Multivariable Calculus

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Lecture – 31
Normal Vector and Potential Field

Hello friends. So, welcome to the thirty first lecture of this course, and in this lecture we will learn few more applications of gradient. Basically we will continue from the last lecture. In the last lecture, I was telling about directional derivative. And directional derivative means the rate of change of a function at a given point in the direction of a given vector.

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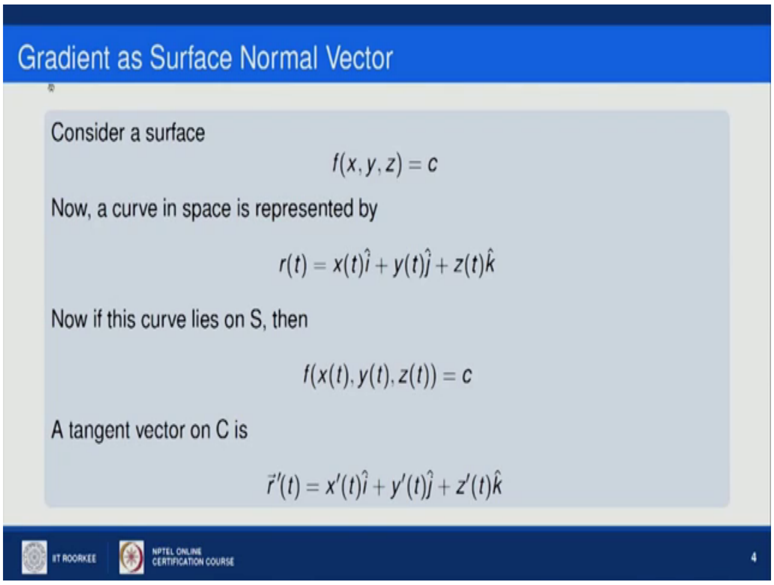

Now, the next application of gradient is the surface normal vector. So, consider a surface f of x, y, z is equals to c. So, for different of values of c, where c is a constant. So, for different value of c, it will be level surface. Now, a curve in the space is represented by this particular vector, that is, vector r t is equal to x t i plus y t j plus z t k. So, this is the parametric representation of the curve. Now, if this curve lie on this surface f of x, y, z equals to c then f of x t, y t, z t will be equal to c now a tangent vector on c is given as r dash t equal to x dash t i plus y dash t j plus z dash t k.

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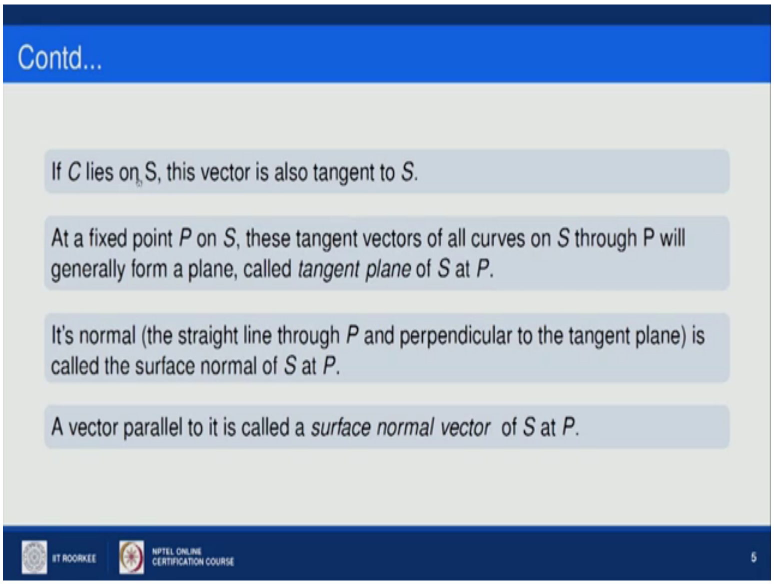

So, if C lies that is the curve, r t lies on S, this tangent vector will also be a tangent vector to surface S. At a fixed point P on the surface S, these tangent vectors, because there will be infinitely many tangent vectors, because at a particular point on the surface, infinitely many curve will be passing. So, the, these tangent vector of all curves on S through P will generally form a plane, because they will be in a plane, and that particular plane is called tangent plane of S at P.

Now, what we need to do? We need to find out, the normal vector to this tangent plane means, a vector which is perpendicular to all those tangent vectors. So, any vector parallel to that particular normal vector is called the surface normal vector of S at a point P. Now, how to calculate it? So, as I told you, the surface is given as, f of x t, y t, z t equals to c that is the constant.

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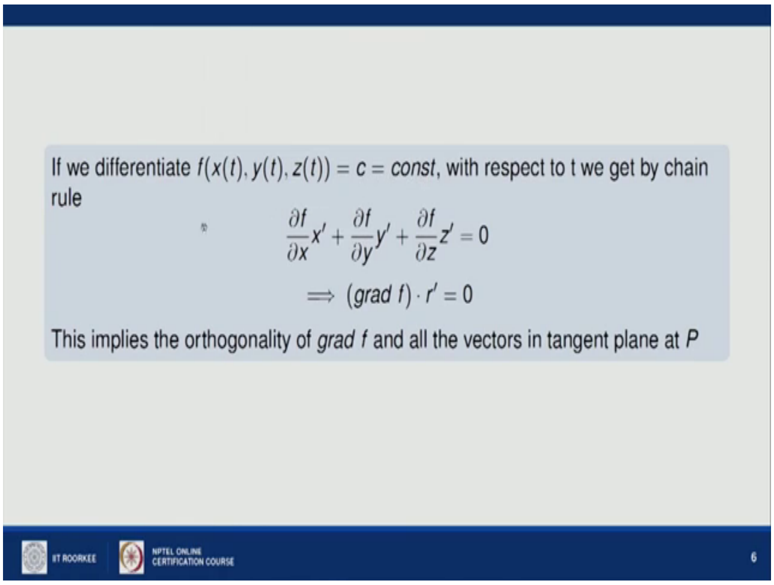

So, now differentiate it with respect to t by chain rule. So, del f over del x into del x over del t plus del f over del y into del y over del t plus del f over del z into del d z over d t equals to 0.

And if you see this, it is the dot product of grade f, because grade f will be del f over del x into i, del f over del y j cap plus del f over del z k cap dot r dash, r dash will be x dash i plus y dash j plus z dash k. So, I can write this equation in this form, in the form dot product of these 2 vectors. Now, r dash is a tangent vector. So, and this tangent vector is having dot product with grade f which is 0. So it means, grade f is perpendicular to r dash. So, this implies the orthogonality of grade f, and all the vectors in the tangent plane at P.

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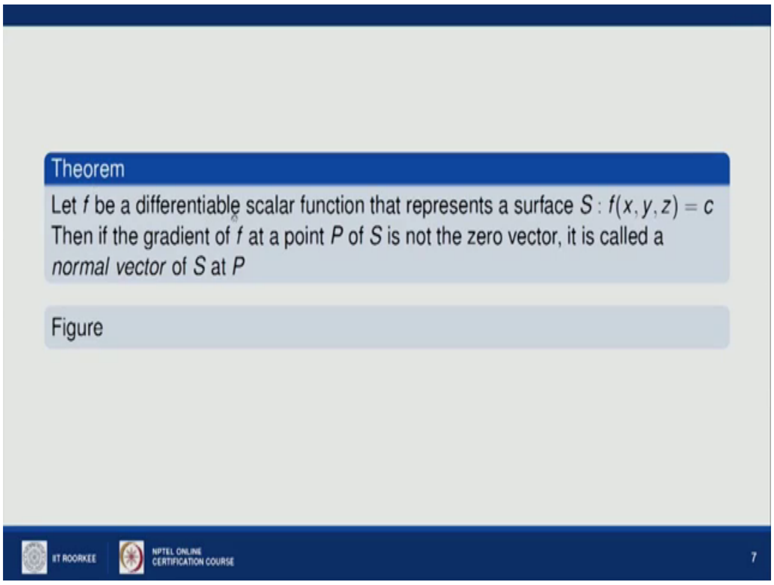

Now, based on this fact we can have this result. So, let f be a differentiable scalar function, that represent a surface S, that is, f of x, y, z equals to c then, if the tangent, gradient of f at point P of S is not zero, means gradient is non zero, then this gradient vector is called normal vector of S at P.

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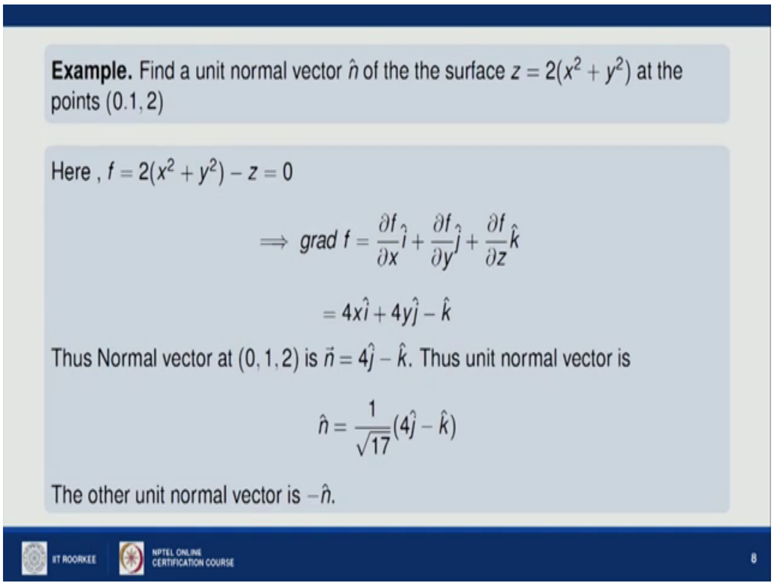

So, let us take an example, based on this particular theorem. So, Find a unit normal vector n cap of the surface z equals to twice of x square plus y square at the point 0, 1, 2.

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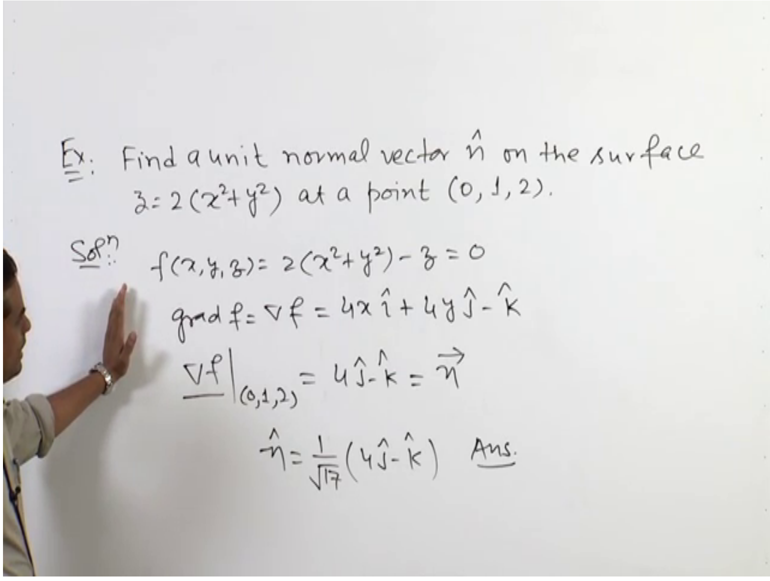

So, Find a unit normal vector n cap, on the surface, z equals to twice of x square plus y square at a point 0, 1, 2. So, the solution of this is given as, so we can write the surface as f of x, y, z equals to twice of x square plus y square minus z equals to 0. Now, we will calculate grade f, that is basically del f, and it will become 4 x i plus 4 y j cap minus k cap. Now this del f, that is the gradient vector at point 0, 1, 2 is given as, 4 j cap minus k cap. So, this vector will be the normal vector to the surface f which is given by this equation at this particular point.

Now, we need to find out unit normal vector. So, n cap will become, 1 upon square root 17, that is the magnitude of this vector into 4 j minus k, and this will be answer for this particular exercise. So, if someone ask you, to find out, the normal vector to a given surface at a point, the, you need to find out del f, you have to calculate del f at that particular point, and that will be the normal vector. If further, someone ask unit normal vector, you divide this vector by the magnitude of it, to make it as a unit normal vector.

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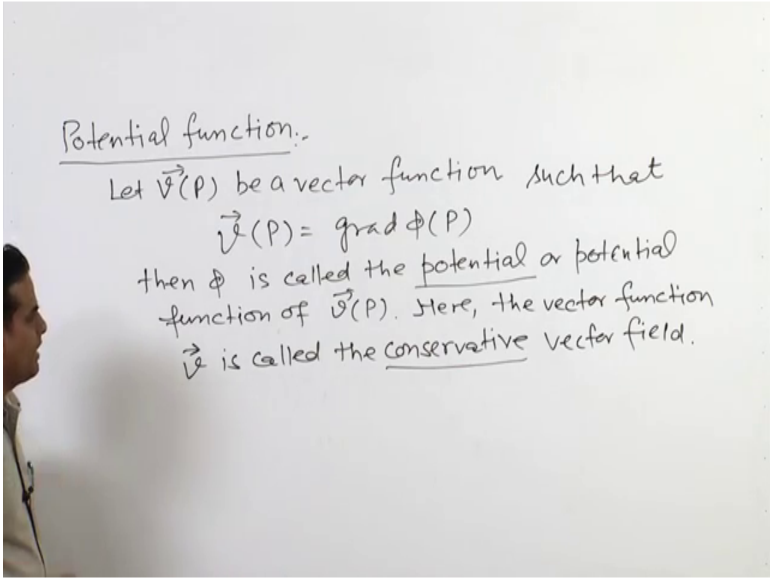

My next definition is Potential function. So, let V be a vector function in the domain d, and P be a point in this domain. So, it be a vector function such that, this V P equals to, gradient of a scalar function phi. Now, if this happens means if for a given vector V, we can write or we can find a scalar function phi such that, the vector function V equals to gradient of that scalar function phi, then the scalar function phi is called the potential or potential function of vector function V.

Here, if you are able to find out a potential function for a given vector function, then, the vector function V is called the conservative Vector field. So, in this way, we are having 2 terms here, one is potential and another one is conservative. And they are related to each other, if you are able to find out, the potential function for a given vector function, then vector function is called conservative. So, let us take some example on this definition, and let us learn how to find out potential function for a given vector function.

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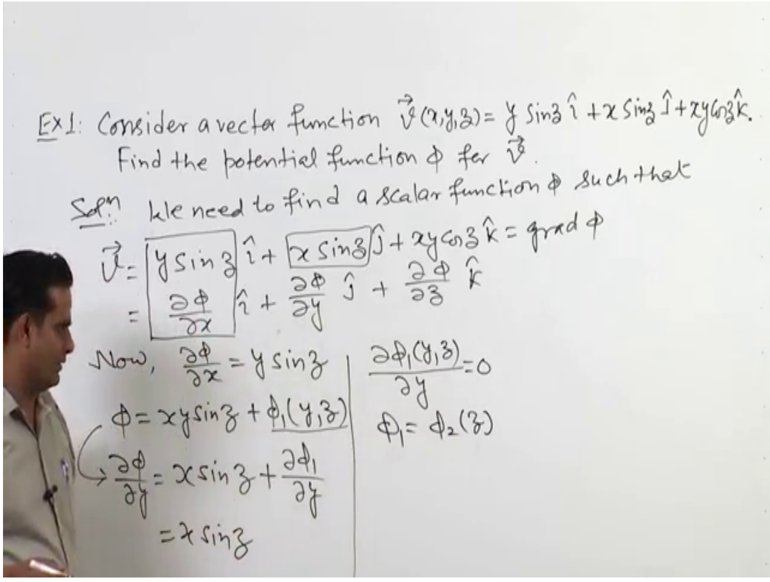

So, let us take first example on it. Consider a vector function V of x, y, z equals to y sin z i plus x sin z j cap plus x y cos z k. Find the potential function phi for V. So basically we need to find out, so we need to find a scalar function phi such that V, so which is y sin z i plus x sin z k sorry j plus x y cos z k equals to grad of phi.

And grad of phi means, del phi over del x in i direction del phi over del y be the component in j direction plus del phi over del z be the component in k direction. So, it means we need to find out a function phi, such that, del phi over del x equals to this quantity, del phi over del y equals to this x z sin z, and del phi over del z equals to x y cos z.

Now, by comparing the component along x i direction, for both of these vectors, I can write del phi over del x equals to y sin z. If I integrate it with respect to phi, sorry, with respect to x, then phi equals to x y sin z plus a function phi 1, which may be a function of y and z. So, this I got by comparing these 2 components. Now, if I calculate del phi over del y, from here means, I am differentiating it partially with respect to y. Then this will become del phi over del y, it will become x sin z plus differentiation of this partially with respect to y so, it will be del phi 1 over del y.

Now, here you can see that, del phi over del y equals to x sin z. So, it means this equals to x sin z from the given vector field. So, if I compare del, this and this, it gives me that del phi 1, which is a function of y and z over del y equals to 0. If this is 0, then phi 1 may be a function pi 2 of z, then only the, because there is no involvement of y in phi 1. So, it may define that, it is the function of only z that is why it is coming out to be 0. So, now substitute this value of phi 1 here, if I substitute this value of phi 1 here, then I can write that phi equals to x y sin z plus phi 2 of z.

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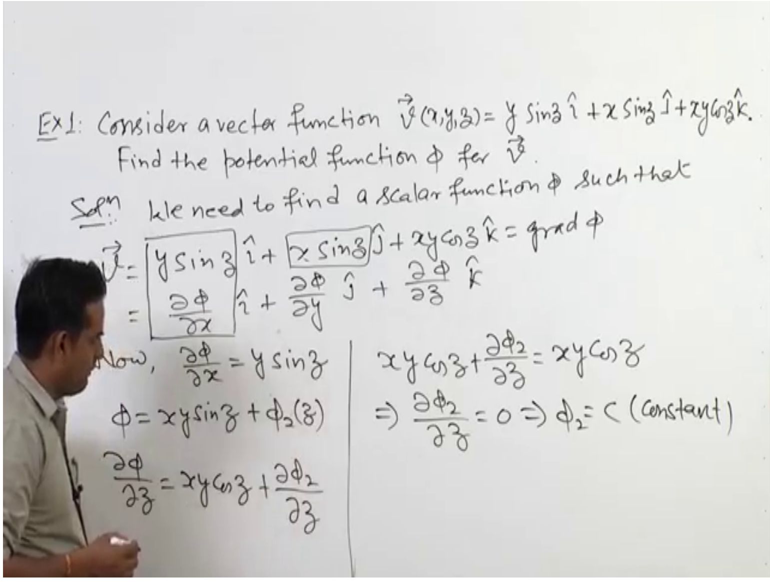

Now let us try to calculate this phi 2. So, we have compare this one, we have compared this one, we have make, made use of this function, particular component, only we did not make any use of component in the direction of unit vector k. So, let us try to do it.

So, calculate del phi over del z from here. So, del phi over del z will become x y cos z plus del phi 2 over del z. So, this is del phi over del z we are getting from this phi. One of the del phi over del z, we are getting from the given vector. So, let us compare these two.

So, it means x y cos z plus del phi 2 over del z equals to x y cos z. So, from here I am getting del phi 2 over del z equals to 0, it means, phi 2 is a constant function. So, we got the value of phi 2, which is equals to C C is a constant.

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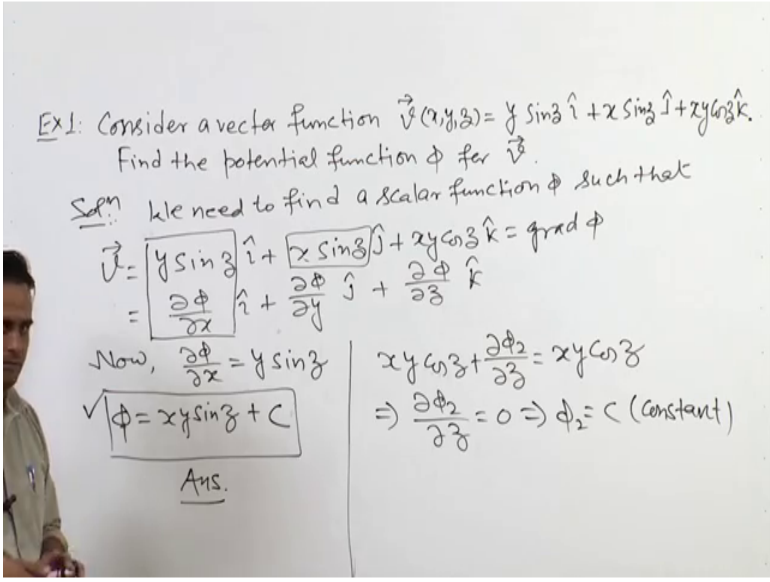

So, let us put this value of phi 2 here so, I get phi equals to x y sin z plus constant C, and this is my final answer. We can verify like this, from here what I can have x y sin z so, del phi over del x will become y sin z, del phi over del y will become x sin z, and del phi over del z will become x y cos z. So, in this way this is my answer, and this is the working procedure to find out a potential function phi for a given vector V.

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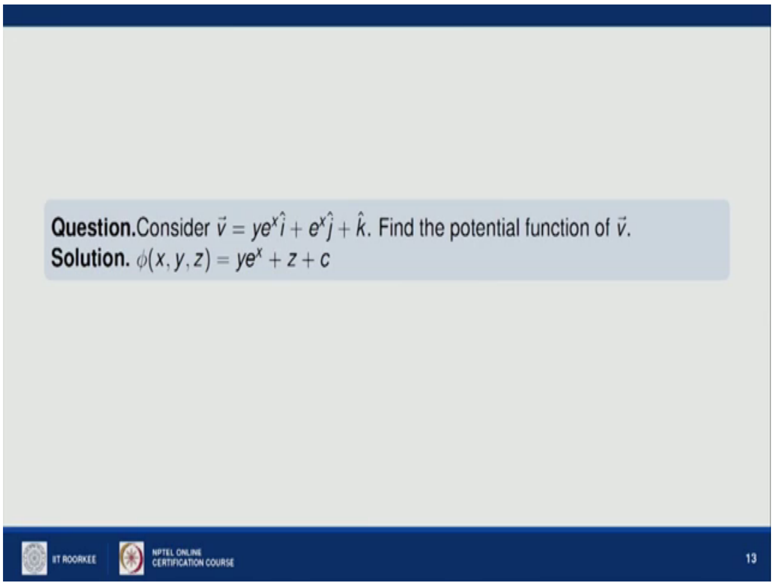

Let us take one more example. So, here I am taking the vector function a. So, let me write that complete example.

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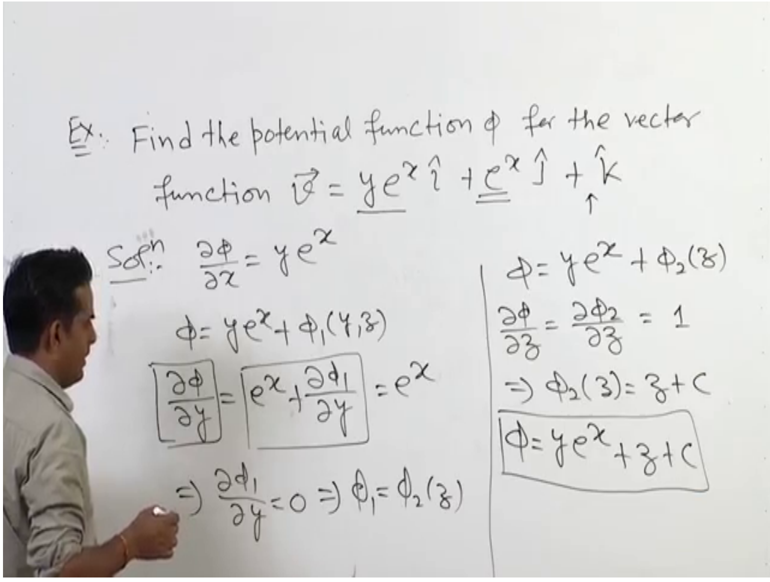

Find the potential function phi for the vector function V equals to y e raised to power x i plus e raised to power x in the direction of j plus k. So, one is there in the direction of k. So, let us do it as we have done in the previous example. So, here del phi over del x is y into e raised power x.

This component del phi over del x will be component of gradient phi in the direction of i. So, I am comparing these 2 components. So, integrating it, I will find out phi equals to y e raised to power x plus a function of phi 1 which may be a function of y and z. Now differentiate partially this phi with respect to y. So, I will get del phi over del y. So, which will become e raised to power x, from here plus del phi 1 over del y.

Now, compare this del phi over del y, which is given by this function with the j component of v. So, which is equals to e raised to power x. So, from here I am getting del phi over del 1 equals to 0, and from here, I can get phi 1 equals to some function of phi 2. So, in phi 2 z and constant are involved. Now, substitute this value of phi 1 here. So, I got, phi equals to y e raised to power x plus a function phi 2 of z. Now, let us make of third component that is the one in the direction of vector unit vector k cap. So, del phi over del z, this will become 0. So, del phi 2 over del z from this and here del phi over del z is 1, so this is equals to 1. So, from here I am getting phi 2 equals to or phi 2 of z equals to z plus a constant C.

So, now put this value of phi 2 in this phi. So, if I put it here, I got phi equals to y e raised to power x plus z plus C. And this is the potential function phi for which gradient of phi equals to this vector V, and you can verify it, if i differentiate it with, partially with respect to x, I got this, if I do it with respect to y, I got this, if I do it with respect to z, I got 1. So, we have taken 2 example, in one example we are getting the function in terms of constant, another one, we are involving one variable in terms of z.

Now, let us take one example where the potential function does not exist.

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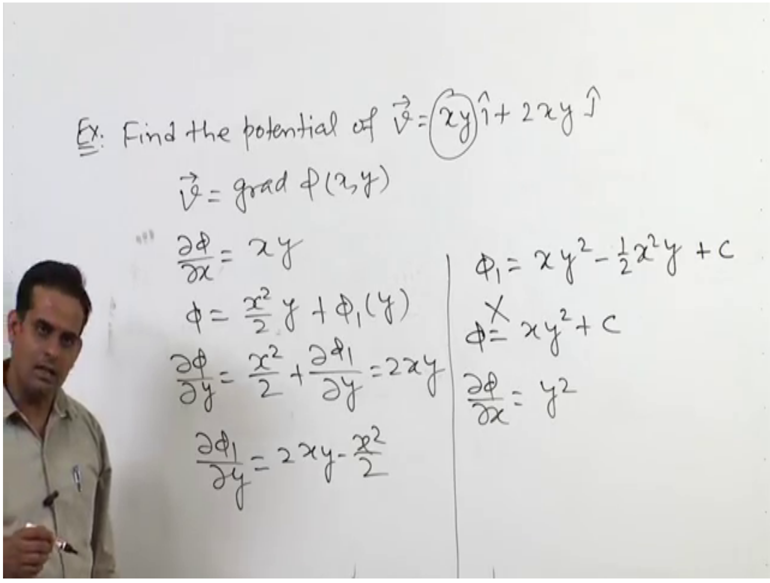

So, Find the potential function or simply potential of vector function V equals to x y is the component in the direction of i, and 2 x y is the component in the direction of j. So, here V is a function of x y. So, what I should f? V equals to gradient of phi x y. So, from here I am getting del phi over del x equals to x y; if I integrate it over x, so I will get phi equals to x square upon 2 into y, because it will become x x square by 2 plus a function phi 1 which is of y.

Now, if I calculate del phi over del y, then what I will get, x square by 2 from here, plus del phi 1 over del y. Now compare this, with this one. So, this is equals to twice of x y. So, from here del phi 1 over del y comes out to be twice of x y minus x square by 2. Integrate it with respect to y over y. So, I will get phi 1 is, x y square minus 1 by 2 x square y, plus some constant C. Now put this value of phi 1 here, so I will get phi equals to, here I am having x square by 2 y, here I am having minus half x square y, so this will be cancel out, I will get x y square plus C. Now, this is my phi which is I am getting according to my working procedure, but is it a correct answer? Let us see, if I calculate del phi over del x here. So, what I am getting? I am getting y square, however it should come out to be x into y.

So, it is not a phi, such that V equal to gradient of phi. And hence, potential function for this vector function does not exist.

That you can verify in one more way. Let me do it. How to check initially whether the vector is conservative or not? Or in other words whether the potential function exists or not? So, here V equals to grad phi.

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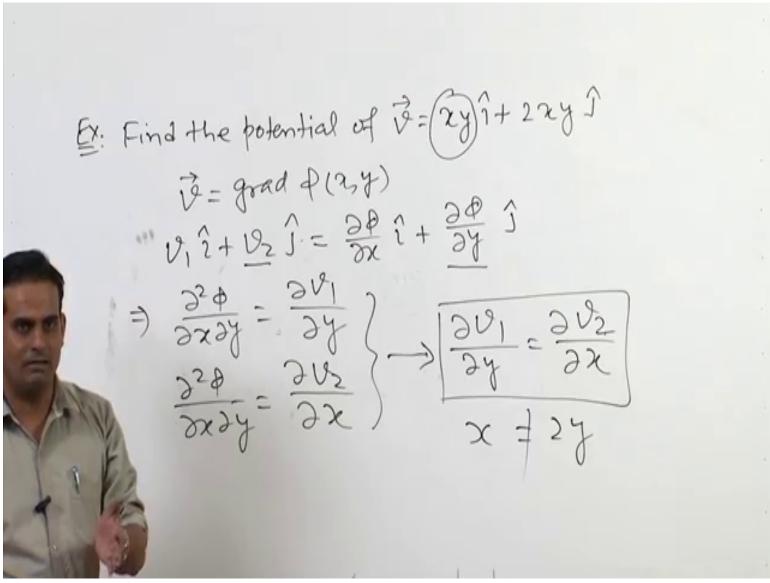

So, V is basically V 1 i plus V 2 j. So, V 1 here x into y, V 2 is 2 x y, this equals to del phi over del x i plus del phi over del y j. Now, if I differentiate partially the i component of these 2 vectors, with respect to x, then this will become del 2 phi over del x, del y sorry, with respect to y, and this will become del V 1 over del y ok. Another one, if I differentiate partially the components along z direction with respect to x. So, that will become del 2 phi over again del x del y, and this will become del V 2 over del x.

So, from here, left hand side are equal. So, it should give us at del V 1 over del y should be equals to del v 2 over del x. So, if this condition holds, then only potential function phi exists, otherwise not. Or in other word, if this condition hold, then the vector function is a conservative vector field, otherwise not. Let us check this condition here.

So, here V 1 is x y. So, del V 1 over del y will become x, del V 2 over del x will become 2 y, which is not equal, however you can verify this particular identity in the previous cases ok. And this is in case of x and y, if V is a function of x and y, vector function of x and y, in for 2 dimensional vector. If it is 3 dimensional, this condition can be extended. So, there will be 3 such kind of relation, we will discuss it later on ok. So, we have taken 3 example, in one, phi is coming in a simple way, in such a way that phi 2 is coming out to be the constant, in another example, phi 2 is coming out to be a function z, and in the third one, there is no potential function at all.

So, with this I will close this particular lecture, and in this lecture we have seen some applications of gradient.

Thank you.