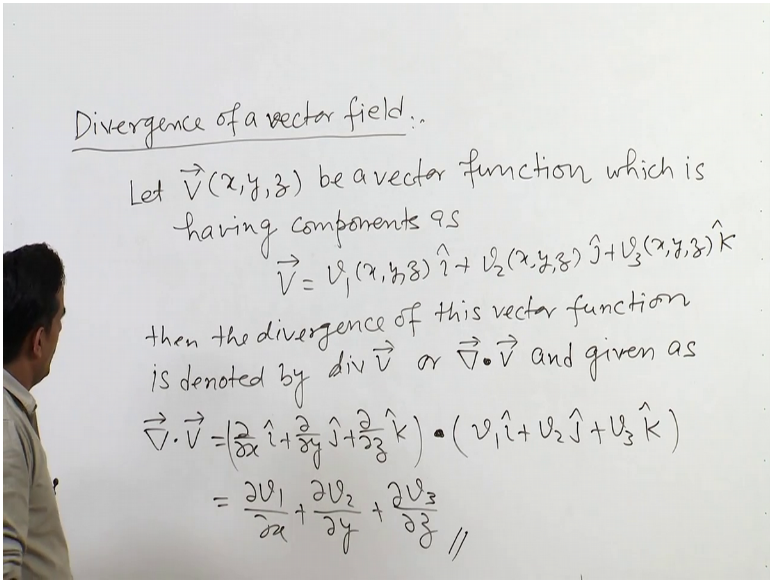

Now, our next definition is, divergence of a vector point function or vector field. So, let a V arrow x y z be a vector function, which is having component has. So, v equals to v 1 x y z i plus v 2 x y z j plus v 3 x y z k. Then, the divergence of this vector function is denoted by divergence of V or the dot product of V it del of vector. And given as, as you can understand from this particular, notation it is the dot product of del of vector with vector v. So, del of vector is del by del x in the direction of i del over del y in the direction of j plus del over del z in the direction of k dot product with v and v is v 1 I, plus v 2 j, plus v 3 k.

So, this will become del v1 over del x plus del v 2 over del y plus del v 3 over del z. So, basically we can say the divergence of a vector function is the sum of their components with respect to partial derivatives of their components with respect to x y and z respectively means, del v 1 over del x plus del v 2 over del y plus del v 3 over del z. And, physically it represent the flux in per unit volume means if a material, is moving through for per unit volume then, the divergence will measure the flux of that means, in flow minus out flow in per unit volume. Let us take some example how to find out it ok. So, let us take an example.

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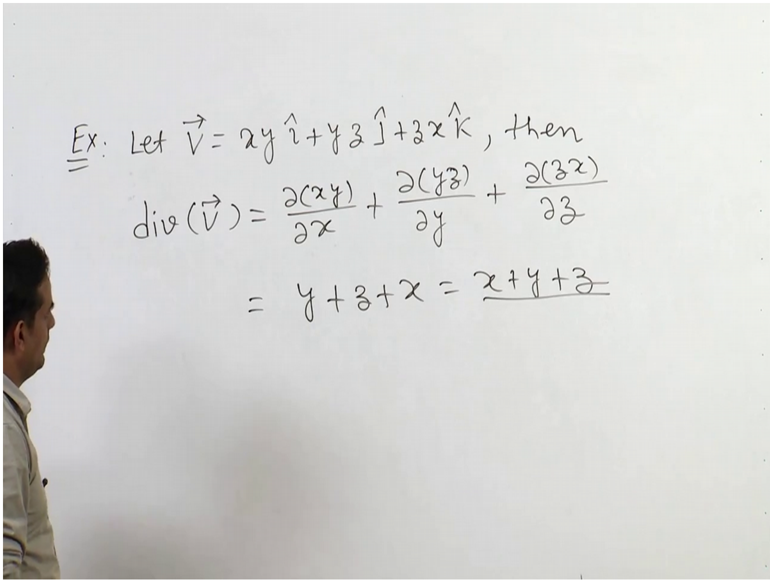

Let V equals to x y i plus y z j plus z x k then, divergence of v is this quantity. Because, this is del of vector dot v and this comes out to be y plus z plus x that is basically, x plus y plus z. So, you can calculate this in very easy manner. Moreover, the another point you can note down that divergence of a vector function is a scalar quantity it is a scalar function, not quantity.

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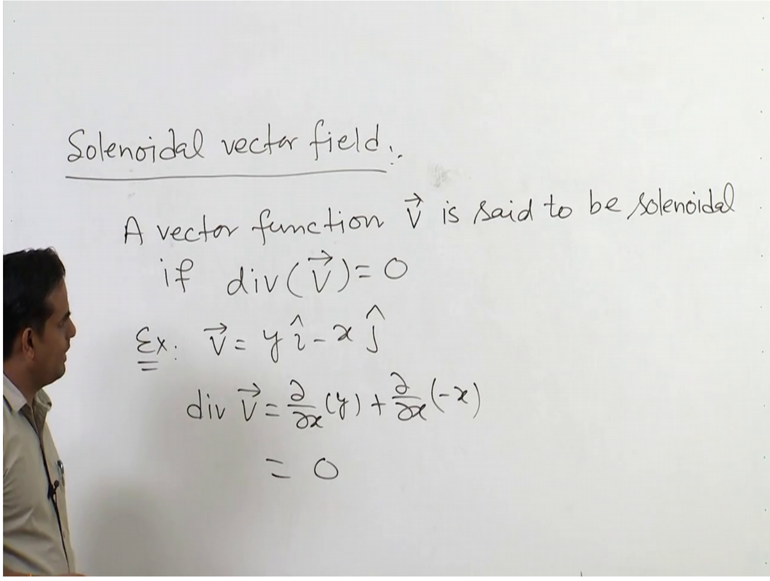

Another termination let me take Solenoidal vector field. So, a vector function, solenoidal if divergence of V equals to 0. And, the physical example or practical example of it is the magnetic field must be solenoidal. Another example if you want to take let us take v cap as y i minus x j then divergence of v will be del by del x of y plus del by del x of minus x and which is obviously, 0 ok. So, you can make other examples based on this particular definition.

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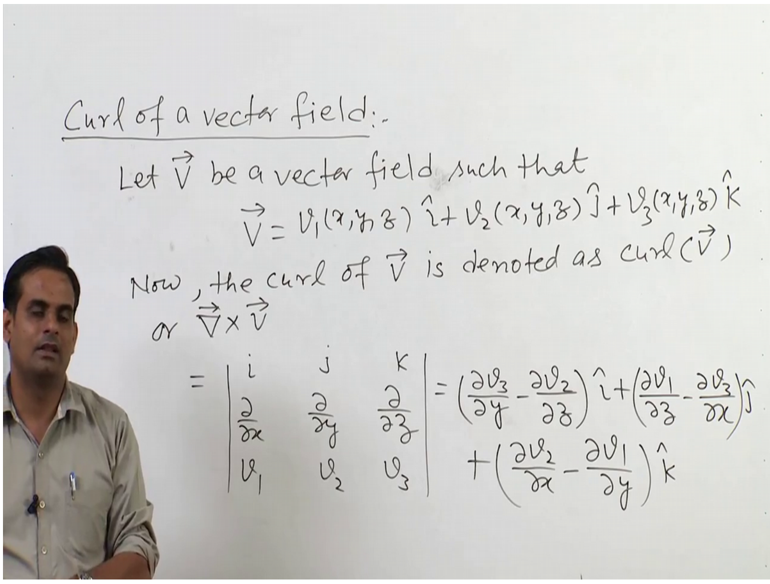

My next definition is curl of a vector filed. So, again let V be a vector field such that, it is given as V equals to v 1 x y z i plus v 2 x y z j plus v 3 x y z k. Now, what we are having? Now, the curl of vector function V is denoted as, curl of V or cross product of del of vector and V. And it is given as, as I told you cross product of del of vector and v. So, it will be something i j, k del by del x del by del y del by del z v 1 v 2 v 3.

So, let us calculate it. So, i component will become, del v 3 over del y minus del v 2 over del z plus, now we will calculate j component. So, it will be del v 1 over del z minus del v 3 over del x plus k component will be del v 2 over del x minus del v 1 over del y. So, this is the curl of v. So, again you can notice the curl of a vector field is again a vector function while the divergence of a vector function is a scalar function.

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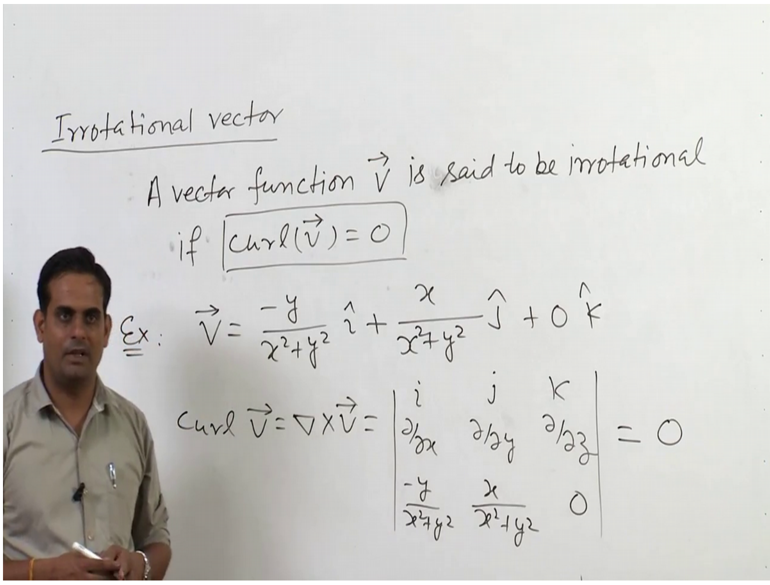

And, the gradient of a scalar function is a vector function. My next definition is irrotational vector. Basically curl of a vector represents its verticity. So, here from there I can define the irrotational vector. So, a vector function V is said to be irrotational, if its curl is 0. For example, take a vector function as, minus y upon x square plus y square i, x upon x square plus y square in the direction of j and plus 0 k. So, if I find out curl of this vector so, this will be del cross V and it will become i j k del by del x and then, minus y upon x square plus y square x upon x square plus y square and 0. And, when I do it, it comes out to be 0. Hence, it is a irrotational vector field.

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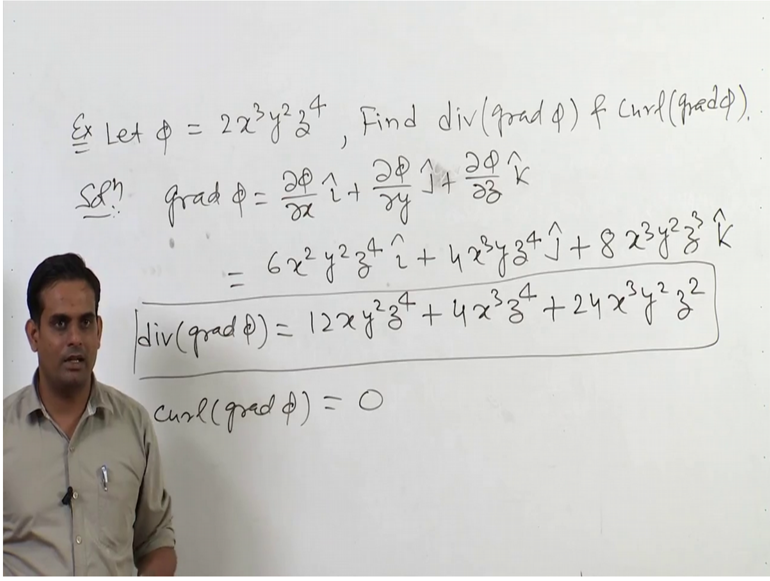

Now, take one more example. let, phi is given as 2 x cube plus y square into z raised to power 4, find divergence of grad phi and curl of grad phi? As you know phi is a scalar function so, grad phi will become a vector function.

So, here grad phi will become del phi over del x i plus del phi over del y j plus del phi over del z k. So, here what I will do? So, it will become del phi over del x. So, 6 x square y square z raised to power 4 i plus del phi over del y j. So, it is basically so, it will become 4 x cube y z raised to power 4 j, plus 8 x cube y square z raised to power 3 j sorry k. Now, divergence of this so, divergence of grad phi will become 12 x y square z raised to power 4 plus 4 x cube z raised to power 4 plus the that is, del of grad phi over del z. So, it will become 24 x cube y square z raised to power 2. So, this will be the divergence of grad phi.

Now, we will find out the curl of grad phi, and this will comes out to be 0. Why it will be the 0? That we will discuss in the coming lecture. So, with this I will close this particular lecture. So, in this lecture we have done few identities is on gradient, and then we have seen the definition of divergence and curl for a given vector function.

Thank you very much.