Multivariable Calculus
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Lecture – 29
Vector Differentiation

Hello friends, so, today we are going to start vector calculus, and in the first lecture of very a vector calculus we are going to introduce vector differentiation. Basically, in this lecture we will talk about what we mean by vector function then we will talk about limit continuity and then differentiability. We will see some applications of differentiability for measuring the length of the curve for a given interval of the parameter, and then we will introduce the concept of arc length.

There are 2 types of quantity scalar quantity and vector quantity. In the scalar quantity, we use to have only magnitude; however, in a vector quantity together with magnitude we used to have direction also. So, based on those vector quantity here we are going to define vector function.

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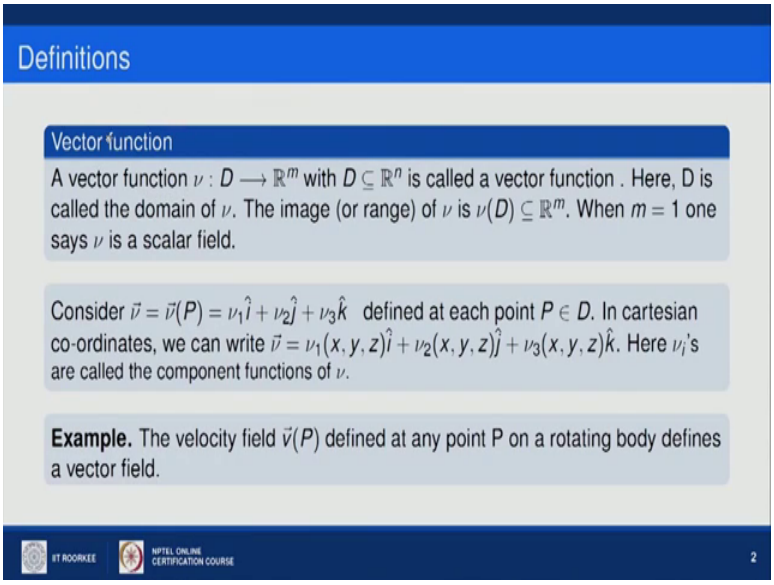

So, a vector function nu that is from domain d to Rm where Dj subset of Rn is called a vector function. Here these the domain of nu, the image of nu is nu of d which is a subset of Rm, if we are having m equals to 1, then we say that nu is a scalar field, means a scalar function. So, if we define vector function on r 3, then we can have this nu, nu is a function of point p, it is given a nu 1 i nu 2 j and nu 3 k; which is define at each point p belongs to the domain d.

In Cartesian coordinates, we can write vector nu equals to v 1 x y z i v 2 x y z j v 3 x y z k. So, here v 1 v 2 and v 3, all these 3 are is scalar functions of x y z, and they are the components along I direction, along j direction, and along k direction. For example, the velocity field define at any point p on a rotating body defines a vector field.

So, if I am having a function like this f of x y z, let us say something x square y cube z.

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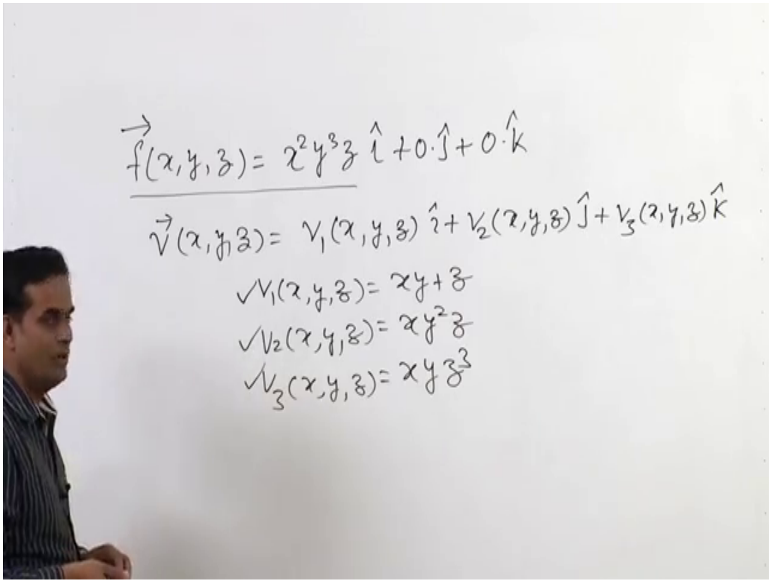

Then it is a scalar function; however, if I am having nu 1 x y z i cap nu to x y z j cap plus nu 3 x y z k cap, then it is a vector function here, nu 1 x y z for example, you can take some function of x y z. So, let us x y plus z. Nu 2 x y z is again a function of x y z. So, let us say x y square z, and nu 3 x y z equals to x y and let us say z cube ok.

So, as I told you these 3 are the components along the different direction i j and k here i j and k are the unit vectors, those are mutually perpendicular to each other, and they are not coplanar, they cannot be contained in a plane.

Now, if we talk about this scalar function f, and if I had a direction here with this function, then I will say it as a vector function. Because here component in j direction will be 0, and component in k direction will also be 0. So, these are some examples of vector function.

Now we will talk about domain of a vector function. So, for that we can explain with this example.

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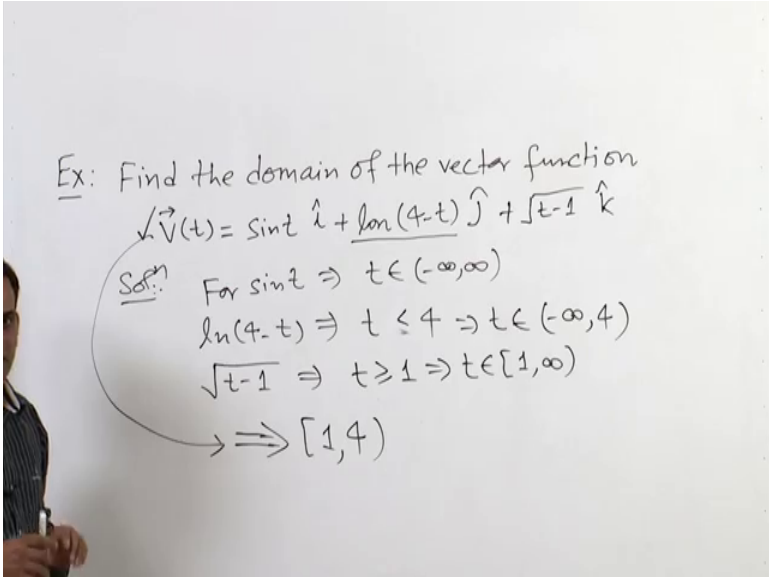

So, find the domain of the vector function so, let us take function is nu t equals to sin t i plus log 4 minus t, this is the component along j direction. And finally, a square root t minus 1 along k direction.

So now we need to find out the domain of this vector function. So, here sin t is defined for all real values of t. So, for sin t, domain is d belongs to minus infinity to infinity. Now take the second component of this vector function so that is log 4 minus t. So, as you know, here this t is defined when t is less than less than 4, because if t will become equals to 4, it will become 0, or if t is greater than 4, this will become negative where log is no defined.

So, it means here t belongs to minus infinity to 4. And finally, we are having a square root t minus 1. So, for this I am having this would be greater than equals to 1, then only this square root is defined. So, it means t belongs to one to infinity. Now the domain of this vector function is the f co intersection of all these 3 regions. And that intersection is given by the interval 1 to 4 so, this is close from 1 and open from the side 4. So, this is the domain of this vector function.

So, in this way we can calculate the domain of a given vector function; means, you need to find out the domain of each and every component of the vector function, and then find the intersection of all those. Now we will talk about the parametric representation of a curve.

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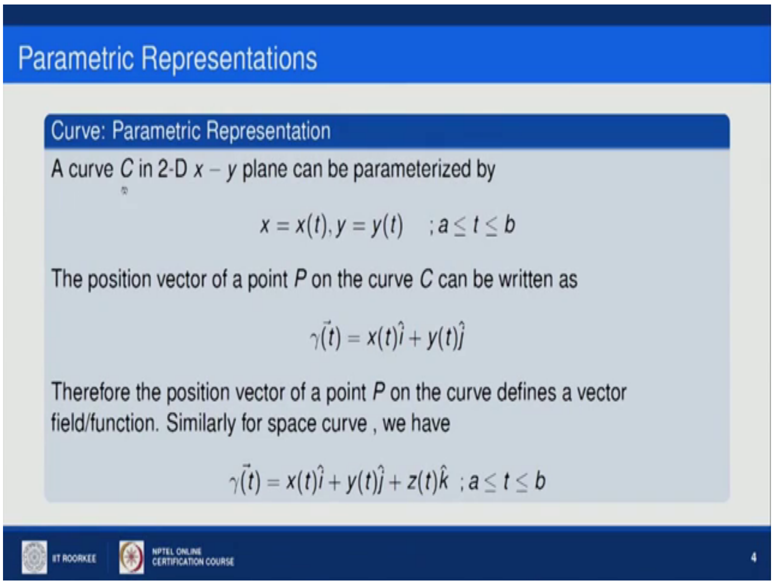

. So, let us say a curve c in 2 dimensional xy plane can be parameterized by x equals 2 x t and y equals to y t. So, here we are defining that x and y are the functions of t, and where is between a to b. The position vector of a point b on the curve c can be written now in the form of a vector function, and it is given as gamma t equals to x t i plus y t j.

So, please note that now this x t and y t can be written explicitly in terms of t, and hence, this vector function gamma become a function of t. Therefore, the position vector of a point p on the curve defines a vector function. Similarly, for a space curve we are having the position vector of a point p h x t i plus y t j plus j z t k and t is between a to b.

Let us take few examples of this parametric representation.

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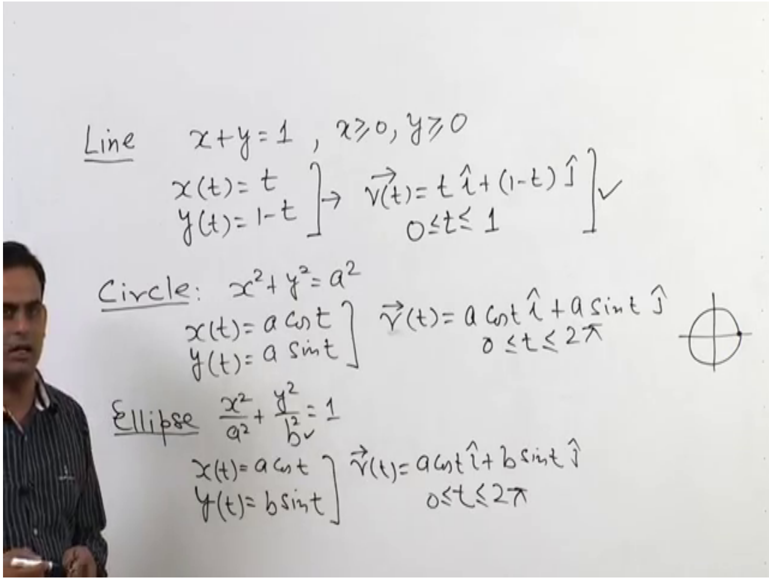

So, for example, in case of line; so consider a line x plus y equals to 1 in the first quadrant. So, here I can write a parametric representation as xt equals to t. So, if x become t, then y will become 1 minus t. So, yt will be 1 minus t, and then the equation of line can be given by the vector function as t i plus 1 minus t j. The interval of t because here x is greater than equals to 0, y is greater than equals to 0. So, t will be greater than equals to 0, and the same time the maximum value of t will be less than one. So, t will be between 0 to 1. So, this is the representation of a line in first quadrant.

Second thing we are taking a circle so, let us take the simple equation of circle x square plus y square equals to a square. So, it is a circle having center at origin and having the radius a. Now parametric representation of the circle can be given as xt equals to a cos t, and y t equals to a sin t. And now the vector function for this circle is the vector r that is a function of t the function x t plus a sin t j. And here since it a it is a circle. So, it will move t equals to 0, and it will come at 2 pi so, 0 to 2 pi. If we are having a semicircle above x axis, then p will be 0 to pi, if we are having a semicircle below x axis, then it will go minus pi 2 2 minus pi to 0 you can say.

The next one is ellipse. So, let us take the standard equation of an ellipse. So, x square upon a square plus y square upon b square equals to 1. Now find a parametric representation of this ellipse. So, let me write x t as a cos t; which is the same as in the circle; however, the different thing will come here. So, yt was a sin t in case of circle, but here due to this it will become b sin t. So, whenever a equals to b the ellipse will become circle as you know.

So now, the parametric equation is rt equals to a cos t i plus b sin t j. And t is again as I told you 0 to 2 pi, the same as in case of circle. So, these were the parametric representations of few conics, now if we talk about the space curve.

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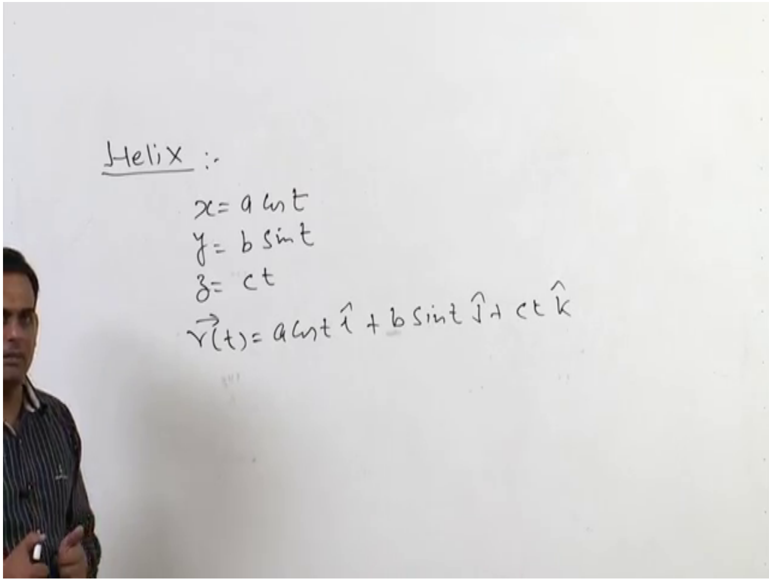

So, let us take the parametric representation of a helix. So, the parametric representation is given as x equals to a cost, y equals to b sin t and z equals to c t. So, here equation will become r t equals to a cos t i plus b sin t j plus c t k.

So, this is the vector function representing the position vector of a point p on an helix at some given value of t. And here note that if a equals to b it will become circular helix. So, in this way we can represent any curve either in 2 d or 3 d by it is parametric representation. And in the parametric representation the position vector of a point lie on the curve will be given by a vector function.

In the similar way, we can represent the parametric surfaces or I will say the parametric representation of surfaces; so in case of surface.

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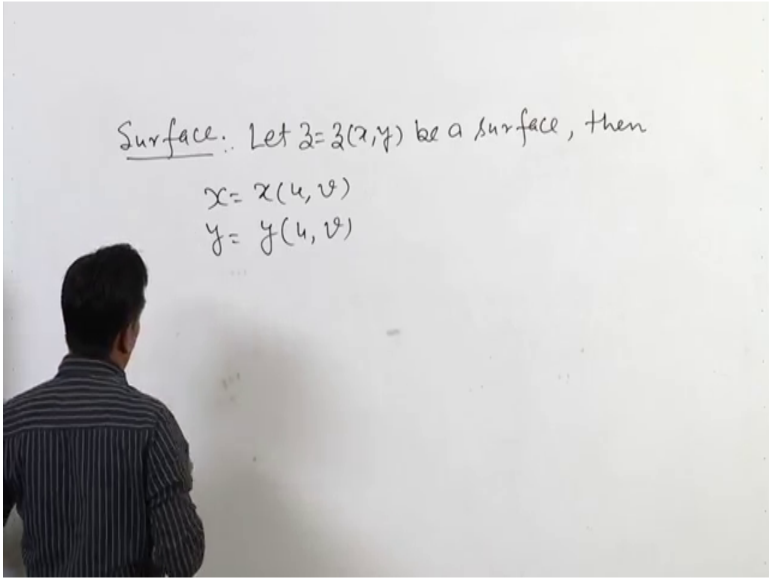

So, let z equal to 0 of x y be a surface, then the parametric representation of the surface is given by the 2 parameters family.

So, I am taking here u and v x equals to x u v y equals to y u v, or better I will write x equals to u y equals to v.

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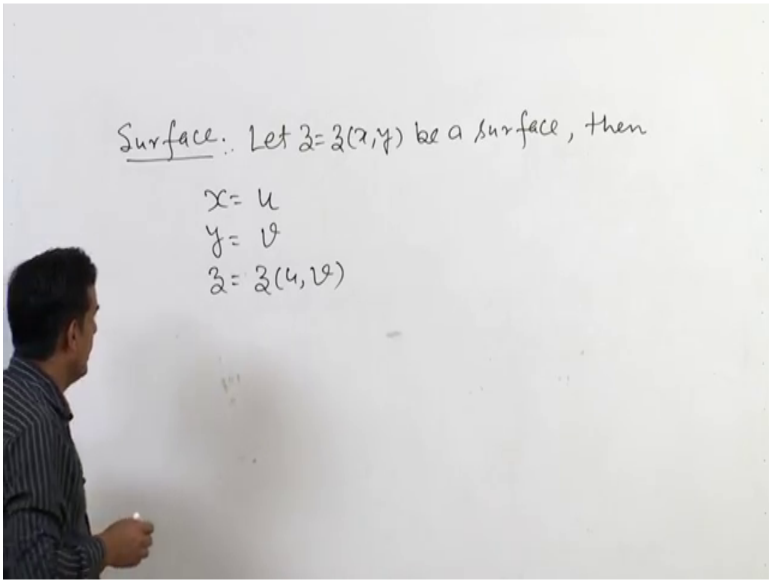

And then my z will become z of uv. That is one of the way we can use for some surfaces, or the more general one will be as I told you x is x of u v y of u v and z of uv.

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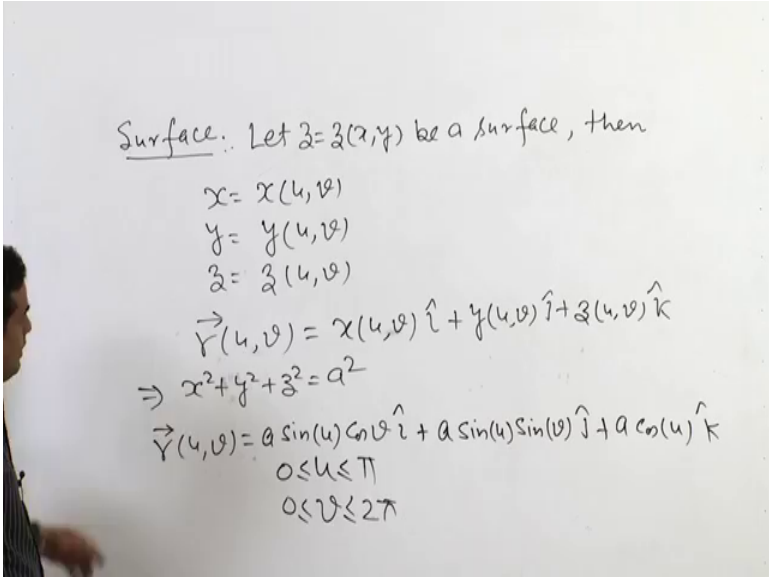

And here surface will become the position vector of a point p on the surface is given by x u v i plus y u v j plus z u v k.

For example, if we are having a surface let us say this so, this is we know that it is a sphere of radius a. So now, parametric representation of this surface is given by r u v a sin u cos v. And this is the component along I direction plus, a sin u sin v, this is the component along j direction, and then finally, a cos u that will be the component along k directions. So, here you will lie between 0 to pi for the given sphere. And we will be between 0 to 2 pi, n. So, this is I am writing just from the transformation from Cartesian coordinates to a spherical coordinates; so x component y component and z component. So, in this way we can represent any surface by parametric representation ok.

Now, we will come to some basic definitions from calculus.

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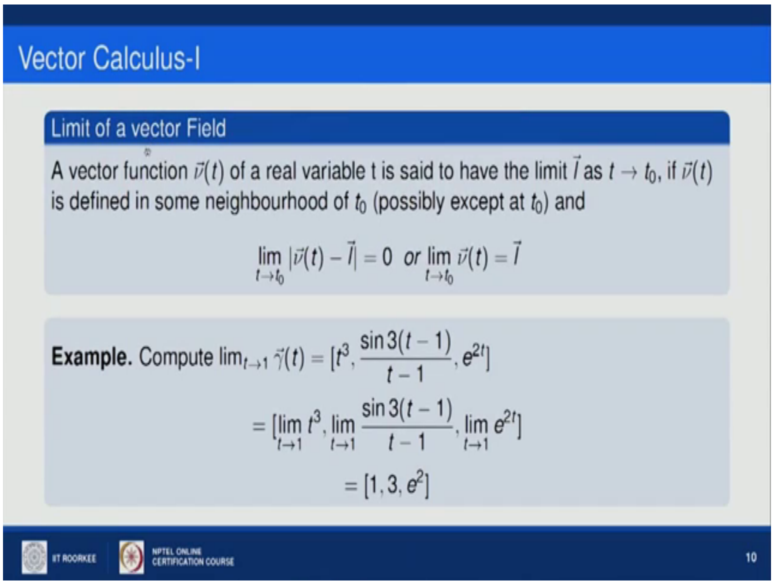

For vector functions, so, limit of a vector function, a vector function nu t of a real variable t said to have a limit l. So, please note that here l is a constant vector. So, l is a vector quantity here, as t tending to t 0, if nu t is defined in some neighborhood of t 0, possibly except at t 0, and limit t tending to t naught absolute value of nu t minus l equals to 0, or this I can write in this way that limit t tending to t naught nu t equals to l.

So, this is the definition of limit, and if I take a particular example like finds a limit.

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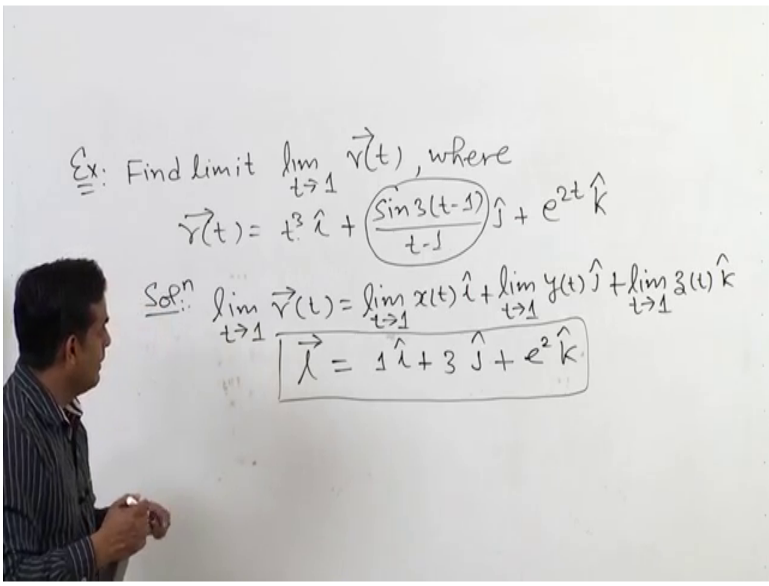

Limit t tending to 1 of a vector function rt where rt is given as t cube i plus sign 3 t minus 1 upon t minus 1 j plus let us say e raised to power 2 t k.

So, a polynomial we are having the component along I direction, a trigonometric function along j direction and an exponential function along k direction. So, here solution of this will be. So, the limit t tending to 1 rt is a vector having limit t tending to 1 x t I; where x t is t cube plus limit t tending to 1, y t along j direction. So, y t sin 3 t minus 1 upon t minus 1, and plus limit t tending to 1 z t k. So, z t is e raised to power twice t. Now limit t tending to 1 t cube will become one. So, one in to i plus when t is tending to 1 let us see the limit of this function.

So, what will happen when t is tending to 1? It is not defined means it is having an indeterminate form. So, what we need to do we have to use l hospital rule and here, when I will use l hospital rule here. So, it will become 3 cos 3 t minus 1 upon 1. So, it will become 3 cos t minus 1, and when t is one it will become 3 code 0. So, it will be cos 0 is 1. So, 3 j plus finally, this will become e square k. So, this is the limit of this function.

So, please note that here limit is again coming as a vector quantity. So, in this way we can compute the limit of a vector function.

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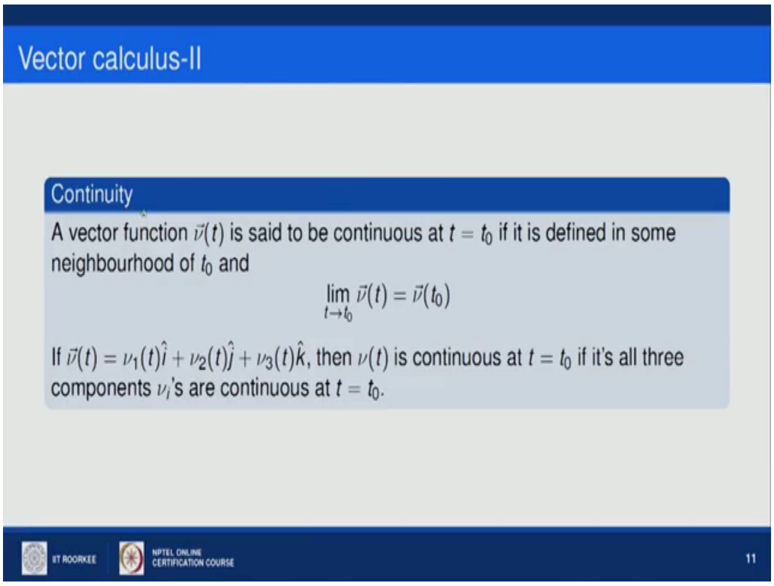

Similarly, we can define the term continuity for the vector functions. So, a vector function nu t said to be continuous at t equals to t naught if it is define in some neighborhood of t naught and limit t tending to t naught nu t equals 2 nu e 0. So, in terms of component, we will say a vector function nu t is continuous at t equals to t naught, if all of it is components are continuous at t equals to t naught.

Now we will come to next term, that is the differentiable t. So, differentiation of a vector function.

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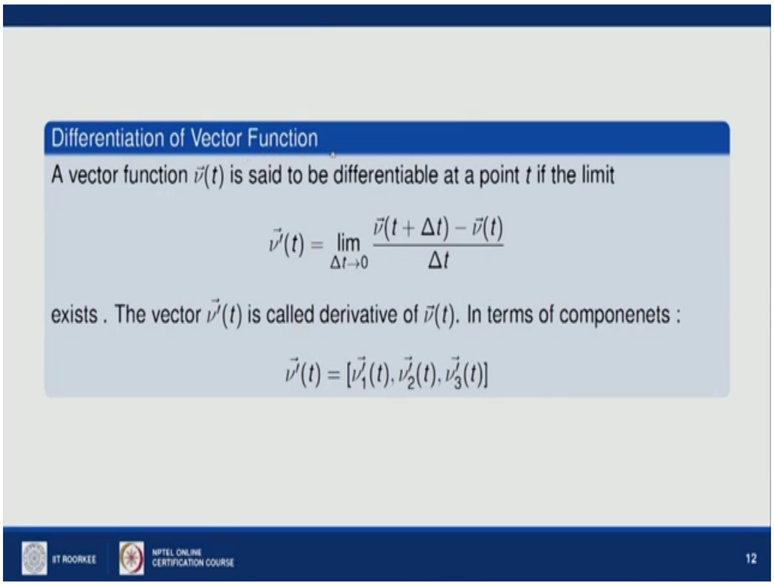

So, a vector function nu t said to be differentiable at a point t if the limit nu dash t that is equals to limit del t tending to 0 nu t plus del t minus nu t upon del t exist, and if this limit exists this will be equals to derivative of nu t at t.

So, in terms of components we can write nu dash t that is the derivative of nu t again it will be a vector function, it is the I component will become nu 1 t j component will become nu 2 dash t and k component will become nu 3 dash t some rules for differentiability.

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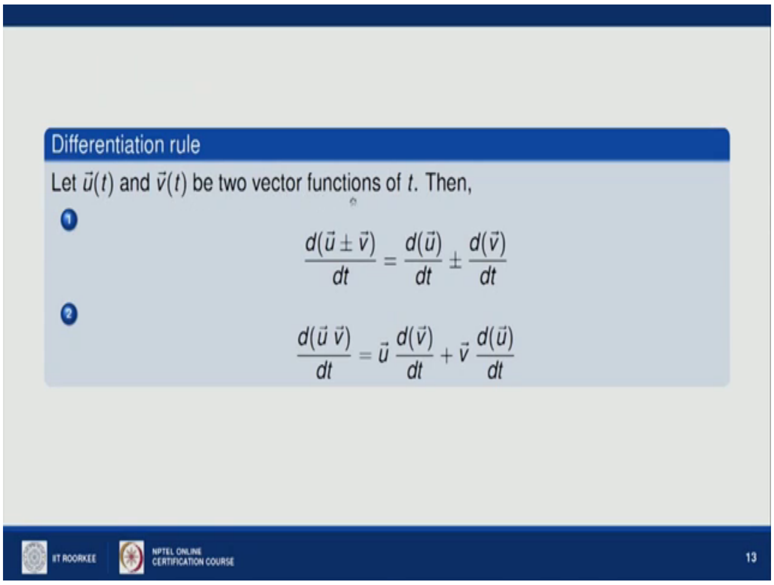

So, if u and v be 2 vector functions of t then d u v upon d t will become d u over d t plus d v or d t. Similarly, d u minus v upon d t will become d u over d t minus dv over dt. So, the rule as we the same do as we are having in case of scalar functions.

If we are having product of 2 vector functions u and v, then differentiation of this with respect to 2 e is given by the first function as such d v by d t plus second function as such differentiation of first. We are having few more remarks about differentiation. So, the necessary and sufficient condition for the vector v t to be constant is that it is first derivative should be 0 vector.

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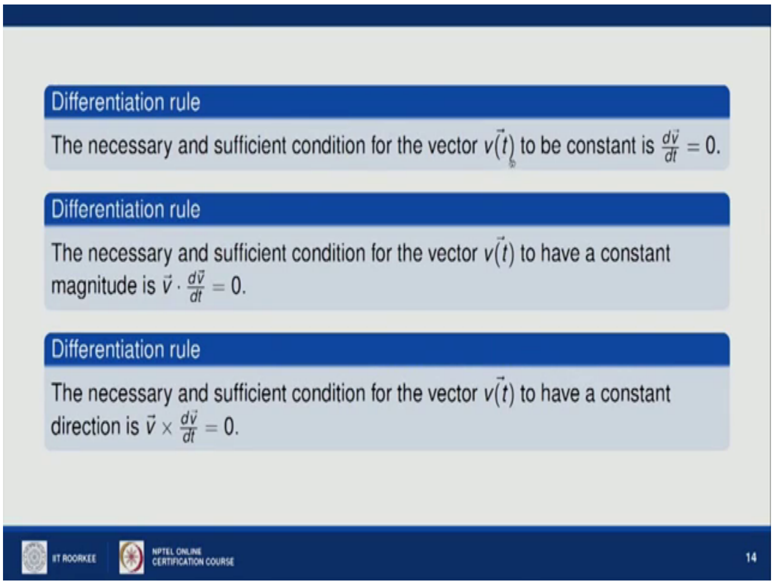

So, it is a 0 vector.

The other remark is the necessary and sufficient condition for the vector v t to have a constant magnitude. So, here we are talking about constant function, here we are talking about constant magnitude, is that the dot product of that particular vector function together with it is derivative equals to 0.

And the final remark is the necessary and sufficient condition for the vector v to have a constant direction is the cross product of vector v together with dv or dt equals to 0. So, we can prove it very easily just by taking the constant magnitude, and then by taking the constant direction. Now one more important application of differentiation of a vector function; so if rt be the position vector of a moving particle with respect to the origin, then the first derivative of r with respect to t denotes the velocity of the particle at time t.

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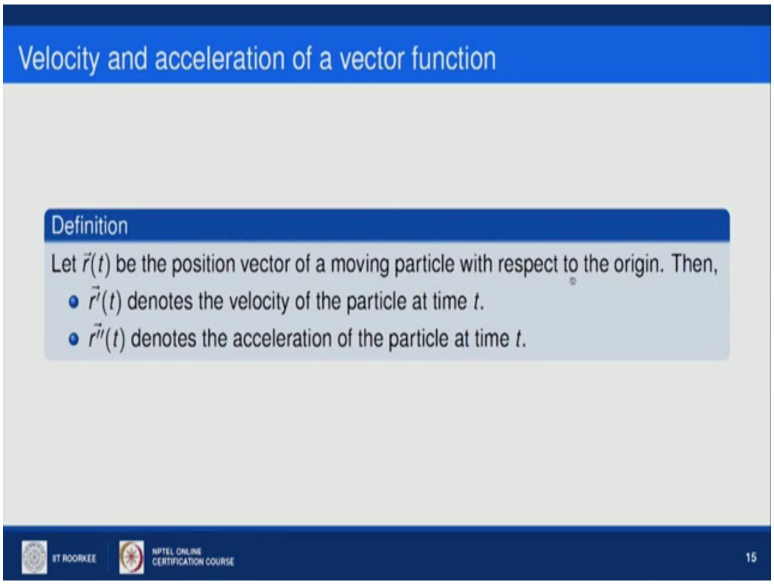

And the second derivative of r with respect to t denotes the acceleration of the particle at time p.

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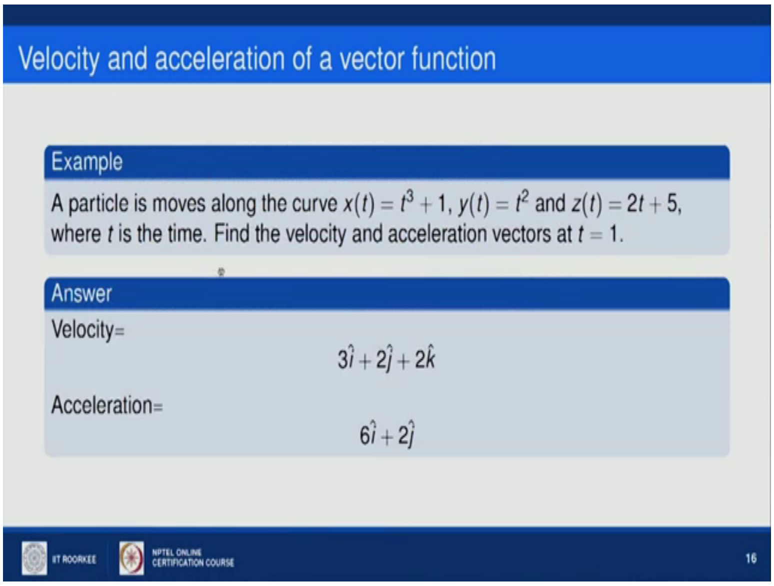

So, let us take this example. So, a particle is moving along the curve x t equals 2 t cube plus 1, y t equals 2 t square and z at t equals to 2 t plus 5. So, basically the position vector of a point p on this curve will be r t equals 2 t cube plus 1 i plus t square j plus 2 t plus 5, where t is the time find a velocity and acceleration vectors at t equals to 1. So, r dash t will become 3 t square i plus 2 t j plus 2 k. So, at t equals to 1 3 I plus 2 j plus 2 k.

So, that is the velocity. Again if I will differentiate the velocity one more time then it will become 6 t i plus 2 j plus 0 k and at t equals to 1 it will become 6 I plus 2 j that is the acceleration vector.

Now, another application of this particular concept that is the differentiation is the tangent to a curve.

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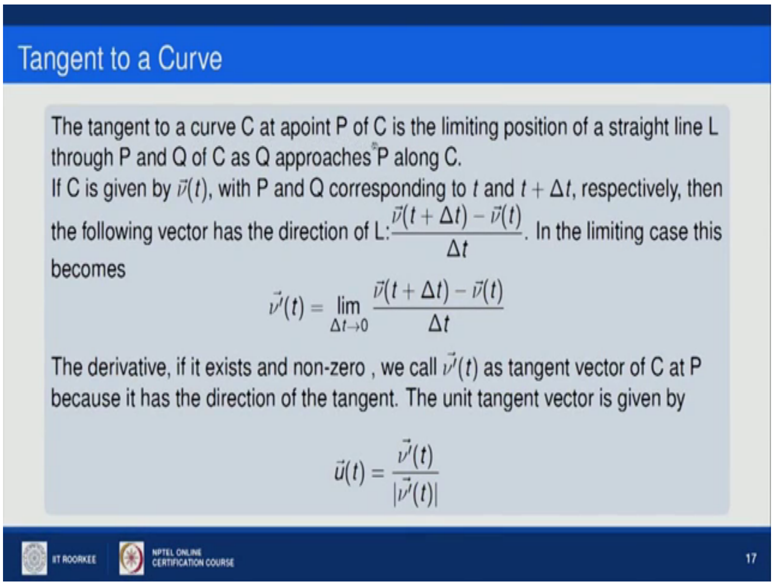

So, the tangent to a curve c at a point p of c is the limiting position of a straight-line l through p and q of c as q approaches b along c. So, this is the limiting case and if we put let us assume that nu t be the position vector of a point on the curve c and p and q are the points corresponding to value t and t plus del t, then this particular tangent will become nu dash t which is limiting condition that is limit del t tending to 0, nu t plus del t minus nu t upon del t, and you know it is the derivative also.

So, basically derivative gives the tangent on a given point. The unit tangent vector is given by just u t that is nu dash t over magnitude of nu dash.

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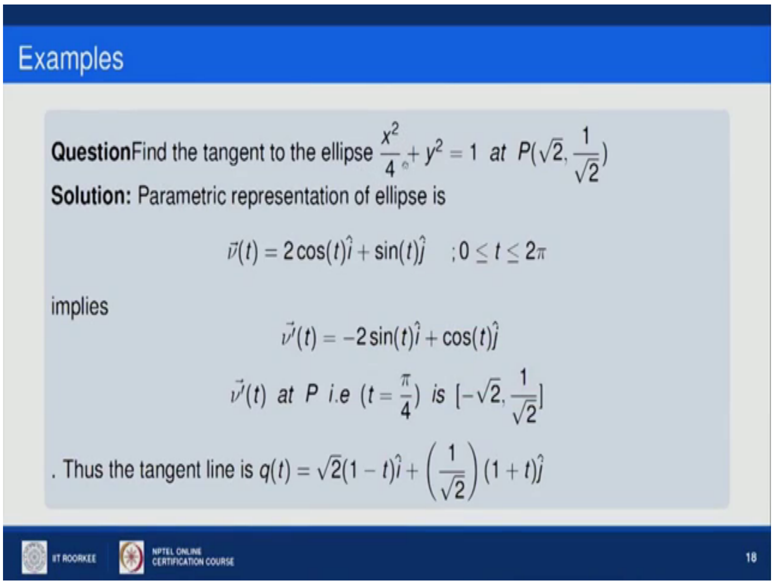

So, let us take this particular example. So, find a tangent to the ellipse x square upon 4 plus y square equals to 1 at p equals to root 2 1 upon root 2. So, first of all, we need to find out the parametric representation of this ellipse. So, parametric representation is given by 2 cos t I plus sin t j, where t is between 0 to 2 2 pi. Now we will calculate nu dash t. So, minus 2 sin t the differentiation of 2 cos t, and sin t will come cos t. Now nu dash t at p it means p is root 2 n 1 upon root 2. So, according to this the value of t satisfy this point on this curve is pi by 4. So, at pi by 4 it will become minus root 2 upon 1 upon root 2. So, in this way we can calculate the tangent line.

Finally, a very important concept, that is how to measure the length of a given curve.

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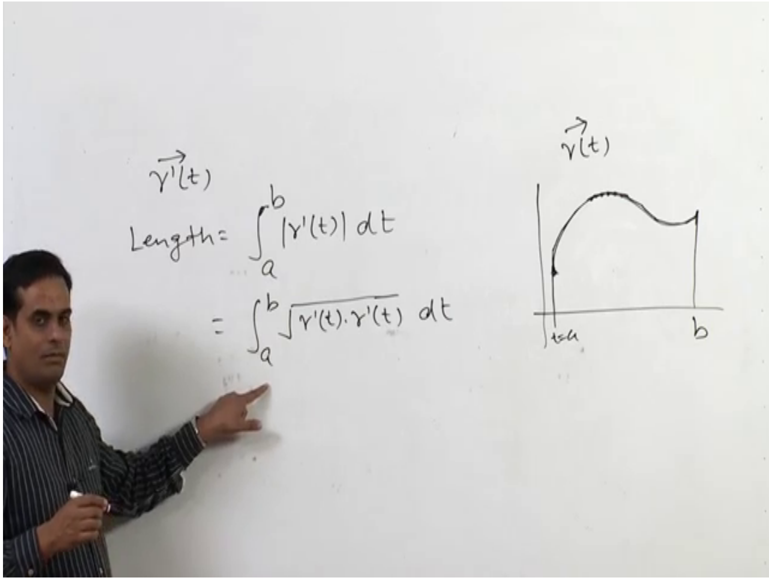

So, let us I am having a curve, let us take a curve in plane, which is at any point position vector is given by rt. Now I need to find out length of this curve. So, let us say t equals to a and t equals to b. So, length of this curve, between when t is between a to b.

. So, how to find this length, the basic idea of finding the length of a curve is plot the tangent at each point find the length of those tangent vectors and add all of them; however, it will be an approximation like if I am finding like this. So, how to refine this particular approximation? It means, take the tangent at infinite number of points on the curve between a to b so that we will find out tangent at each point and add the length of all those tangent.

So, in this way, as you know if curve if given by rt then tangent is r dash t and length or tangent. So, length of the curve is given by integration over a to b. Why I am taking this integration? Because I am adding all the tangents, and then r dash t dt. That is basically a to b a square root r dash t dot product with r dash t dt so, this is the length. So, this particular thing will give the length of the curve between t equals to a to t equals to b.

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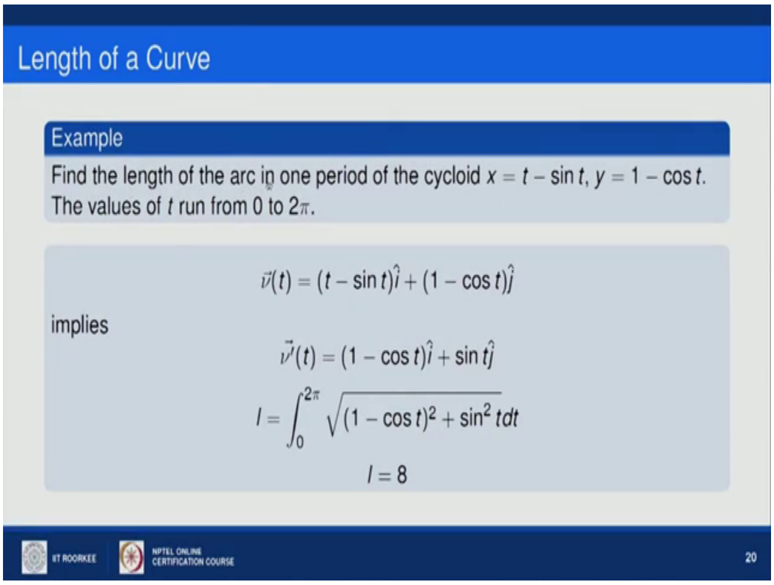

. So, is find the length of the arc in one period of the cycloid given by x equals to t minus sin t and y equals to 1 minus cos t. The value of t run from 0 to 2 pi. So, here are the parametric representation of a point p on the cycloid is given by the position vector nu t v t minus sin t I plus 1 minus cos tj. Now find out the nu dash t so, this will become 1 minus cos t, and this will become sin t. So, 1 minus cos t I can sin t j.

N ow the length of this is I equals to 0 to 2 pi integration over 0 to 2 pi is square root of 1 minus cos t plus sin square t. So, when I will do it it will become 1 plus cos square 2 t minus 2 cos t; so cos square t plus sin square t will become 1. So, it will become square root 2 minus 2 cos t so, I can take is is square root 2 out. So, it will become 1 minus cos t that I can write 2 sin square t by 2. And in this way, I can calculate this particular integral, and value will come out to be 8. So, the length of this one period of the cycloid is 8.

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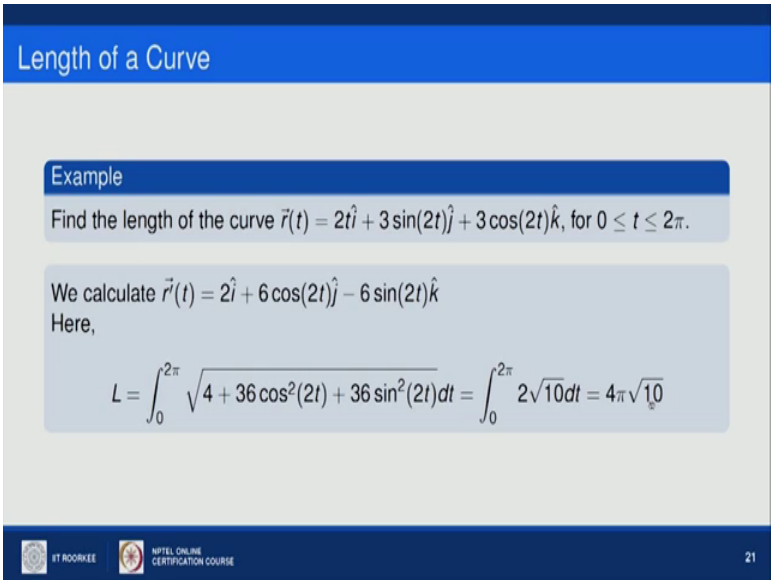

Similarly, this example here the equation of curve is given by this t 0 to 2 pi. So, if I calculate the length it is coming out 4 pi square root 10; finally, the arc length s of a curve.

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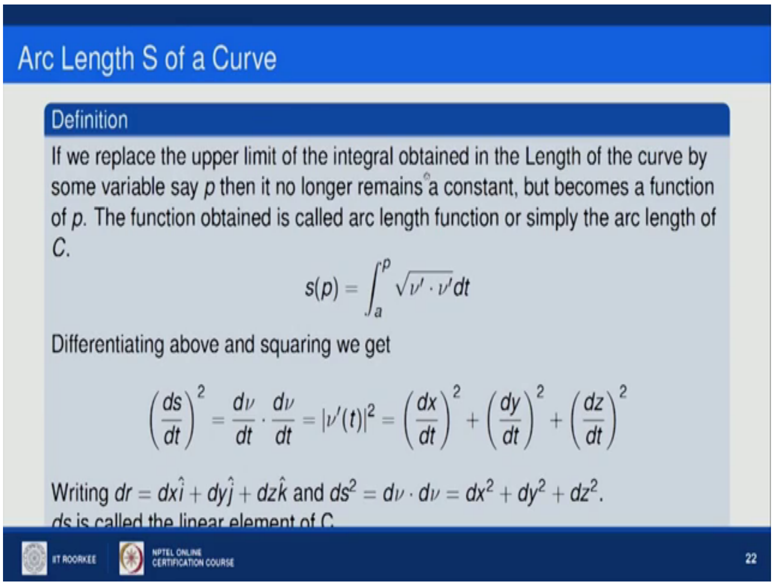

So, if we replace the upper limit of the integral that is basically the length was a to b square root r dash t dot product with dash t dt.

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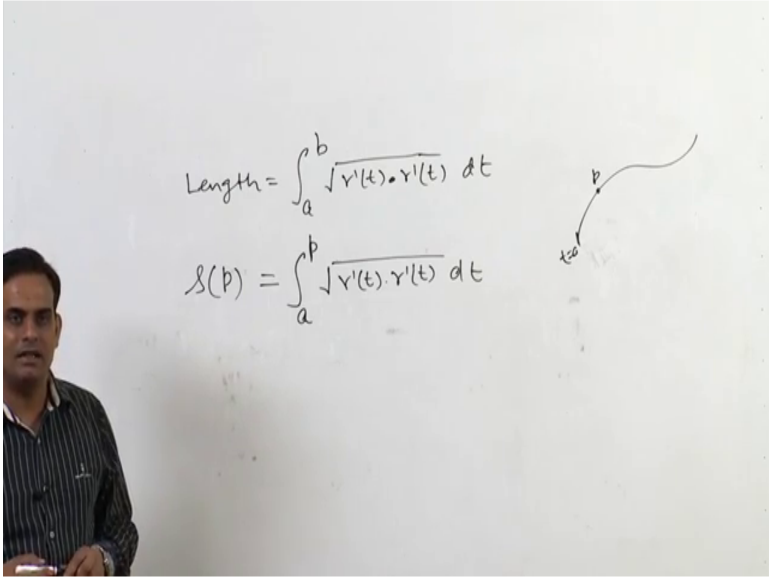

So, what I am doing? If this is the curve so, let us say this is t equals to a. So, I want to find out the length in terms of a parameter p, so that by putting a specific value p, I can get the let at any point. So, I want to find out length in terms of p. So, here arc length is s which is a function of p. So, initial point is a, and the upper limit I am replacing with this p. And then rest of the thing will be same. So, once I will get this, it will come out as a function of p, and then I can write p in terms of s, and then I can change the equation of the curve from t to p in terms of s; that is the parametric representation in terms of arc length. Let us see it with the help of an example.

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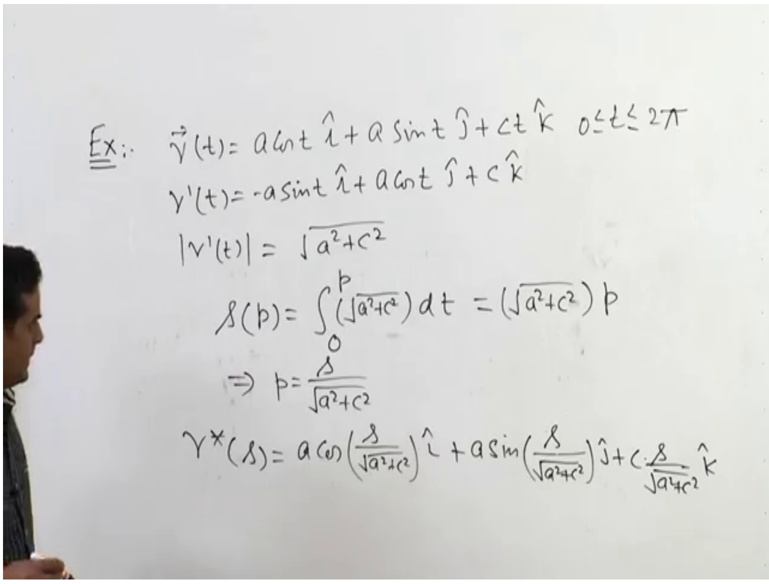

So, find arc length representation of a helix. So, as you know that for an helix, I am having the parametric representation as r t equals to a cos t i, and let us take the circular helix a sin t j plus c t k. Now here r dash t will become minus a sin t i plus a cause t j plus c k. The magnitude of r dash t will become square root a square plus c square. So, here arc length is sp; that is, and here let us take t is between 0 to 2 pi in the helix in this one shows 0 to p square root a square plus c square into dt. So, this comes out to be square root a square plus c square into p.

So, from here I can write p equals to s upon square root a square plus c square. And from here I can write the arc length representation of the curve. So, it will be a in terms of s. So, a cos s upon square root a square plus c square which will be the component in I direction plus, a sin a upon square root a square plus c square j plus c times s upon square root a square plus c square in term in the direction k. So, this is called the arc length representation of a given curve. So, what you have to do you have to find out the arc length represent a arc length, then write the parameter which we have chosen for arc length as a function of s, and then replace that particular thing with that this particular value.

So, in this nature we have learned differentiability continuity limits and various other concepts of vector functions, and then we have seen some applications of differentiability in terms of tangent vector in terms of finding the length of a curve. And finally, we have learn how to write the arc length representation of a curve; which is quite useful to find out a particular position at the curve where we are at a particular value of t.

So, with this I will end this lecture.

Thank you.