Multivariable Calculus
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Lecture – 30
Gradient of a Scalar Filed and Directional Derivative

Hello friends. So, welcome to the second lecture from Vector Calculus. And in this lecture, I will talk about gradient of a scalar field, and then we will learn what we mean by directional derivative. So, first of all let me define the gradient of a scalar field, but before that let me introduce you the del operator.

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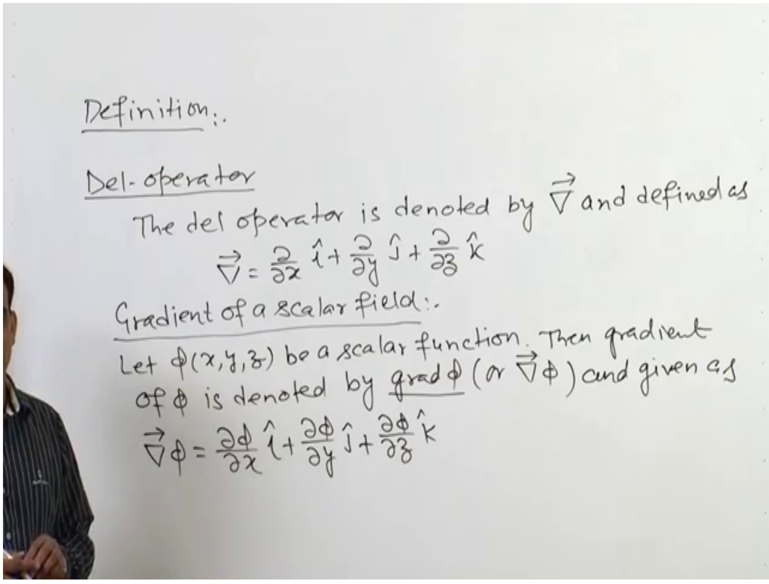

So, here I am doing few definition. So, the del operator is denoted by this symbol that is nabla, and it is a vector and defined as ok.

So, with this operator, we will define our gradient of a scalar field. So, my next definition is so, let phi be a scalar function of xy and z, then gradient of phi is denoted by simply grad phi or del phi. And given as so, please note that phi is a scalar function, but when I am operating it with del operator, it will become del phi over del x, that is the partial derivative of I with respect to x, I plus del phi over del y into j, plus del phi over del z into a means the component in the direction of k.

So, phi is a scalar function, but gradient of phi that it is also we say del phi it is a vector function. So, gradient of a scalar function will be a vector function, and it is defined in this way.

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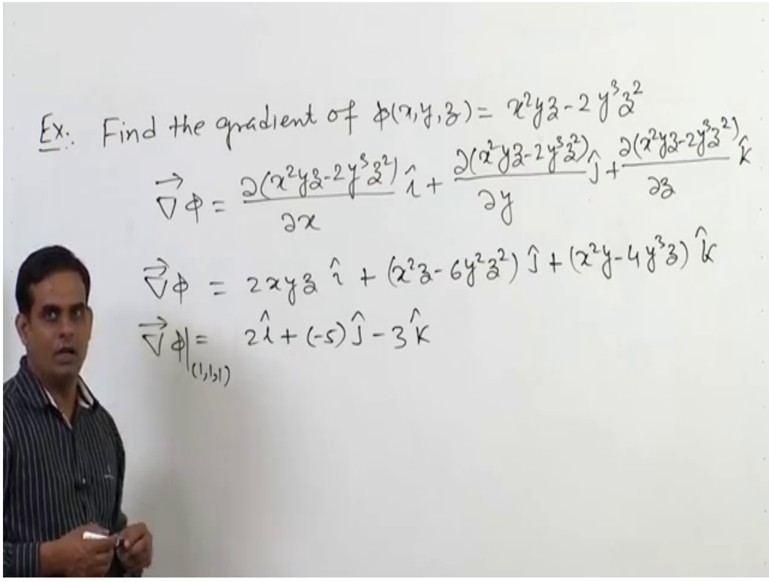

So now, we will take an example that is find a gradient of phi xyz equals to x square yz minus 2 y 3 z square. So, here phi is a scalar function. Now we will find out the gradient of this. So, gradient of phi is given by del x square yz minus 2 y cube z square upon del x in the direction of i plus, then again, the function phi and the same in the direction of k.

So, it will become the i component will become 2 xy z, plus the i component will become x square z minus 6 y square z square j. And k component will become x square y minus 4 y cube z. So, in this way, we can calculate the gradient of a scalar function. If someone ask you what will be the gradient at a point x equals to 1 y equals to 1 z equals to 1. So, at point 1, 1, 1; it will be given as a vector 2 i plus minus 5 j minus 3 k, by this vector, constant vector.

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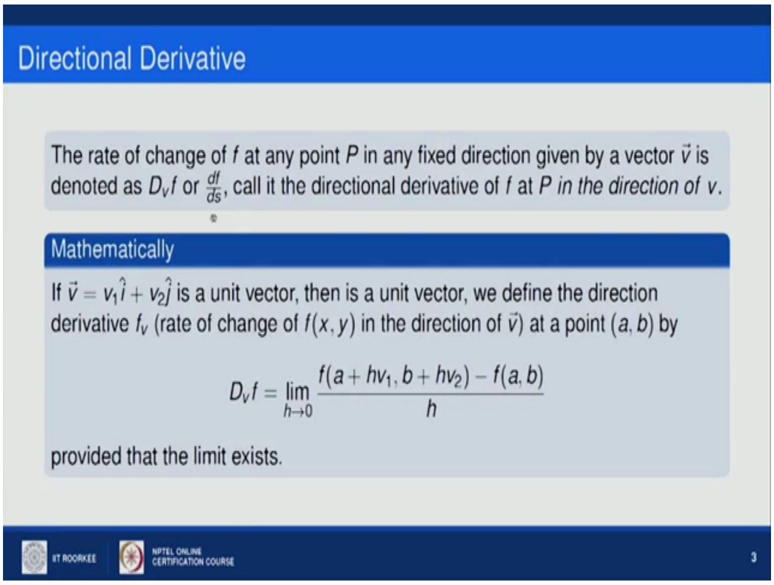

Another important concept where we will make use of gradient and I will make use of this gradient later on in the concept of directional derivative; so the directional derivative of a function f is the rate of change of f at any point p in any fixed direction given by a vector v. So, I will denote it as capital d v in the subscript and f so, dvf or df over ds. So, this means the directional derivative of f at a point p in the direction of vector v. So, this is called as the directional derivative.

So, directional derivative of a function, in the direction of a given vector is the rate of change of that particular function in that direction. For example, the partial derivative of a function f with respect to x that is del f over del x is the directional derivative in the direction of vector i. Del upon over del y the other partial derivative is the directional derivative in the direction of j or along y axis del f over del z gives the rate of change in the direction of z axis. That is the directional derivative in the direction of vector k.

However, we are having many other vectors those are having all components non-0 it means all components means ij and k. Hence, we need to find out a process to find out the rate of change of the function in the directions of those vectors. Those are not limited in the direction of ij or k only ok. So, mathematically if v be a vector having component v 1 and v 2 along I and j directions respectively and it is a unit vector v defined a direction fv that is the rate of change of f in the direction of v at a point ab by this limiting definition.

So, here mathematically we will define the directional derivative as.

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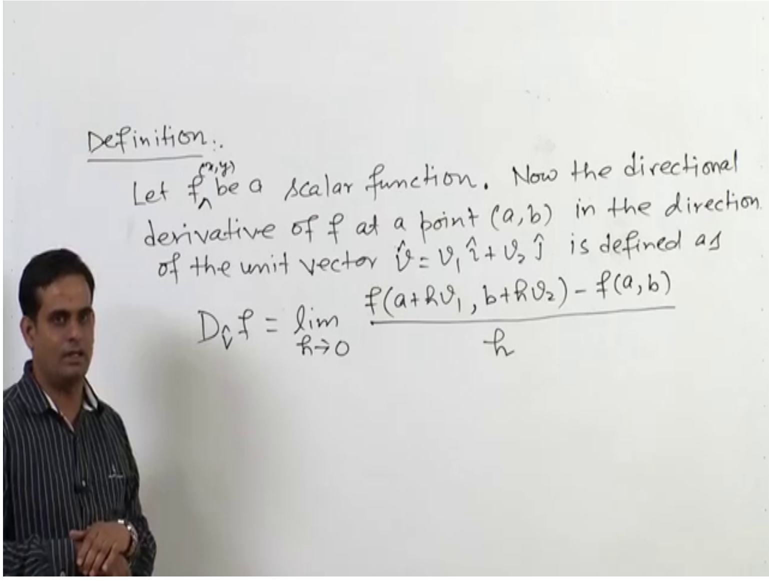

So, definition so, let f be a scalar function of xy let us say, I am defining it in a plane only. So, f be have a scalar function, now the directional derivative of f at a point ab. So, better you mention here f of xy, because we are talking about function of 2 variables. So, at a point ab in the direction of unit vector, let us say v; which is given as v 1 I plus v 2 j is defined as so, d in the direction of v. So, here v is a unit vector limit s tending to 0 f of a plus h v 1 b plus h v 2 minus fab upon h.

So, this is the mathematical definition of directional derivative. So, directional derivative at a point ab in the direction of unit vector v exist, if this limit exist. Otherwise they do not exist. Now let us take an example, and then we will generalize this concept in terms of gradient.

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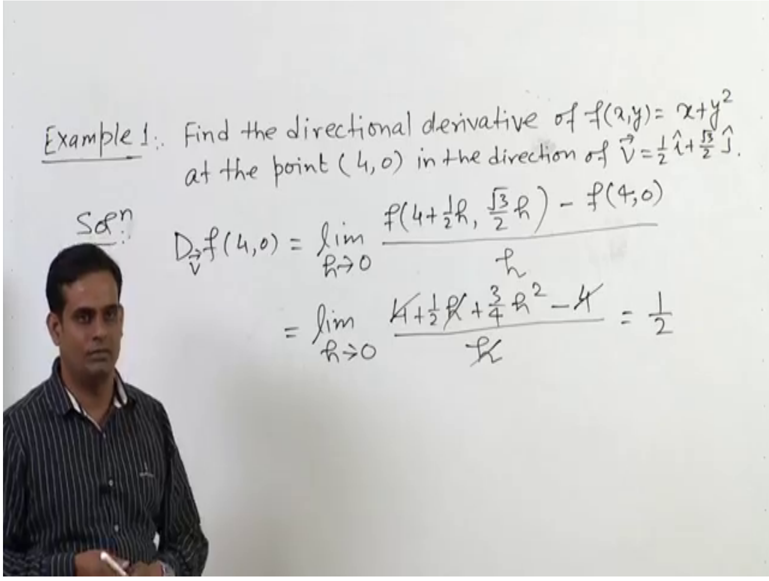

So, I will say example 1. So, find a directional derivative of f of xy which is defined as x plus y square at the point 4 0 in the direction of vector 1 by 2 i plus let us say root 3 by 2 j.

So, basically, we need to find out the rate of change of this function at this point in the direction of this vector. So, let us try to obtain the solution of this example; so, here this one. So, please note that here vj unit vector ok, so, we no need to make it as a unit vector. If it is not a unit vector first of all, we need to find out unit vector, because in the definition we have used the unit vector. So now, directional derivative of f in the direction of v at a point ab, so point is here 4 0 is given as limit h tending to 0 f of a plus hv 1 b be 0 here.

So, root 3 by 2 h. So, b plus hv 2 minus f 4 0, that is fab upon h. This equals to limit h tending to 0. So, 4 plus half h plus 3 by 4 h square, x plus y square, minus f of 4 0 will become 4 upon h. So, this comes out to be 1 by 2. Because 4 will be canceled out so, it is half. So, the directional derivative of this function at a point 4 0 in the direction of this vector v is 1 by 2.

So, let us define another example. So, I will write example 2.

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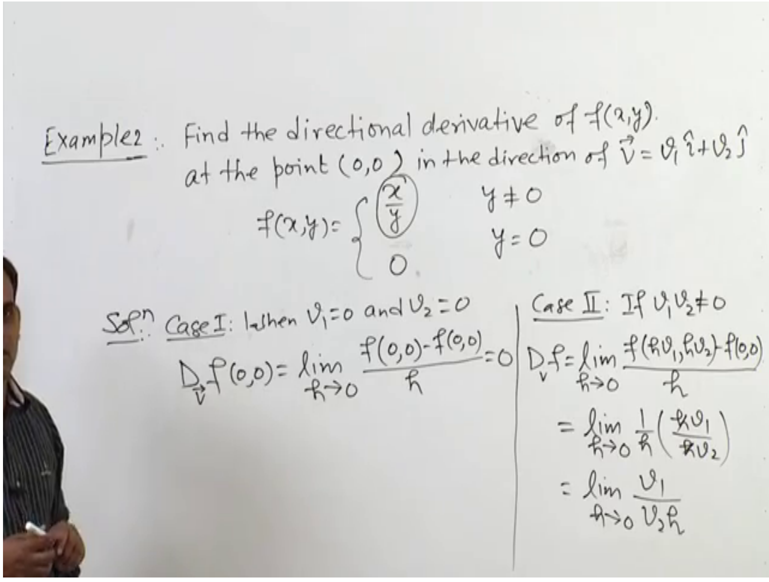

So, find a directional derivative of a function f of xy. So, I will define this function explicitly at the point, let us write make it simple 0, 0, in the direction of vector v, and let me take the a general case. V 1 I plus v 2 j, and function is defined as f of xy it is x upon y when y not equals to 0, and it is 0 when y equals to 0. Now find a directional derivative. So, here we will take 2 separate cases. So, case one, when v 1 not equals to 0 and v 2 not equals to 0.

So, we are not along x axis as well as y axis ok. In this case what will happen? Directional derivative of f in the direction of v at 0 0 will be or let us take more simple case first v 1 0 and v 2 is also 0, v 1 is 0 as well as v 2 is 0. So, in this case limit s tending to 0 f of a a 0 so 0 plus 0 0 0 minus f 0 0 upon h. And this comes out to be 0. Now take case 2, if the product of v 1 and v 2 those are the components of the unit vector v along I and z direction, they are their product is not equals to 0. It means either v 1 or v 2 cannot be 0. If one of them is 0, product will be 0, if both of them are 0, product will be 0.

So, both are non-0, so, this is the general case, we are not going along axis x or y axis. So, in this case the directional derivative is limit h tending to 0 f of hv 1 hv 2 minus f 0 0 upon h. So, as you know v 1 is not 0 v 2 is not 0. So, we have to consider this case of the definition of f of xy. So, it will become limit h tending to 0 1 upon h I have taken this h out hv 1 upon hv 2. So, limit is tending to 0 v 1 upon v 2 h. And you know this limit does not exist.

Hence, the directional derivative of this function in the direction of a vector v 1 I plus v 2 j, where, v 1 and v 2 both are nonzero component does not exist ok. Now I want to generalize the concept of directional derivative, and I want to relate it with the gradient of a function. So, if the function f is differentiable, here differentiable means because f is a function of several variable. So, if f is having that total derivative ok, then we can write the directional derivative as the gradient of f and dot product of these 2 with the unit vector v, and that is coming from this limiting definition only.

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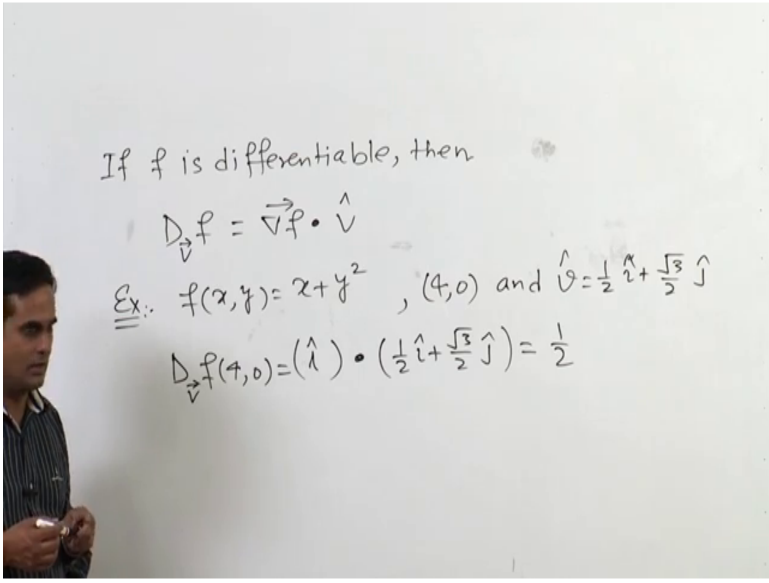

So, what I want to say? If f is differentiable, then directional derivative of f in the direction of a vector v is given by del f and dot product with unit vector v. And hence, if it is differentiable, it is directional derivative exist in each and every direction. So, for example, take the earlier example itself where f of xy was x plus y square, I have taken the points as 4 0, and vector was 1 by 2 I plus root 3 by 2 j.

So, find a directional derivative of this function, at this point in the direction of this vector so, apply this definition. So, here d f at 4 0 will be grade f, and grade f will become i plus 2 i j, and this is at the point 4 0. So, at the point 4 0, it will become simply i only ok, because y is 0. So, i dot with vector 1 by 2 i plus root 3 by 2 j, and this comes out to be 1 by 2.

So, the same answer we obtained which we have brought earlier using the limiting definition. So, another example you can take f of xy as a x square, upon y square where xy belongs to in this interval.

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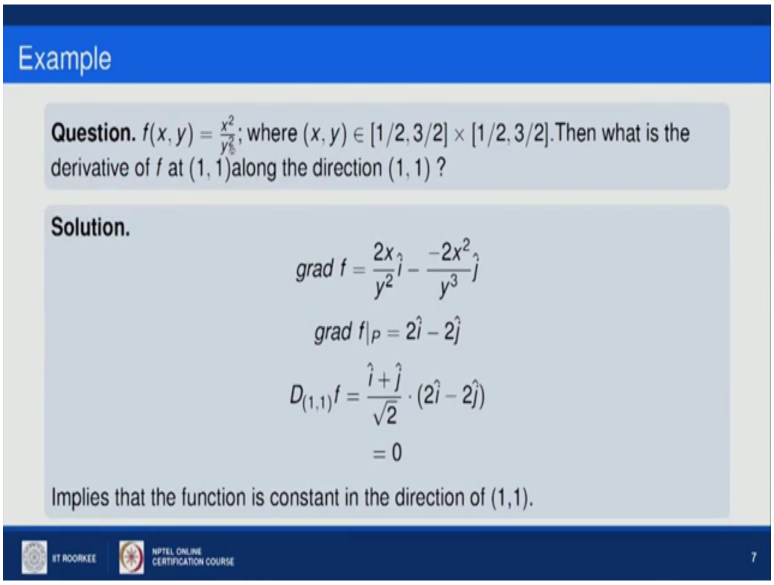

So, please note that in this rectangular domain, I am not having 0, 0 that is the y axis is not included here, because there this function is not defined. And what is the derivative of f at 1 1 along the direction i plus j.

So, if the vector is i plus j I need to make it unit vector. So, it will become I plus j upon square root 2. That is my v cap, and grade of f will become this vector at this point 1 1 this will be 2 I minus 2 j, now the dot product of this it comes out to be 0, and this implies that the function is constant in the direction of the vector i plus j there is no change in a ah rate of change is 0.

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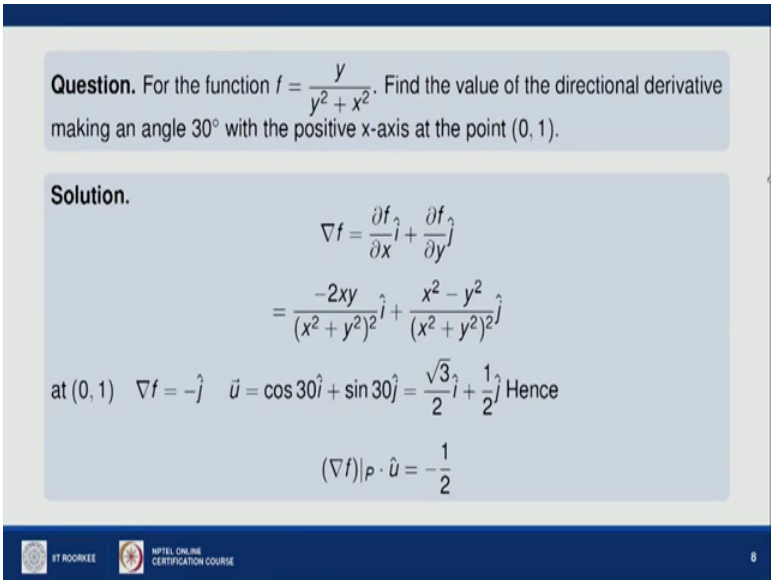

This is another example so, find the value of directional derivative of this function making an angle 30 degree with the positive x axis at the point 0 1. So, again I will calculate del f del f is given by this 1 at point 0 one del f comes out to be minus j. Now vector is given as because it is making an angle 30 degree with the x axis the x component will be cos 30 degree. And y component will be sin pi by 6. So, this comes out to be root 3 by 2 I plus 1 by 2 j, and hence directional derivative is minus 1 by 2.

Now one more important remark about directional derivative means in which direction.

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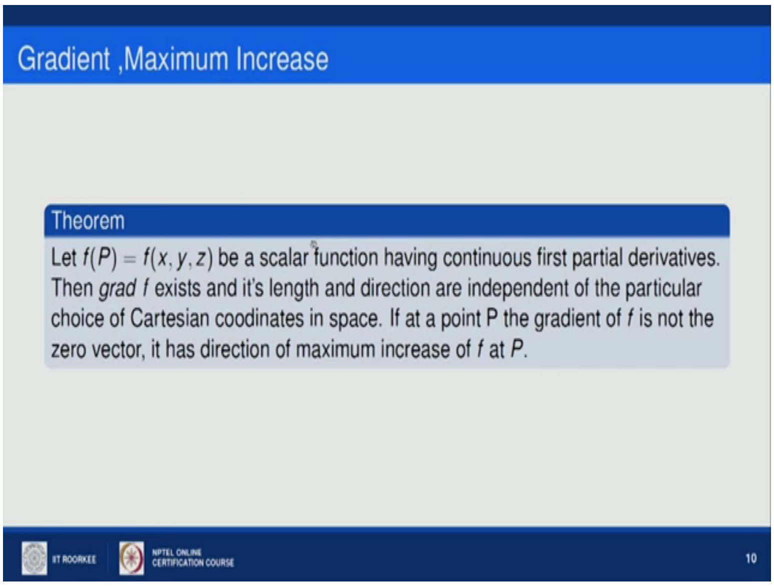

The function is having the maximum rate of change. So, let f be or f of xyz be a scalar function having continuous first order partial derivatives, then gradient of f exists and it slant and direction are independent of the particular choice of Cartesian coordinate in the space. If a point p the gradient of f is not a 0 vector it has the direction of maximum increase of f at p.

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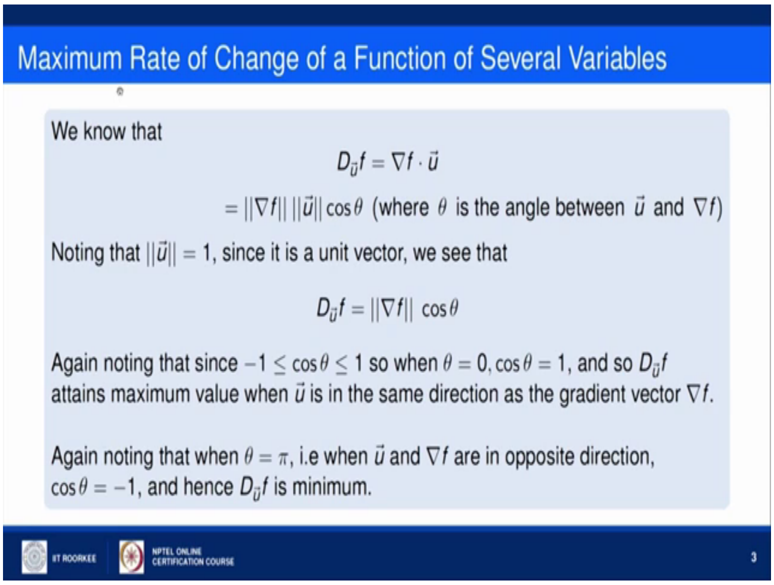

So, let us see the proof of this. So, here we are taking f as a function of several variables. So, we know that the directional derivative of f in the direction of a unit vector u is given by gradient of f dot product with unit vector u. And as we know that and the dot product of 2 vectors a and b is given by magnitude of a multiply with magnitude of b multiply with cos theta, where theta is the angle between these 2 vectors a and b.

So, in the same way we can write this particular thing as the magnitude of gradient of f magnitude of u into cos theta we know that u is a unit vector. So, the magnitude of u will become 1. So, this can be written as a magnitude of gradient of f multiply with cos theta. So now, we have to see that where we are having the maximum value of this particular thing, because the maximum value will give you the maximum rate of change of a function f. So, we can notice that that the cos theta is between minus 1 to 1. So, the maximum value of this will be a magnitude of gradient of f, and minimum value will be minus magnitude of gradient of f.

So, when theta equals to 0, cos theta will be a cos theta will become one and we will get the maximum value of this. So, maximum rate of change will occur in the direction of that gradient of f. Again, when we take theta equals to pi that is when u and gradient of f are in opposite direction, then cos theta will become minus 1, and hence the rate of change will be minimum in this case. So, let us take this example. So, find a gradient in which the function f of xy sin x plus e raised to power y minus 1 is the greatest rate of change at the point 0 1.

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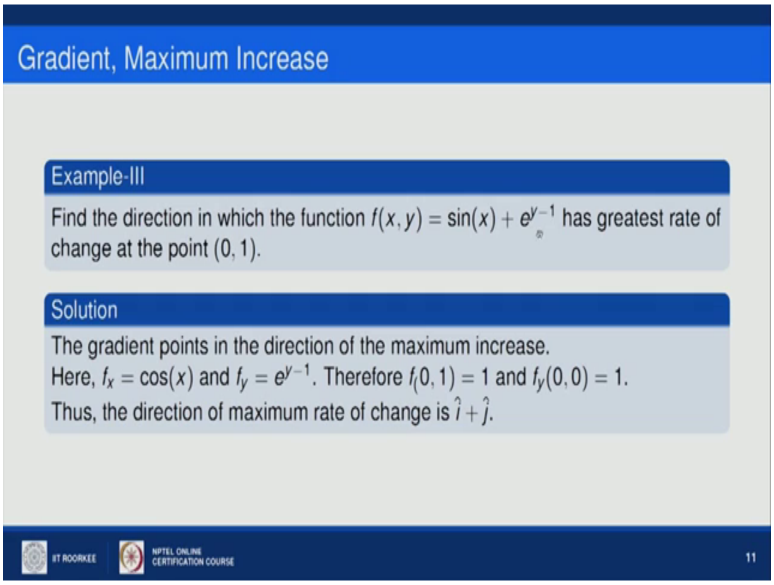

So, as I told you it will be in the direction of del f. So, here del f is given at point 0 1 h I plus j. So, answer will be the maximum or greatest rate of change of this function at this point will be in this direction.

In the next lecture, we will see few more properties of gradient and some other applications of gradient.

Thank you very much.