Multivariable Calculus
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Lecture - 25
Properties of Beta and Gamma Functions – I

Hello friends, so welcome to the 25th lecture of this course. And in this lecture, I will introduce some more properties of beta and gamma functions. So, in the past couple of lectures, I told about beta functions and gamma function and then I have taken some identities on these two functions. So, let us come across few more properties of these two functions.

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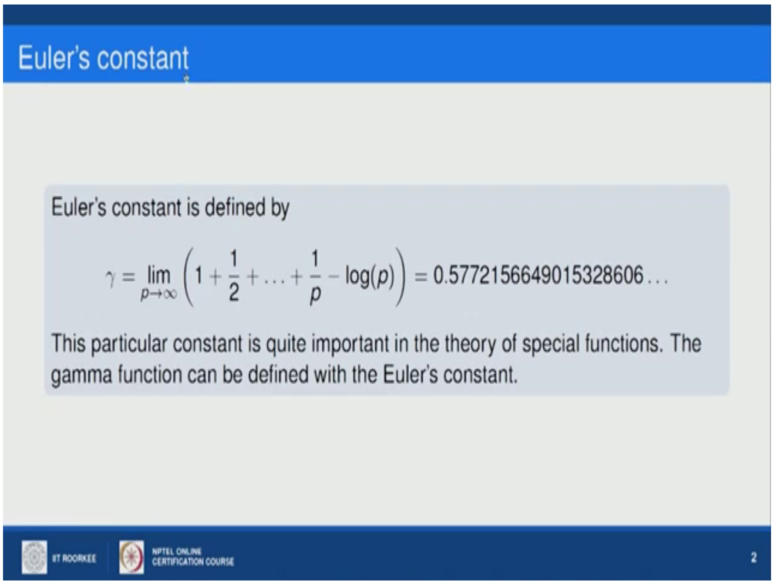

So, before going to those properties, I would like to introduce Euler's constant. So, Euler's constant is defined by gamma equals to limit p tending into infinity 1 plus 1 by 2 plus up to 1 by p minus log p. And if we if we find out the value of this limiting function, then we are getting this value, so 0.5772 and so on. So, this particular constant is quite important in the theory of special functions. We can relate gamma function also with this particular constant.

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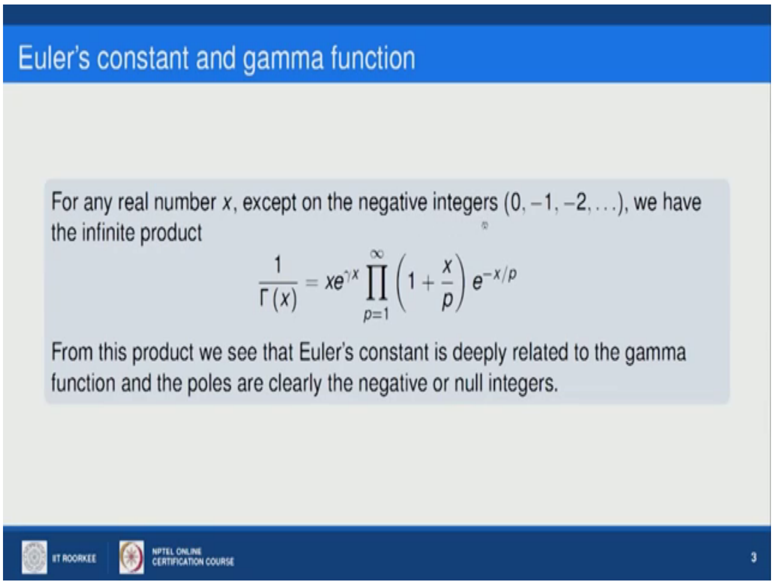

So, for any real number x except on the negative integers that is a null integer 0 minus 1 minus 2 and so on. We have the infinite products x into e raise power gamma x, here gamma is the Euler's constant, and then a product of the infinite series where p equals to 1 to infinity 1 plus x upon p. So, it will be like 1 plus x, 1 plus x by 2, 1 plus x by 3 and so on, e raise to power minus x upon p. So, and this particular expression will be equals to 1 upon gamma x. So, this relation between the gamma function and Euler's constant was given by v r starts.

So, from this product we see that Euler's constant is deeply related to the gamma function and the poles are clearly the negative or null integers. As we are not going to define it x 0 minus 1 minus 2 and so on, because these are the poles.

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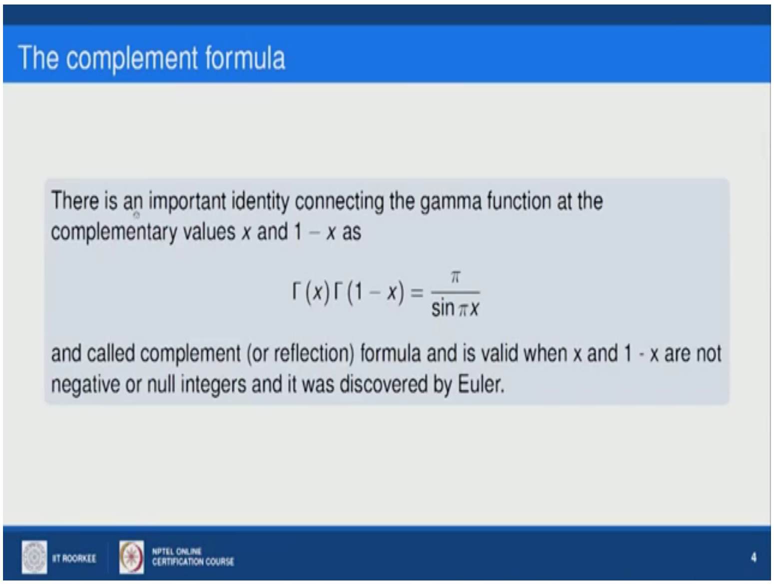

Based on this definition, we will see a very important property of gamma function that is called the complement formula. So, there is an important identity connecting the gamma function at the complementary value complementary values means x and 1 minus x; x, gamma x into gamma 1 minus x equals to pi upon sin pi x. And this formula is called complement or reflection formula and it is valid when x and 1 minus x are not negative or null integers.

So, please because we are not going we have not defined gamma function for these negative integers and null integer and it was discovered by Euler. So, let us try to prove this relation from the relation between gamma function and Euler's constant.

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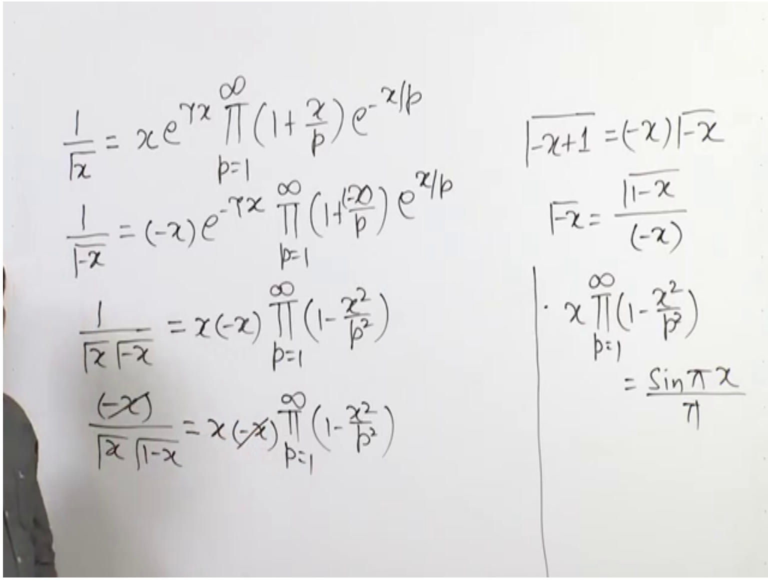

So, we know that that 1 upon gamma x equals to x into e raise power gamma x and then product from p equals to 1 to infinity 1 plus x upon p into e raise to power minus x upon p. Similarly, if I define it for minus x so 1 upon gamma minus x will be minus x into e raise to power minus gamma x then p from 1 to infinity infinite term product 1 plus x upon p. And here x is minus x so I am writing minus x here e raise to power minus x is here, so minus, minus will become plus x upon p.

Now, if I multiply these two relations then in left hand side I will be having gamma x into gamma minus x in the denominator. So, here I am having e raise to power gamma x, here I am having e raise to power minus gamma x. So, it will become e raise to power 0 and that is 1.

So, I will be having x into minus x for x form here minus x from here. And then p equals to 1 to infinity 1 minus x square upon p square, because here you are having 1 plus x upon p, here 1 minus x upon p. So, 1 plus a into 1 minus a will become 1 minus a square. And then e raise to power minus x upon p into e raise to power x upon p, it will become e raise to power 0 that is 1. So, after multiplying this we are having this relation.

Now, we know that that minus x plus 1 the gamma function value of this particular thing that it is minus x plus 1 will be minus x into gamma minus x from the lesson that gamma n plus 1 equals to n gamma n. So, the same thing here n is minus x. So, from here I can write gamma minus x equals to gamma 1 minus x that is the term; and here please note that 1 minus x is positive we have assumed it in the beginning upon minus x.

So, substitute this value of gamma minus x in this relation. So, what I will be having minus in the numerator upon gamma x gamma 1 minus x equals to x into minus x. Now, there is a famous relation between this particular infinite product (Refer Time: 07:36) and sine function that is pi x p equals to 1 to infinity 1 minus x square upon p square will be sin pi x, and this we can prove.

So, what I am having it means x into this term will be sin pi x upon pi. Now, this I am having p equals to 1 to infinity 1 minus x square upon p square. Now, this will be cancel out. Now, see this x p equals to 1 to infinity 1 minus x square p square. So, this value I can put from here.

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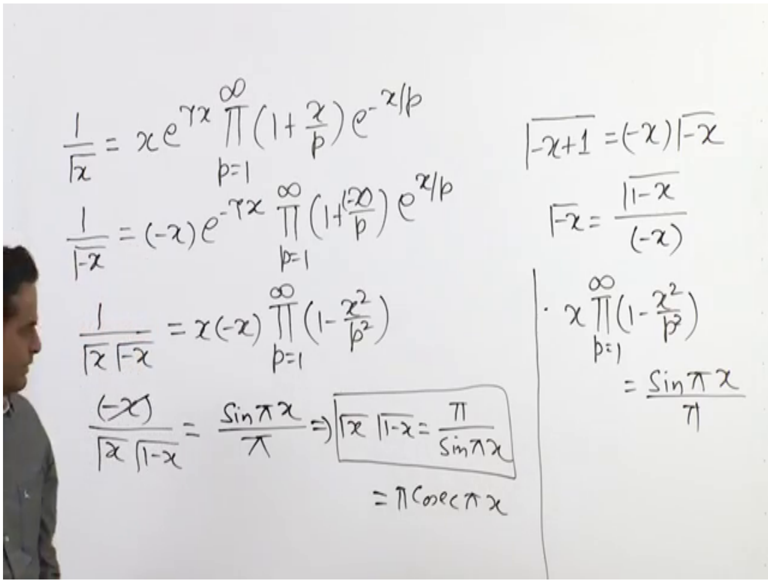

So, it will become sin pi x upon pi. And from here I can write gamma x gamma 1 minus x equals to pi upon sin pi x. And this formula is called reflection formula or complement formula. Also in some references it will be written as pi cosec pi x which is oblivious 1 upon sin pi x will become cosec pi x.

So, this is the proof of complement formula for gamma function. And here we are using the relation between gamma x and Euler constant which I have just introduce you in the beginning of this particular lecture.

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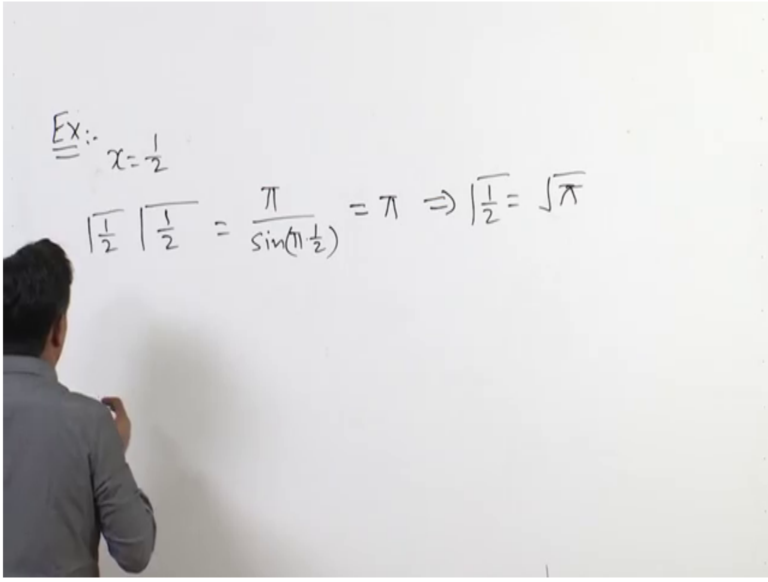

Now, take few values of x and see what we are getting using this formula. So, if I take x equals to half, then by the complement formula it will be gamma x that is gamma half gamma 1 minus x, so 1 minus 1 by 2 again will become gamma 1 by 2 equals to pi upon sin pi into 1 by 2 because x is 1 by 2. So, it means pi upon sin pi by 2 and sin pi by is 1. So, it equals to pi. So, here gamma half square equals to pi. So, from here I am getting directly gamma half equals to root pi, which is the result which we have dive in previous classes also.

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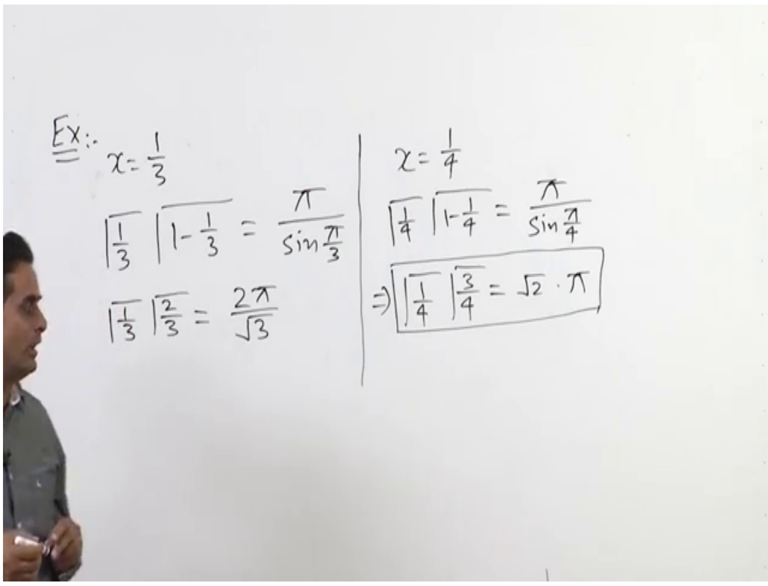

Now if I take x equals to 1 by 3, so x is 1 by 3. So, let us see what will be the value or what lesson we are getting using the complement formula. So, it will become gamma 1 by 3 that gamma x into gamma 1 by x, so 1 minus 1 by 3 equals to pi upon sin pi upon three. So, this becomes gamma 1 by 3 into gamma 2 by 3 and that is equals to sin pi by 3 is root 3 by 2. So, it will become 2 pi over root 3.

Similarly, if I take x equals to 1 by 4 then I can get the value of gamma 1 by 4 into gamma three by four because it will be gamma 1 by 4 1 minus 1 by 4 equals to pi into sin pi by 4. So, from here I will be having gamma 1 by 4 gamma 3 by 4 and you know that sin pi by 4 each 1 by root 2. So, it will become root 2 into pi. So, in this where we can get the value of the gamma functions operating on the complement pair, pair of two numbers that is x and 1 minus x; only thing we need to take care that x as well as 1 minus x what should be positive ok.

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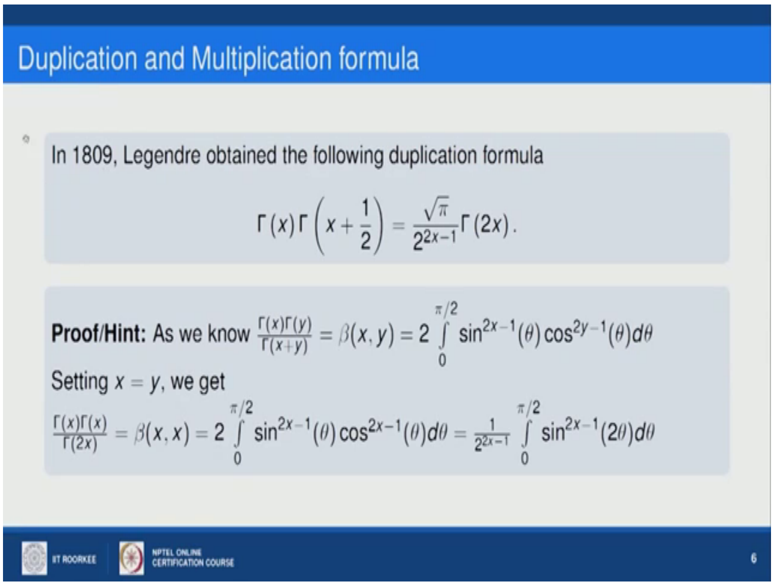

So, our result is given by Legendre in 1809 that is called duplication formula. So, it is given as gamma x into gamma x plus 1 by 2 and product of these two will be root pi upon 2 raise to power 2 x minus 1 gamma 2 x. So, let us try to prove this result also.

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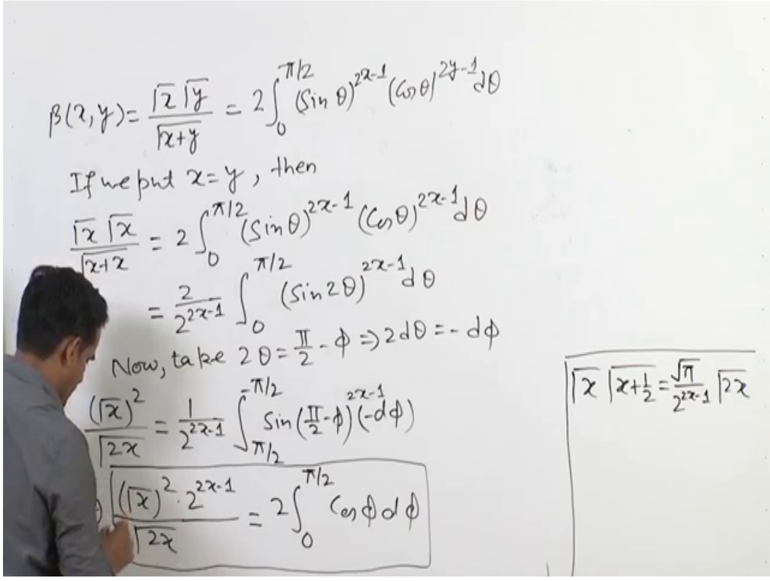

So, result is given as gamma x gamma x plus half, and these I need to prove root pi over 2 raise to power 2 x minus 1 into gamma 2 x as I told you this formula is called duplication formula and the proof will be something like this. We know that gamma x or basically I will start with the definition of beta function. So, I know that beta of x y equals to gamma x gamma y upon gamma x plus y. So, this is the relation between beta and gamma function and you know that beat x y can be written as two times 0 to pi by 2 sin theta raise to power 2 x minus 1 cos theta raise to power 2 y minus 1 d theta. So, this is the famous identity of beta function.

Now, if we put x equals to y then this formula I can write as gamma x into gamma x upon gamma x plus x equals to 2 0 to pi by 2 sin theta raise to power 2 x minus 1 cos theta raise to power 2 x minus 1 d theta; or this I can write as 2 over 2 raise to power 2 x minus 1. So, what I did I have multiplied and divide this particular right hand side by 2 raise to power 2 x minus 1. So, it is coming in denominator, whatever multiplied in numerator will be inside.

So, it will become two sin theta into cos theta raise to power 2 x minus 1 and you know that two sin theta cos theta equals to sin 2 theta. So, it will be 0 to pi by 2 sin 2 theta raise to power 2 x minus 1 d theta. Now, take 2 theta equals to 1 by 2 minus phi then this will give you 2 times d theta equals to minus d phi. So, what I will be having gamma x whole square upon gamma two x equals to put it here. So, 2 times d theta I will write at minus d phi. So, it will be 1 upon 2 into 2 raise to power x minus 1 sin sorry take it pi by 2 minus phi. So, sin pi by 2 minus phi, and then minus d phi what will be the limit, when theta is 0, phi will become pi by 2, when theta is pi by 2, so 2 times pi by 2 will be pi. So, phi will become minus pi by 2.

So, from here I can write gamma x square upon gamma 2 x I am taking this term also in left hand side. So, it will become 2 raise to power 2 x minus 1 into equals to so this is minus here. So, I can inter change the limit ok. And then sin pi by 2 minus phi will become cos phi and cos is an even functions. So, I can write limit from 0 to pi by 2 by taking 2 outside. So, I can write it 2 times 0 to pi by 2 cos phi d phi. So, this is my relation one.

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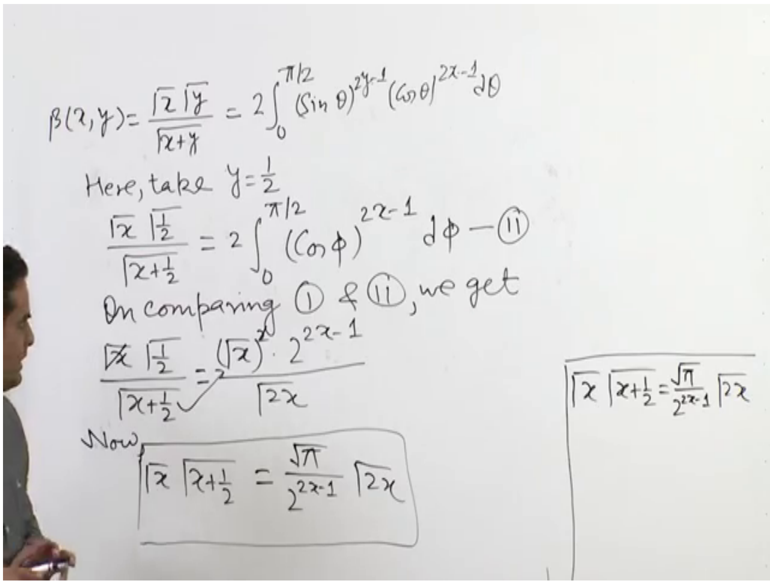

Now, we will start from this relation only. Here take y equals to half because I need some time like gamma x plus half. So, if I take y equals to half from the denominator, I will get x plus half. So, it will become gamma x into gamma half upon gamma x plus half equals to 2 times 0 to pi by 2. And here I am making I am writing because you know beta is a symmetric function. So, beta x y with equals to beta y x, so I can inter change the role of x and y here, it will not affect these particular relation.

So, from here I can write, so this I am taking y and this I am taking x x. So, when y is half sin theta 1 minus 1 0, so this term will become 1 and this term will be as such. So, cos theta raise to power 2 x minus 1 and then d theta or I can put theta equals to phi. So, if I take theta equals to phi, then this will become cos phi d phi. So, from here I will be having this I am writing relation second. So, now, you notice that the right hand side of relation one and relation two are same. So, I can compare the left hand side.

So, when I comparing left hand side of one and second, so that is on comparing one and two, we get that is gamma x into gamma half upon gamma x plus half, this equals to left hand side of one that is gamma x square into 2 raise to power 2 x minus 1 upon gamma 2 x. Now, I need to prove this relation. So, this is coming from here I will take gamma x plus half here.

So, gamma x will be cancel this. And then this gamma half will become root pi, I will be having the gamma x into gamma x plus half because this I have taken here this I have taken in the other side. So, it will come in denominator gamma half is here. So, this will become root pi into gamma 2 x and that is the duplication formula which we need to prove. So, this is the proof of duplication formula.

And here what we have done we had use this particular definition of beta function; also we have used the relation between beta function and gamma function. The proof sketch means I have done the complete proof, but if you want to remember, it is easy. In first step in this definition put x equals to y; then you need to make one more substitution that is 2 theta equals to pi by two minus phi that is quite important. Then you will get a relation. Again you start from this formula take y equals to half and you got another formula. If you compare these two formulas, you will get the duplication formula.

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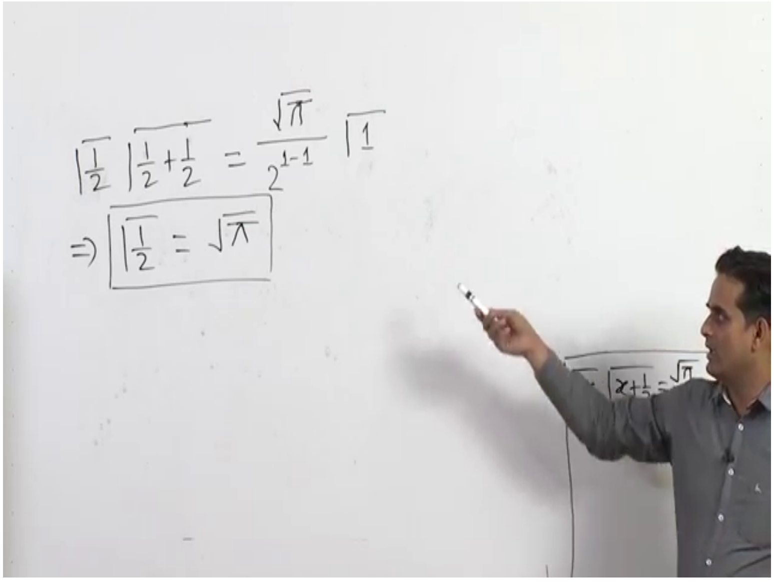

So, again from the duplication formula, if I put x equals to half in this formula what I will get gamma half into gamma half plus half that is these two terms I have written equals to 1 upon 2 raise to power 1 minus 1 because x is half. So, 2 x will become 1 into gamma 1, and only thing I have left it out root pi ok. So, from here this will be gamma half, gamma 1 will be 1 equals to again we got our famous relation that gamma half equals to root pi and this valid that is the relation given by the duplication formula ok.

So, in this lecture, we started with the definition of Euler's constant, then we learn a relation between Euler's constant and gamma function. After that we have seen two important formulas related to gamma function, one is called complement formula and the other one is called duplication formula. We have learned how to prove these two formulas, and what properties we need to use to obtain the proof of these formulas and then we have seen in both of the cases that gamma half equals to root pi.

In the next lecture, we will take few more properties of beta and gamma functions, we will take a few important examples also related to these two functions. So, I will end this lecture here.

Thank you very much.