Multivariable Calculus
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Lecture – 27
Dirichlet's Integral

Hello friends. Welcome to the 27th lecture of this course. And in this lecture, I will introduce another important result from the multiple integral that is called Dirichlet integral. So, basically it is a means if the multiple integral is defined on a specified domain and it is having some specific type of integrand, and then we can write that particular integral in terms of gamma function directly. And this result is called Dirichlet integral.

So, let us start with the main result. So, it is saying that if v is the region such that.

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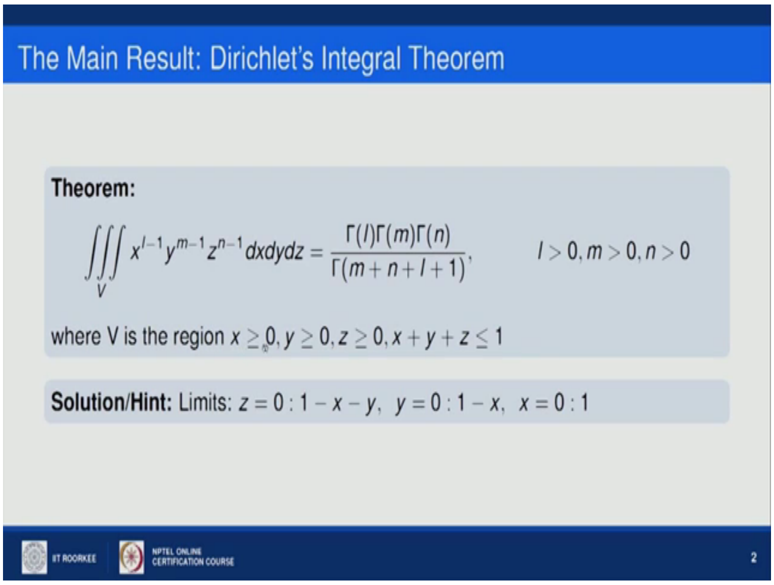

Xyz all are positive greater than equals to 0. And x plus y plus z is less than equals to 1, then the triple integral define on this region v having integral as xl minus 1 ym minus 1 zn minus 1 dx dy dz equals to gamma l into gamma m into gamma n, divided by gamma m plus n plus l plus 1. Here lm n and n all are positive.

So, let us try to prove this theorem that is called Dirichlet integral theorem.

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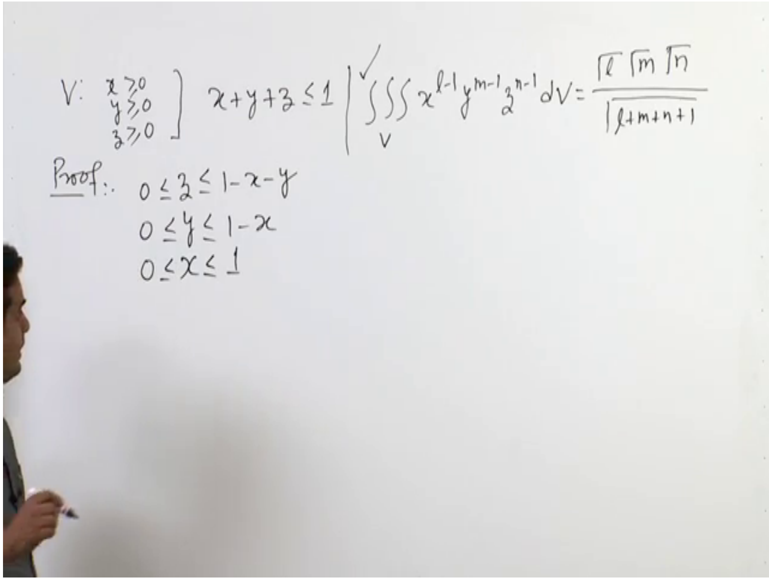

x is greater than 0, y is greater than equal to 0, z is greater than equal to 0; it means, we are there in the first obtained. Together with a condition that x plus y plus z is less than equals to 1. And then we are having the triple integral over this region v, such that x raised to power l minus 1, y raised to power m minus 1 z raised to power n minus 1 dv here dv stands for dx dy dz equals to gamma l, gamma m, gamma n, upon gamma l plus m plus n plus 1. So, we need to prove this result.

So, let us try to obtain the limits over different variables in this region v. So, here I can write that z is greater than 0, less than 1 minus x minus y, from this relation. So, I can take x and y in the right-hand side. Then I am having y is greater than 0 from here, and less than 1 minus x in xy plane; where z is 0. So, in this way x will become between 0 to 1.

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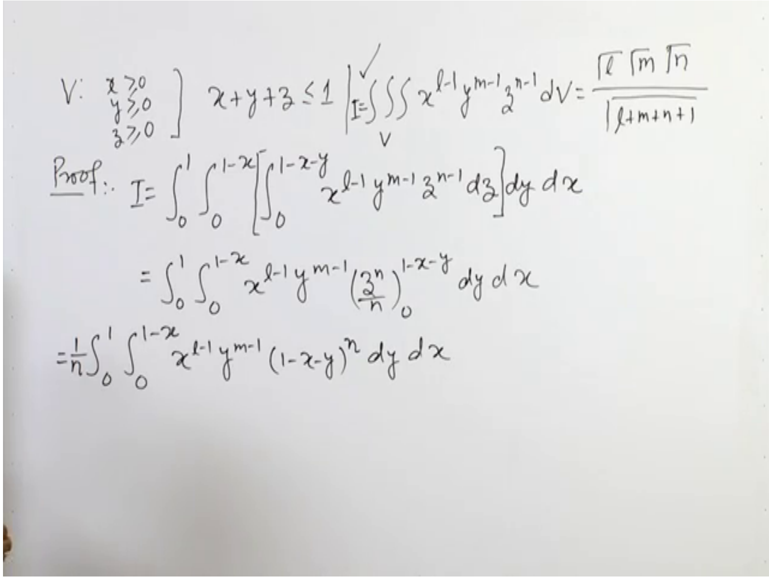

Hence if this is I, then I can write, I equals to 0 to 1 0 to 1 minus x, 0 to 1 minus x minus y, and then x raised to power l minus 1, y raised to power m minus 1 z raised to power n minus 1, then I am having this limit for dz dy and dx.

So, let us do this integral first that is in the square bracket it is an integral over z limits are from 0 to 1 minus xy. So, I can write this 0 to 1 0 to 1 minus x, and then I am having x raised to power l minus 1 y raised to power m minus 1. And when you will integrate it it will become z raised to power n upon n. So, I can write z raised to power n upon n limits are 0 to 1 minus x minus y, and then dy dx, this thing equals to 0 to 1, 0 to 1 minus x and I will take 1 by n out, this term x raised to power l minus 1, y raised to power m minus 1, and then 1 minus x minus y for z raised to power n, when you will put 0 it will come minus 0. So, it will remain same dy dx. So, what we have done? These triple integrals now become double integral.

Now, let us try to solve this double integral. So, for doing this I need to make some substitution and here I making a substitution like put y equals to 1 minus x times t.

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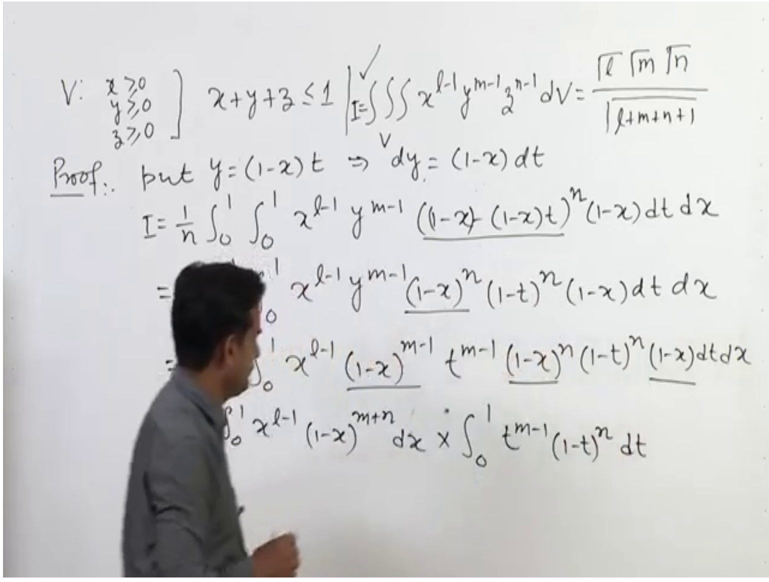

And this substitution I am assuming that I want to make this limit 0 to 1. So, from this when y 0, t will become 0, when y it is 1 minus x t will become 1. So, that is my idea to make the constant limits for both the integral. So, from here I am having dy equals to 1 minus x dt.

Now, substitute here in this integral. So, I will become 1 by n 0 to 1, and then as I told you limit will become 0 to 1 x raised to power l minus 1, y raised to power m minus 1, 1 minus x minus 1 minus xt. So, for this y I have written 1 minus x into t raised to power n. And then dy will become 1 minus x dt and finally dx. This equals to 1 by n 0 to 1 0 to 1 x raised to power l minus 1 y raised to power m minus 1, then this particular thing I can write 1 minus x raised to power n into 1 minus t raised to power n, because I can take common 1 minus x.

So, I will get one from here minus t into 1 minus x dt dx, 1 by 1 upon n then 0 to 1. So, by the property of double integral, since limits are constant, I can write as the product of 2 definite integrals, 2 single integral so, that I am doing here. So, 0 to 1 x raised to power l minus 1, 1 minus x one more change I need to do I did not replace this y. So, I need to replace this y also. So, this will be 1 by n 0 to 1 0 to 1 x raised to power l minus 1, y is 1 minus x raised to power m minus 1 t raised to power m minus 1, because y is 1 minus x times t, and then 1 minus x raised to power n 1 minus t raised to power n 1 minus x dt dx.

So now I will write it as the product of 2 single integral. So, 1 by n 0 to 1 x raised to power l minus 1 into 1 minus x raised to power m minus 1 from here, and from here, and one from here so, m minus 1 n 1. So, it will become 1 minus x raised to power m plus n, into the another integral, and this is our dx into 0 to 1, t raised to power m minus 1, and then 1 minus t raised to power n dt.

Now, we will try to write these 2 integral in terms of beta function, and then from beta functions, I will try to obtain the results in terms of gamma function.

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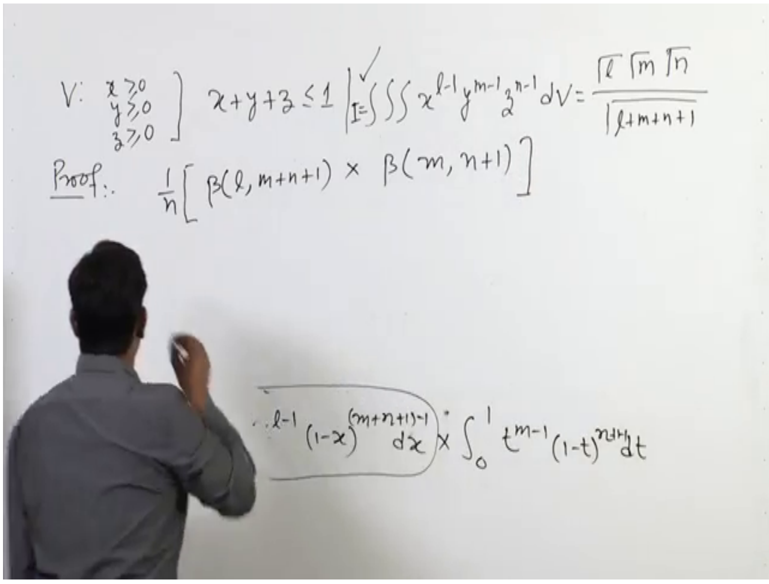

So now, from here 1 by n if you look this I can write this m plus n plus 1 minus 1. So, this I can write beta of l m plus n plus 1, this particular integral the first one, into the second integral I can write beta of m minus 1 will be m, and this I can write n plus 1 minus 1. So, n plus 1, and this is I equals to I means this particular integer.

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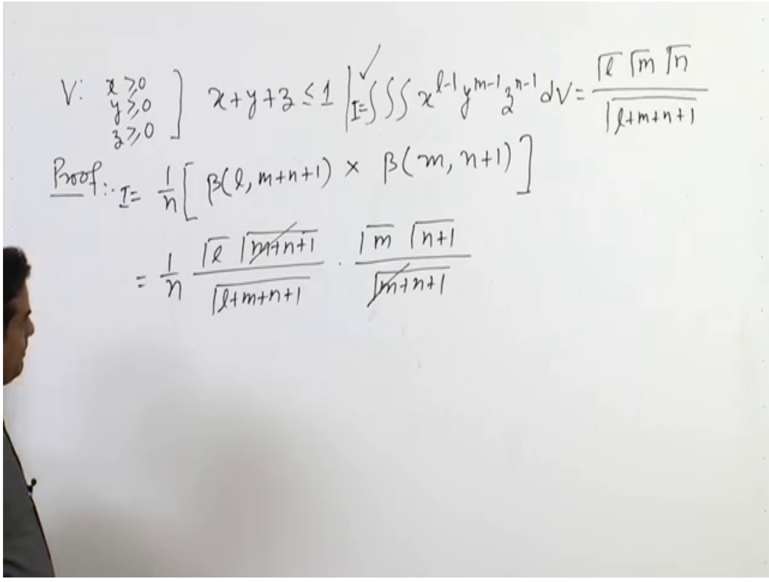

So, this equals to 1 by n beta this is I can write gamma l gamma m plus n plus 1 upon gamma l plus m plus n plus 1 into gamma m gamma n plus 1 upon gamma m plus n plus 1. This cancel, this gamma n plus 1 I can write as n gamma n.

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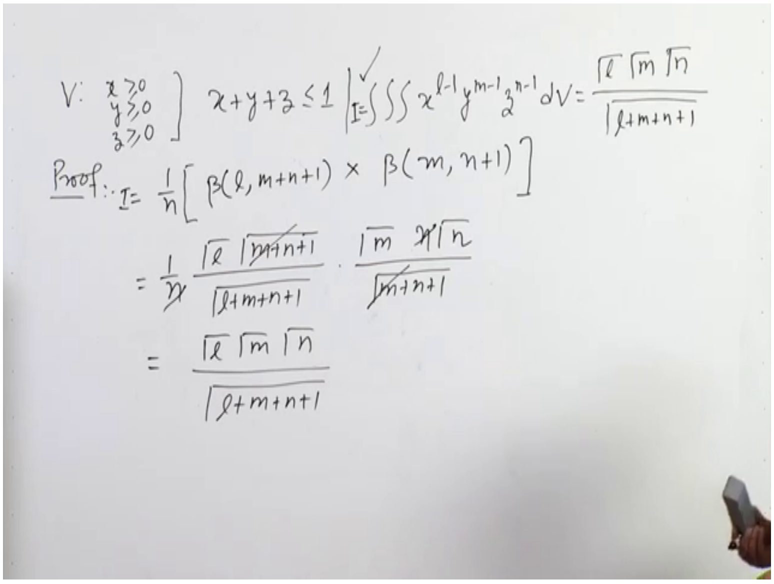

So, n will be cancel out. So, from here I will be having gamma l gamma m gamma n upon gamma l plus m plus n plus 1. And this is what we need to prove.

So, this is the proof of Dirichlet integral theorem. Now let us take an example based on this integral formula.

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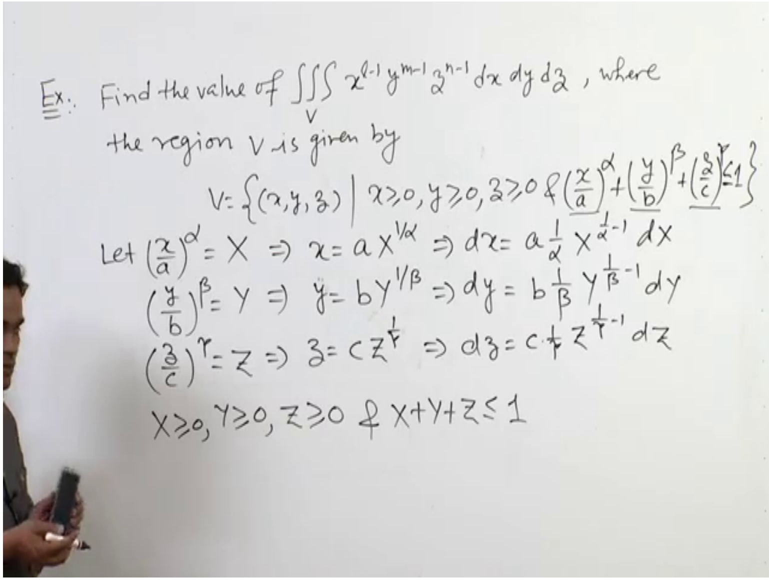

So, find the value of over a region v x raised to power l minus 1 y raised to power m minus 1 z raised to power n minus 1 dx dy dz; where the region v is given by v equals to xyz, such that x is greater than 0, y is greater than equals to 0, z is greater than equals to 0. And x by a raised to power alpha plus y raised to power b beta plus z raised to power c gamma equals to 1, less than equals to 1. And all lm and abc and alpha beta gamma are positive.

So now let us try to obtain solve this example. So, here we can apply the Dirichlet integral formula in this example by looking on the problem; however, our region is different. So, first of all what we need to do; we need to change our region according to the Dirichlet integral formula. So, here first term is x upon a raised to power alpha. So, let x upon a raised to power alpha equals to capital X; this will give me x equals to a times x raised to power 1 upon alpha, or dx will be a times 1 by alpha x raised to power 1 upon alpha minus 1 dx.

Similarly, I will assume that y upon b raised to power beta equals to capital Y. So, from here I will get y equals to b times capital Y raised to power 1 upon beta, and then dy equals to b into 1 upon beta y raised to power 1 upon beta minus 1 dy. So, please note that in left hand side I am having small xy in the right-hand side I am having capital Xy. Similarly, I will put z upon c raised to power gamma equals to Z; so, here capital Z. So, this will give me small z equals to c times capital Z raised to power 1 upon gamma, and then from here I will get d small z equals to c into 1 upon gamma capital Z raised to power 1 upon gamma minus 1 dZ.

Now, if I am having this kind of thing since a alpha b beta c gamma all are positive so, it means capital X is greater than equals to 0, capital Y is greater than equals to 0, and capital Z is also greater than equals to 0. So, my new region will become x is greater than 0, capital Y is greater than 0, capital Z is greater than 0. And X plus Y plus Z is less than equals to 1. Because this is my capital X, this is capital Y, this is capital Z.

Now, so, region is now similar to the Dirichlet integral, only thing I need to make substitution. So, once I will make these substitution, then we will see what formula we will get.

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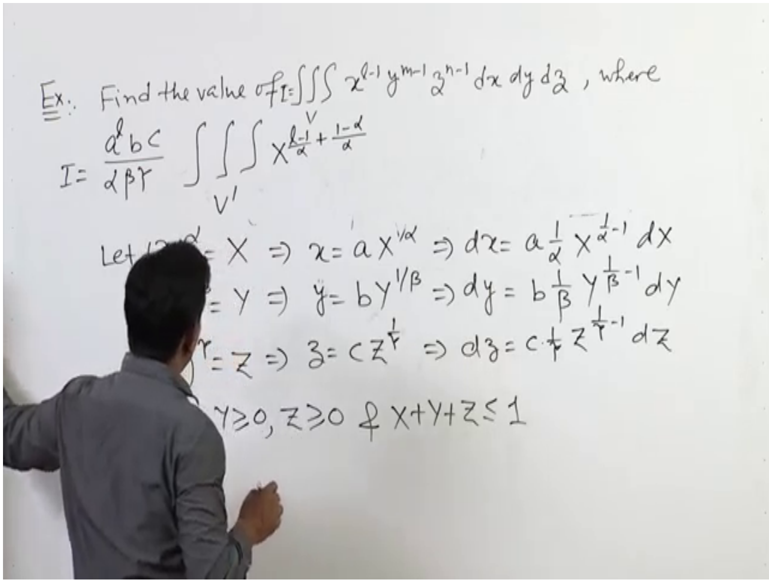

So now, let us assume this is my integral I so now, I equals to after making all these substitution. So, you can notice here abc upon alpha beta gamma alpha beta gamma. This is I am getting from that dx dy dz integral over new region v dash which is given by this one, x raised to power l minus 1. So, x raised to power l minus 1 will become a raised to power l minus 1 into x raised to power capital X raised to power l minus 1 upon alpha.

So, capital X raised to power l minus 1 upon alpha and since a raised to power l minus 1 is here so, a raised to power l ok. And then what I am getting something from dx also, x raised to power 1 upon alpha minus 1. So, it will become plus 1 minus alpha upon alpha, because it is 1 minus alpha upon alpha. So, if I simplify it so, it will become minus 1 plus 1 will cancel. So, it will become x raised to power l upon alpha minus 1.

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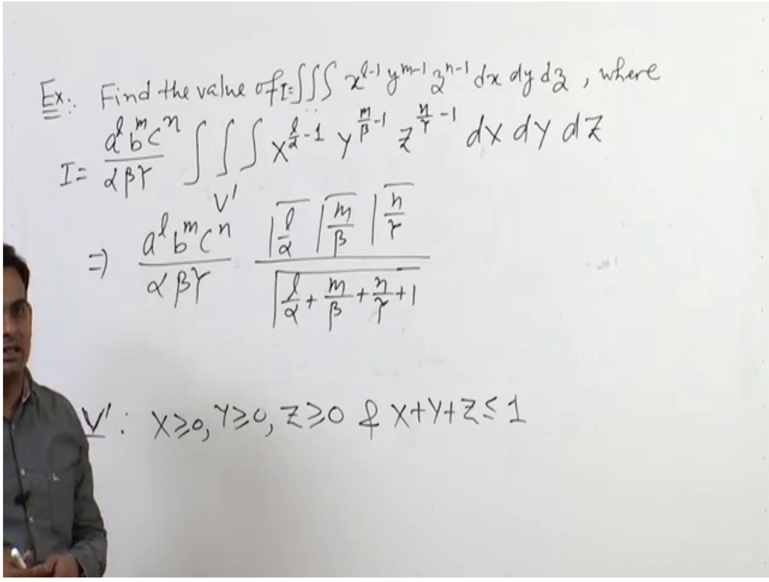

Similarly, for beta I can write y capital Y raised to power m upon beta minus 1. And then capital Z, raised to power n upon gamma minus 1 and then dx d capital Y d capital Z. So now, I need to solve this integral over this region. So, by the Dirichlet formula what I will get? So, I this will become a raised to power l, b raised to power n, c raised to power n alpha beta gamma gamma l by alpha gamma m by beta gamma n by gamma a small gamma upon. So, this is the value of this integral over the earlier mentioned region. So, this is the solution of this example.

So, another example I am taking that is quite simple.

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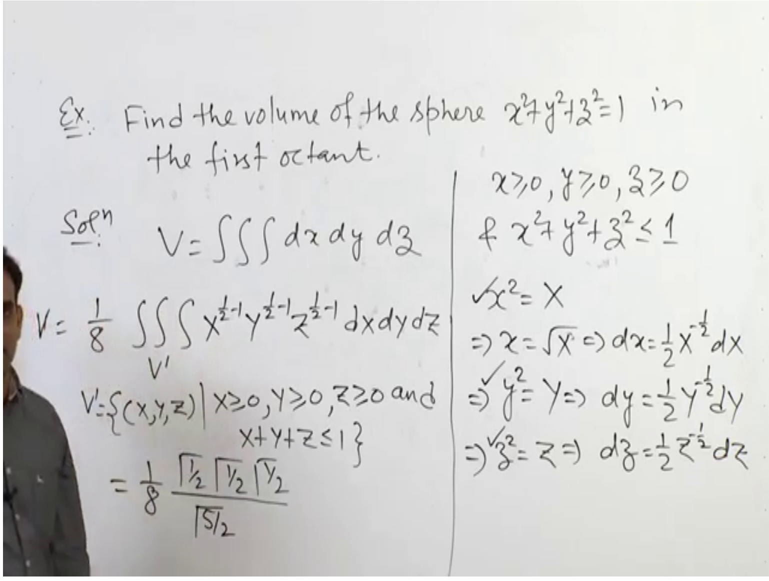

So, example is find the volume of the sphere x square plus y square plus z square equals to 1 in the first octant. Why I am writing first octant? Because I need to put the condition xyz should be non-negative. So, as you know that volume is dx, dy, dz. Now here area is or region is x is greater than 0, y is greater than 0, z is greater than 0, all these are coming due to this condition first obtained. And x square plus y square plus z square less than equals to 1.

Now, it is not in the form as we need in the Dirichlet integral formula. So, we need to make some substitution. So, let us make like earlier example x square equals to capital X ok. So, here x will become root X, root capital X. So, from here dx will be 1 by 2 x raised to power minus 1 by 2 dx. Similarly, let us say y square equals to capital Y. So, this will give you dy equals to 1 by 2 capital Y raised to power minus half dy. And finally, z square equals to capital Z. So, it will give me dz equals to 1 by 2 capital Z raised to power minus half dz. So, from this, this and this a situation this region can be written as capital X greater than 0, capital Y greater than equals to 0, capital Z greater than equals to 0, and then capital X plus capital Y plus capital Z is less than equals to 1.

So now this integral will become over a new region v dash, dx is 1 by 2, 1 by 2, 1 by 2. So, that I am taking out. So, 1 by 8, and then capital X raised to power minus half. So, this I can write capital X raised to power half minus 1, capital Y raised to power half minus 1, capital Z raised to power half minus 1 dx dy dz, and region v dash is given by capital X capital Y capital Z, such that capital X is greater than equals to 0, capital Y is greater than equals to 0, capital Z is greater than equals to 0, and x plus y plus z less than equals to 1.

So, by the Dirichlet integral formula, this will be equal to 1 by 8 gamma half gamma half, gamma half upon gamma half plus half plus half plus 1. So, it will be gamma 3 by 2 plus 1 that is 5 by 2. You can simplify it.

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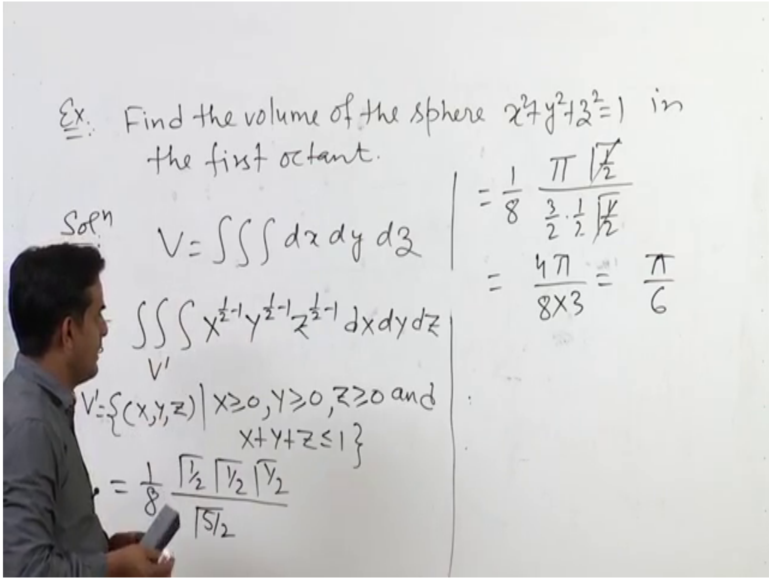

So, this will become 1 by 8. So, gamma half into gamma half will become pi into gamma half upon 3 by 2 into 1 by 2 into gamma half. So, this will be canceled out. So, this will become 4 pi 8 into 3, because this 4 will go in the numerator. So, this will be pi upon 6. So, this is the solution of such an example.

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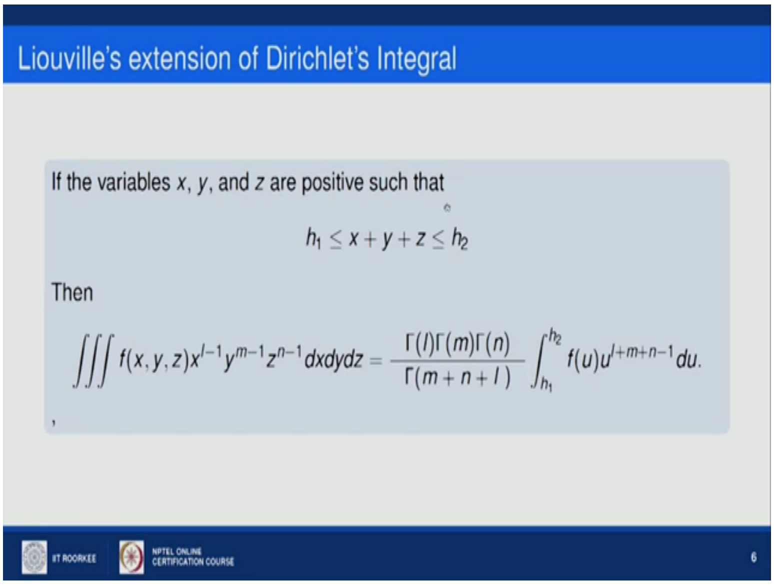

Now, one more important result in this lecture that is liouvilles extension of the Dirichlet integral; so it is saying that if the variable xy and z are positive such that x plus y plus z is bounded by h 1 and h 2. So, h 1 is greater than h 1 is less than equals to x plus y plus z, which is less than equals to h 2, then the triple integral f of xyz at x raised to power l minus 1 y raised to power m minus 1 and z raised to power n minus 1 dx dy dz over this particular region is given by this particular expression, that is gamma l gamma m gamma n upon gamma l plus m plus n integral h 1 to h 2 fu u raised to power l plus m plus n minus 1 du.

So, let us take an example on this particular result also.

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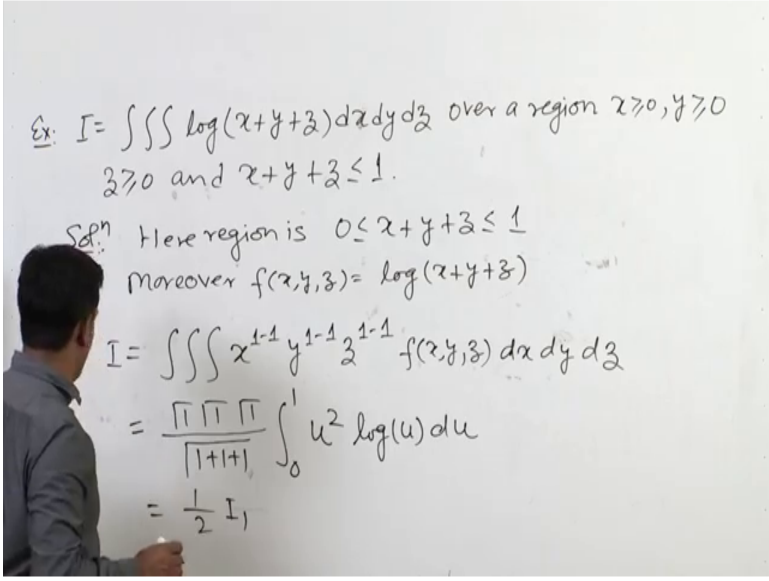

So, here we need to evaluate log x plus y plus z dx dy dz. Over a region that is less than equals to 1. So, solution so, here region is something like this, moreover hero f of xyz equals to log x plus y plus z. So now, by the liouvilles extension of Dirichlet integral, I can write this triple integral as x raised to power 1 minus 1, y raised to power 1 minus 1 z raised to power n 1 minus 1 f of xyz where f of xyz is given by this one dx dy dz. So, by the formula it will be equals to gamma l, gamma m gamma n, all are 1 upon gamma l plus m plus n.

So, 1 plus 1 plus 1 integral 0 to 1 u raised to power l plus m plus n minus 1. So, here it will become u square because 1 plus 1 plus 1 minus 1, and then f of u will be log of u du. So now, this will be 1 upon factorial gamma 3 gamma 3 factorial 2 that is 2 into, I 1, I 1 is given by this definite integral. So now, let us solve this integral what will be the value of this. So, I am having integral 0 to 1, u square log of u du. So, let us do integration by parts. So, this is my second function and first function. So, it will be log u as such u 3 by 3 limit 0 to 1.

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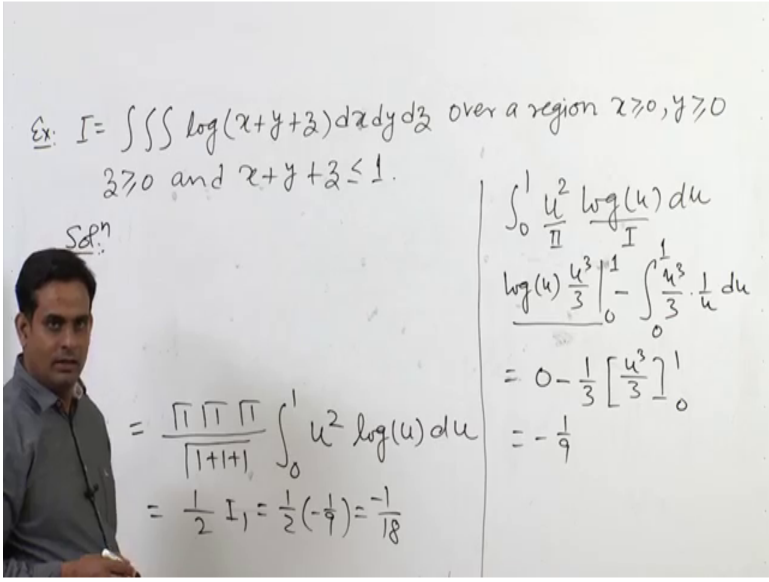

Minus u 3 by 3 that is the integration of second into differentiation of one. So, log u will become 1 by u differentiation of log u du 0 to 1.

So, when I am putting 1. So, log one is 0, when I am putting 0 by rule it will come out 0. So, this is basically 0 this term, minus u square. So, 1 by 3 I have taken here, and then u square. So, u square will become again u 3 by 3, 0 to 1. So, this will be minus 1 upon 9. So, this will be 1 by 2 into minus 1 upon 9; so 1 upon 18 minus. So, this is the answer of this particular problem.

So, with this I will end this lecture. So, in this lecture we have learned the Dirichlet integral formula proof of this formula, and some example based on it.

Thank you very much.