Multivariable Calculus
Dr. Sanjeev Kumar
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 26
Properties of Beta and Gamma Functions-II

Hello friends. So, welcome to the 26th lecture of this course. And this lecture is again just the continuation of the previous lecture. Here, I will introduce few more properties of beta and gamma functions. Again, we will take some example, and we will solve them in terms of beta and gamma function.

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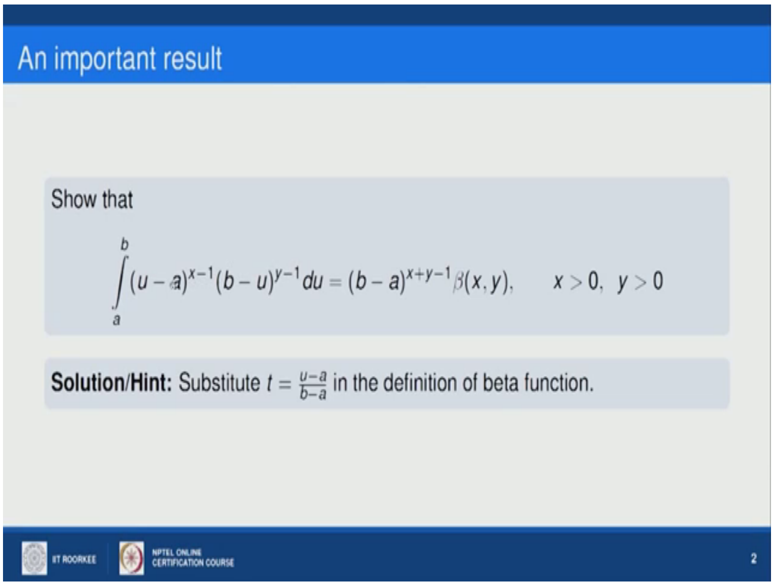

So, first of all, see this important result related to beta function so, it is saying that the integral over a to b u minus a raise to power x minus 1 b minus u raise to power y minus 1 d u equals to b minus a x plus y minus 1 beta of xy; where x and y are positive numbers. So, the result is something like that.

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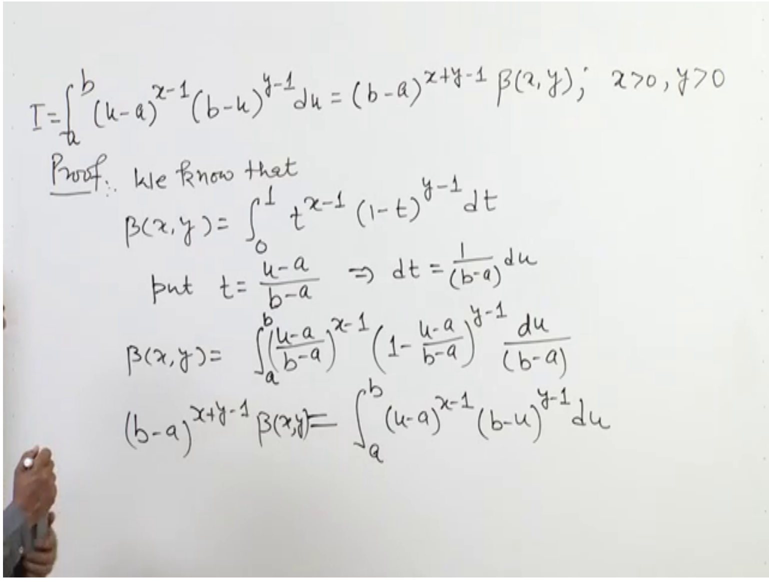

So, integral I equals to integral over a to b u minus a raise to power x minus 1 b minus u raise to power y minus 1 du. This equals to b minus a raise to power x plus y minus 1 into beta xy, x and y are positive.

So, let us try to prove it so, proof is something like that. We know that beta of xy equals to integral over 0 to 1 p raise to power x minus 1, 1 minus t raise to power y minus 1 dt. So, this is the definition of beta function we have seen. Now put t equals to u minus a upon b minus a. So, this will give us dt equals to 1 upon b minus a du. So, from here I can write beta xy equals to t is u minus a upon b minus a raise to power x minus 1. Then 1 minus u minus a upon b minus a raise to power y minus 1. And dt is du upon b minus a.

So, if you notice here in the denominator I am having b minus a raise to power x minus 1 from this expression, b minus a raise to power y minus 1 from this expression; so, it will become x plus y minus 1 minus 1. So, x plus y minus 2, the same time we are having plus one from here. So, so, total I am having b minus a raise to power x plus y minus 1, and I forgot to put limit here, when t 0 u will become a; when t is 1 u will become b, integral over a to b.

So, this will be u minus a raise to power x minus 1. B minus a minus u plus a; so, this will become b minus u because minus a and plus a will be cancelled out raise to power y minus 1 du. And this thing equals to beta of xy. So, if I take this constant term another side I will be having this equals to b minus a raise to power x plus y minus 1 into beta of xy equals to this 1. And this is the relation which we need to show.

So, this is the proof of this particular example. Now let us see the consequences of this particular relation.

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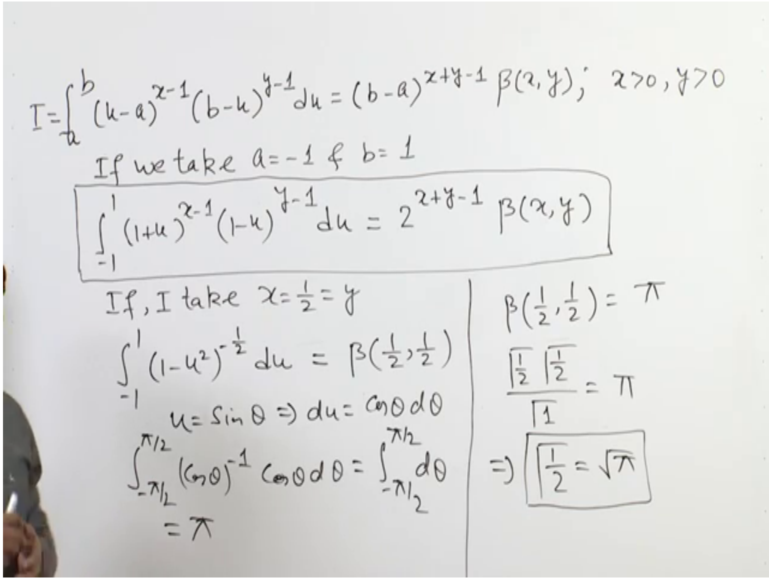

If we take a equals to minus 1 and b equals to 1, then what we will be having? Minus 1 to 1 u plus 1 raise to power x minus 1, and then b is one. So, it will be 1 minus u raise to power y minus 1. So, better let me write it 1 plus u to maintain the symmetry. So, 1 plus u raise to power x minus 1 into 1 minus u raise to power y minus 1 du. And what will be the value of this definite integral b minus a, b minus a means 1 minus 1 minus 1 1 plus 1 2 raise to power x plus y minus 1 into beta xy.

So, we obtain another relation in beta function and definite integral. Now in this relation if I take x equals to half equals to y, then what I will be having here? 1 minus 1 to 1 1 plus u raise to power minus half, 1 minus u raise to power minus half. So, it will become 1 plus u into 1 minus u will become 1 minus u square raise to power minus half du. And in the right hand side, it will become 2 one half plus half minus 1. So, 2 raise to power 0 total; so 1 into beta xy.

So, beta xy will be beta 1 by 2 1 by 2. Now let us see about this particular integral here if I substitute u equals to sin theta, then what I will be having? Du equals to cos theta d theta. So, what integral it will be; when u is minus 1 so, sin theta is minus 1. So, it will become minus pi by 2 theta and when it is one it will become pi by 2, 1 minus u square means 1 minus sin square theta that will become cos square theta. So, cos square theta raise to power minus half. So, basically it will become cos theta raise to power minus 1 into du will be cos theta d theta. So, it will be basically minus pi by 2 to pi by 2 d theta, and this equals to pi. So, what I am having? My left-hand side integral is giving me pi, right hand side it beta half half. So, beta half half equals to pi. Now if we see the relation between beta and gamma function. So, it will be gamma half gamma half upon gamma 1 equals to pi. So, it means gamma 1 is 1. So, gamma half square equals to pi.

So, from here again we are getting our famous relation that gamma half equals to root pi. So, we ended up with this relation. Hence, we can use this formula in many ways to evaluate definite integral by substituting the appropriate values for xy and ab ok. So, now we will see one more important result that is gamma x plus 1 approximately equals to x raise to power x plus 1 by 2 e raise to power minus x into a square root 2 pi.

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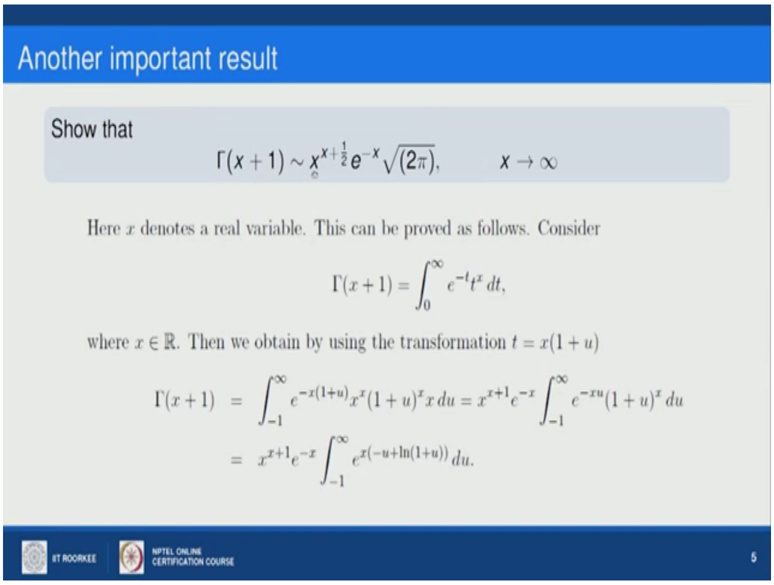

When x tend in to infinity means, this relation is 2 for large values of x not for the small values of x. So, how to prove this relation that is a bit tricky.

Let me give the outline of this so, here x denotes a real variable. And we know from the definition of gamma function that I can write gamma x plus 1 equals to integral 0 to infinity, e raise to power minus into t raise to power x dt. Now what you do in this?

Substitute t equals to x into 1 plus u, then in the left-hand side it will remain as gamma x plus 1. In the right-hand side when t 0 u will become minus 1, when t is infinity u will remain as infinity so, it will become minus 1 to 0 e raise to power minus t. So, t is x times 1 plus u so, e raise to power minus x 1 plus u and then t raise to power x. Means, x raise to power x into 1 plus u raise to power x. And then you know, dt will become x times du. So, this x is coming due to due to dt so, dt equals to x times du. So, after this particul lar transformation or substitution this particular integral becomes in this form. Now as you know that this integral is on variable u. So, I can take the terms involving x out. So, I have taken x raise to power x from here, x from here. So, total will become x raise to power x plus 1 e raise to power minus x I have taken out here.

And then integral over minus 1 to infinity e raise to power minus xu into 1 plus ux du. Or this I can write that 1 plus u raise to power x I can write as e raise to power x into log of 1 plus u. So, that in this way this particular integral can be written in this form. So now, I am having this particular integral. Now the question is how to solve this integral. What sort of transformation we should make to find out the value of this particular integral and on what values of x we can evaluate this particular integral and so on. So now, let us can see this function minus u plus ln 1 plus u.

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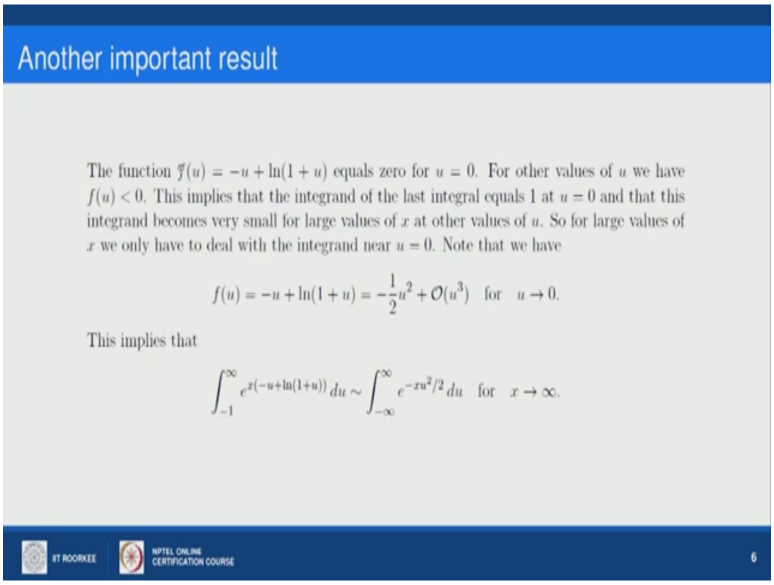

So, let us assume that it is a function of u and I am naming it f of u. So, f of u is minus u plus log 1 plus u. This thing equals to 0 for u equals to 0. Because when u is 0 this term will be 0, and log one will be 0. For other values of u we have the value of u is negative, because always the value of this particular thing will be greater than log 1 plus u. Hence, for other values of u fu will be a negative function; means, it will be below x axis or u axis. This implies that the integrand of the last integral equals to 1 at u equals to 0. And that this integrand becomes very small for large values of x, at other values of u.

So, the large values of x we only have to deal with integrand near u equals to 0. Because it is very small for other values of u. Note that, we have fu is minus u plus ln 1 plus u. So, I can write it minus half u square plus terms having u raise to power 3 and higher order term for u tending to 0. Means, I have explained log 1 plus u. So, this u will be cancelled out. So, I will be having terms of second order. This implies the integral over minus 1 to infinity, e raise to power x times minus u plus log 1 plus u du is approximately equal to minus infinity to infinity e raise to power minus xu square upon to du, for x tending to infinity.

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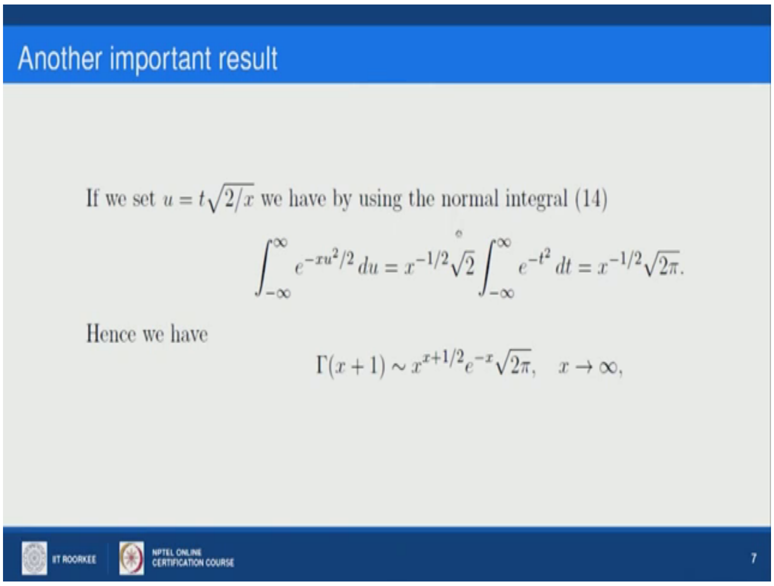

If we set u equals to t times root 2 upon x we have the less and minus infinity to infinity e raise to power minus x is u square by 2 du equals to x raise to power minus half root 2 will be out, minus infinity to infinity e raise to power minus t square dt. And you know that minus infinity to infinity e raise to power minus t square is root pi. So, this becomes x raise to power minus half root 2 pi. Hence, we have gamma x plus one which is approximately equals to this relation equals to x into x plus 1 raise to x plus half e raise to power minus x root 2 pi when extending to infinity so, this is what we need to prove.

So, proof is a bit complicated. You should have the idea of plug 1 plus u how to expand it, and how to check the value of fu for different values of u and x. And how to write this particular integral approximately equals to this one for large values of x. Now let us take one more example, it is given that integral over minus 1 to 1 1 minus t square raise to power ndt.

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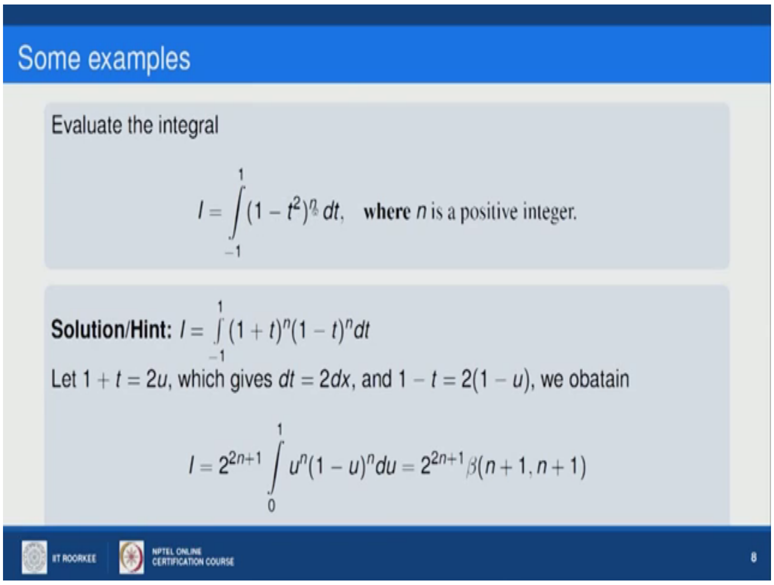

Where n is a positive integer and we have to evaluate the value of this particular integral.

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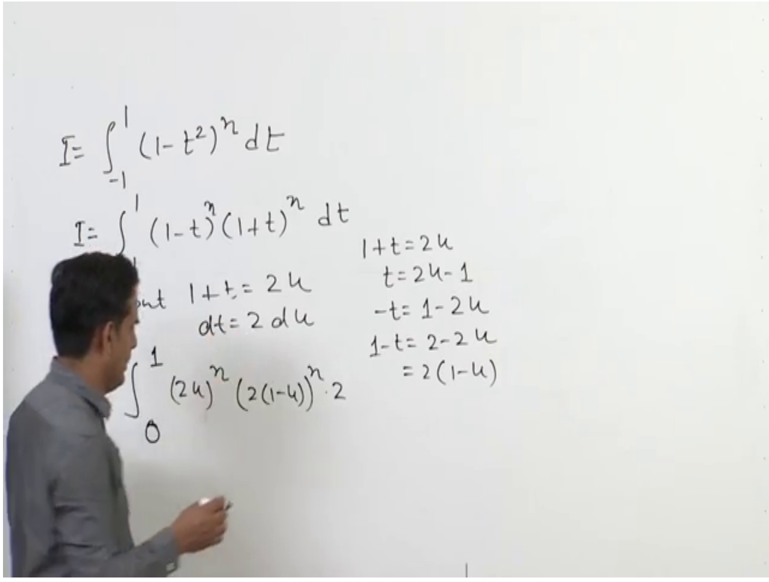

So, integral is I equals to minus 1 to 1 1 minus t square raise to power n dt and is a positive integer. So, I can write I as minus 1 to 1 1 minus t 1 plus t.

So, 1 minus t raise to power n 1 plus t raise to power n dt. Now how to move? Ok, we can do in this way. Put 1 plus t equals to twice of u. This will give me dt equals to 2 times du. So, from here I will be having when t is minus 1 u will become 0 1 minus 1 0, when t is plus 1 u will become 1. 1 plus t is 2 u so, 2 u raise to power n, now what will be 1 minus t. So, 1 plus t equals to 2 u. So, t equals to 2 u minus 1.

So, minus t equals to 1 minus 2 u, and then 1 minus t will become 2 minus 2 u. That is 2 times 1 minus u. So, let me write it here, 2 times 1 minus u raise to power n into 2 is coming from dt du.

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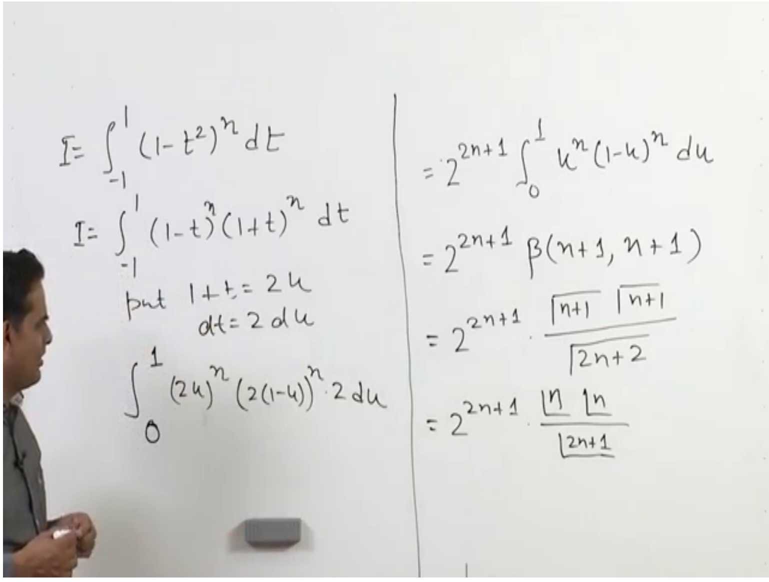

I can take all these 2 outside. So, 2 raise to power n from here. 2 raise to power n from here, and 2 from here. So, it will become 2 raise to power 2 n plus 1, 0 to 1 u raise to power n 1 minus u raise to power n du. Now this thing equals to 2 raise to power 2 n plus 1.

And then see this particular integral it is we can equate it with the definition of beta function. So, it will become beta of n plus 1 n plus 1 or n is a positive integers. So, 2 raise to power 2 n plus 1, we know the relation between beta and gamma function. So, I can write it gamma n plus 1, gamma n plus 1, upon gamma 2 n plus 2. Let us make more simplification in it, 2 raise to power 2 n, plus 1 gamma n plus 1 is factorial n because n is a positive integer gamma n plus 1 is again factorial n n is a positive integer.

So, 2 n plus 2 so, it will become factorial of 2 n plus 1 and later on you can simplify it a bit more. So, this is the value of this particular integral few more examples.

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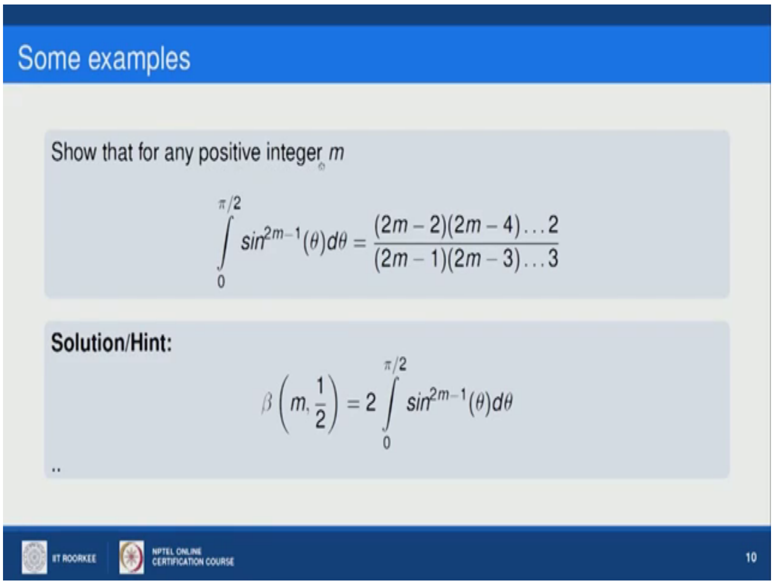

So, that for any positive integer m integral over 0 to pi by 2, sin theta raise to power 2 m minus 1 d theta equals to this particular thing that is in the numerator I am having 2 m minus 2 2 m minus 4 and up to 2, in the denominator I am having to m minus 1 to m minus 3 up to 3. Now how to solve it?

So, basically we need to prove that integral 0 to pi by 2 sin theta raise to power 2 m minus 1 d theta equals to 2 m minus 1 2 m minus 3 up to 1.

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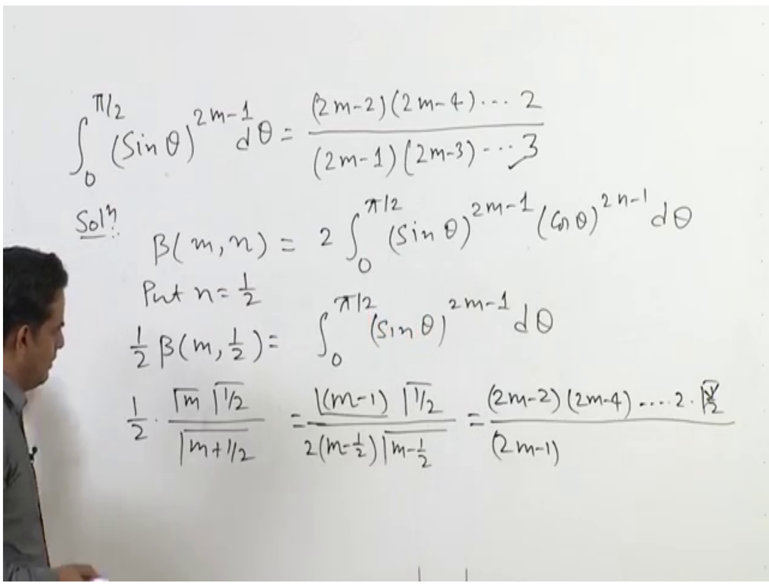

Or one is not make any effect. So, better I will light up to 3. We know that beta of mn can be written as 2 times 0 to pi by 2 sin theta raise to power 2 m minus 1 cos theta raise to power 2 m minus 1 d theta. From where I am getting this feeling because I need to solve this integral. So, here the term is involving sin theta.

So, I have taken this particular identity of beta function. Now put n equals to half here. So, what I am having? Beta m half equals to 2 times 0 to pi by 2 sin theta raise to power 2 m minus 1 d theta and if I take this 2 this side. So, it will become 1 by 2 this one. So, the right hand side of this exactly equal to this one. Now we have to prove that this particular value equals to this one. So, let us try to expand it, 1 by 2 into beta mn will become gamma m gamma n upon gamma m plus n; so gamma m gamma half upon gamma m plus half. Here m is a positive integer.

So, this equals to factorial m minus 1 gamma half upon gamma n minus half. Now what you do? Here if I open this factorial it will become m minus 1 m minus 2, and it will go up to 2 1 finally, 1 will come. So, total terms are n minus 1 terms. So, I multiplied in numerator as well as denominator by 2 raise to power m minus n so, one 2 I forgot here this 2. So, 2 raise to power m minus 1 in numerator 2 raise to power m minus 1 in denominator.

So, in each term I am multiplying by 2. So, when I will multiplying by 2 in m minus 1 it will become 2 m minus 2. When I am multiplying in m minus 2 it will become 2 m minus 4. And then up to 2 into gamma half upon; here also I am multiplying with 2 raise to power m minus 1 and 1 2, I am already having here. So, if I multiplied by this in this 2 in this one it will become 2 m minus 1. And then in this way it will moving on I will get the required denominator and this gamma will be cancelled out.

So, in this way I will get the relation given in the problem. Similarly, if instead of 2 m minus 1 I am having sin 2 m theta in the integral then I will get this result.

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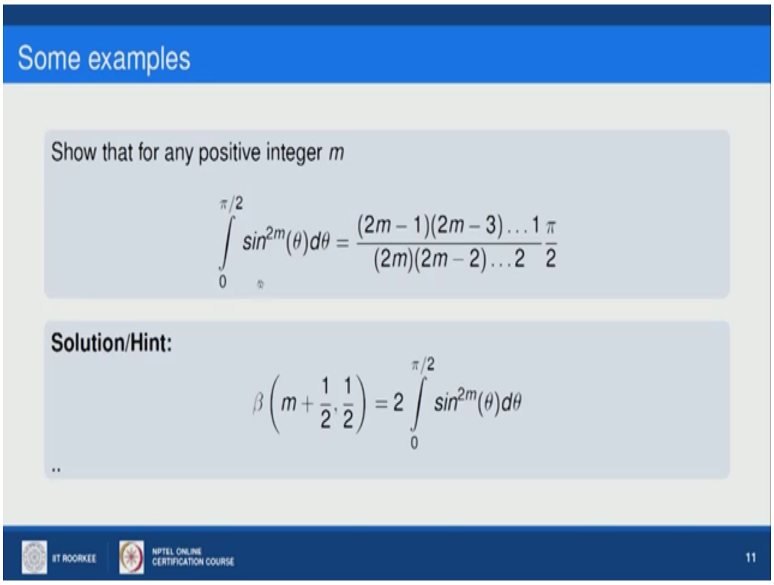

And this will be obtained in a same way, just by taking the value of m plus half and half, and following the same process as we are having in the earlier example.

So, in this way, I will end the beta and gamma functions. Later on we will see the applications of these 2 functions in multiple integral in next few lectures; where the integral can be directly written in terms of these functions.

So, thank you very much.