Multivariable Calculus
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Lecture - 24
Introduction to Beta Function

Hello friends, so welcome to the fourth lecture of this unit. And in this lecture, I will introduce the another special function that is called beta function. And then we will see some relation between beta function and the function which I have introduced to you in the last lecture that is gamma function means relation between beta and gamma functions. Finally, we will take some examples which can be solved with the help of beta function.

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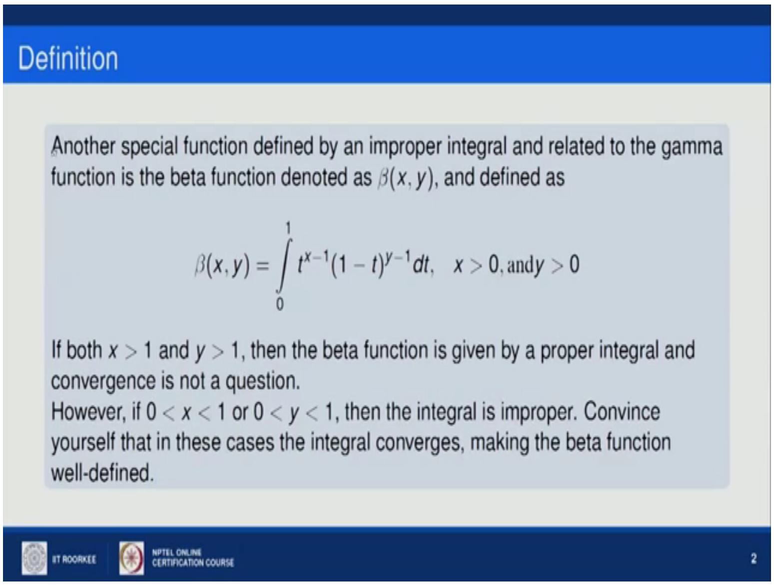

So, first of all let us see the definition of beta function. So, beta function can be defined by an improper integral and denoted by beta of x, y. So, beta is a function of two variables x and y, while the gamma function was the function of only one independent variables. So, beta of x y is given by this improper integral 0 to 1 t raise to power x minus 1 into 1 minus t raise to power y minus 1 d t; here this particular definition whole for positive values of x and y. So, if both x and y are greater than 1, then the beta function is given by a proper integral means this interval will become a proper integral, and hence convergence is not in question about this definition.

However, if x is between 0 to 1 or y is between 0 to 1 or both of them are between 0 to 1, then the integral is improper. And hence the convergence of this integral is in question. However, y breaking this integral 0 to epsilon and n epsilon to 1, in this way we can convince that in this particular case also integral converges. And hence in that way for all positive values of x and y, this definition is well-defined for beta function. Now, based on this definition, now we will take some important property of or I will say the important identities of beta function.

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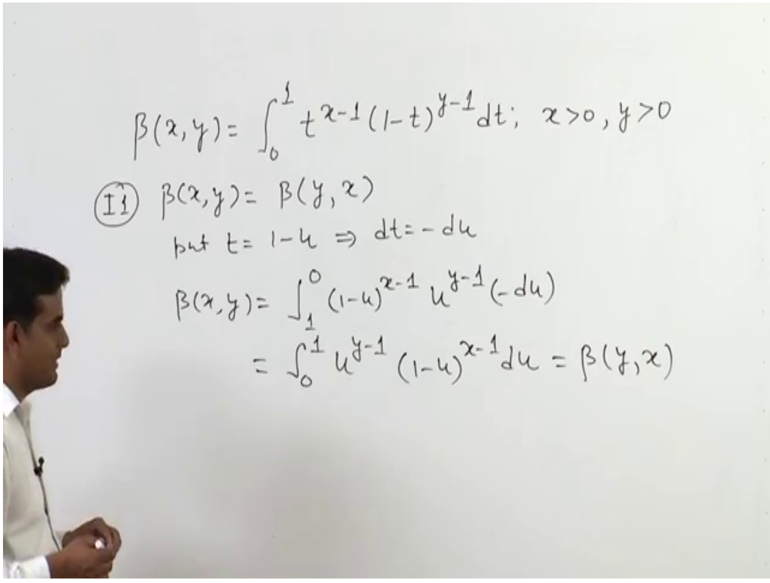

So, we have defined beta function as beta of x y 0 to 1 t raise to power x minus 1 1 minus t raise to power y minus 1 d t. And here x is greater than 0, y is greater than 0. Now, the first identity, so I will write it as I 1, I will prove that the beta function is a symmetric function that is beta of x y equals to beta of y x. So, take the definition. So, in the definition put t equals to 1 minus u. So, this will give me d t equals to minus d u. So, this beta x y will become by this substitution as so when u is 0, t is 1. So, 1 to 0, and then t will be 1 minus u raise to power x minus 1 1 minus t will become 1 minus 1 plus u. So, u raise to power y minus 1 and then minus d u.

This is so I will take this minus and I will interchange the lower and upper limits. So, it will be 0 to 1 u raise to power y minus 1 I have taken this term in the beginning and then this term x minus 1 d u. And according to the standard definition what is this, this is beta of y x. Hence beta function is a symmetric function in x and y. The second identity;

which I am going to take it is quite important identity because we will use this identity in many examples.

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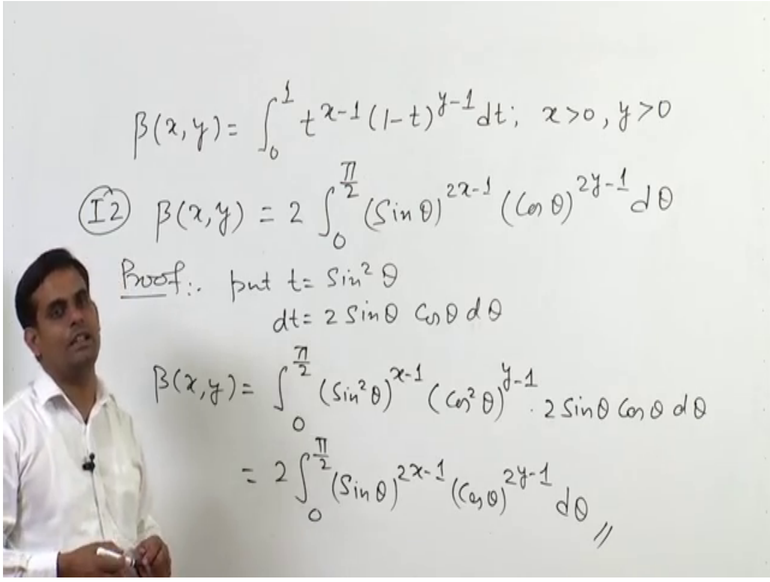

So, it is saying that the beta function also can be written in terms of sine and cosine. And it is given by the relation beta of x y equals to 2 times integral over to 0 to pi by 2 sin theta raise to power 2 x minus 1 cos theta raise to power 2 minus 1 d theta. So, let us prove it. So, in the definition of beta function what you do you just substitute put t equals to sin squared theta. So, from here it will become d t equals to 2 sin theta cos theta d theta.

Now, substitute all these values in the definition of beta function, so beta x y will be become. So, when t 0, theta will be 0; when t is 1, sin square theta equals to 1, so sin theta will be 1, so theta will be pi by 2. So, 0 to pi by 2 t raise to power x minus 1 t is sin square theta. So, sin square theta raise to power x minus 1 then 1 minus t means 1 minus sin square theta 1 minus sin square theta will become cos square theta. So, cos is square theta raise to power y minus 1 and then d t d t is two sin theta cos theta d theta so into two times sin theta cos theta d theta.

So, I have taken two in the beginning outside the integrant. So, two times integral over 0 to pi by 2, here it is sin theta raise to power 2 x minus 2; here I am having plus 1. So, 2 x minus 2 plus 1, it will become sin theta raise to power 2 x minus 1. Similarly, cos theta raise to power 2 y minus 1 and then d theta and this is a thing which we need to prove. As I told you, I will use this identity in several questions.

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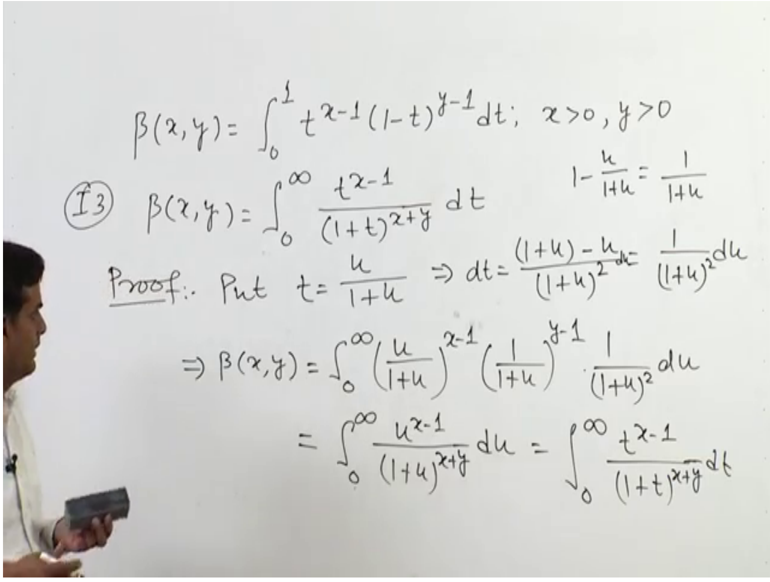

So, the third identity which I am going to discuss here is again and important property of beta function. So, it is saying that beta function can also be written as in the form as a fractional function of t that is 0 to infinity. So, please note here in this particular definition of beta function for identity and integral over 0 to infinity t raise to power x minus 1 upon 1 plus t raise to power x plus y d t. So, let us try to get this identity from the definition of beta function.

So, what you do you make a substitution like t equals to u upon 1 plus u. So, from here what I will get d t equals to 1 plus u as such differentiation of u will be 1 minus u as such upon 1 plus a point 1 plus u whole square. So, this will be 1 upon 1 plus u whole square into d u. When t is 0, u will be 0; when t is 1, what your u should be, u should be infinity.

So, from here I will get beta x, y h 0 to infinity t raise to power x minus 1. So, u upon 1 plus u raise to power x minus 1 then 1 minus t, so 1 minus u upon 1 plus u. So, this will become 1 plus u minus u, so 1 upon 1 plus u. So, 1 upon 1 plus u raise to power y minus 1. So, this is coming for this particular term and then d t. So, d t is 1 over 1 plus u whole square d u. So, this I can write 0 to infinity.

So, u raise to power x minus 1 upon please note that 1 plus u in a denominator raise to power x minus 1 1 plus u raise to power y minus 1. So, it will become 1 plus u raise to power x plus y minus 1 minus 1 minus 2 and plus 2 I am having here, so plus 2. So, this will be cancelled. So, 1 plus u raise to power x plus y and then finally, d u.

And if I put this u as t again 0 to infinity t raise to power x minus 1 1 plus t raise to power x plus y d t. This is the required form in this identity. So, hence we have seen three important identity identities of beta function. Now, we will see another important result of gamma function and beta function what is the relation between beta and gamma functions.

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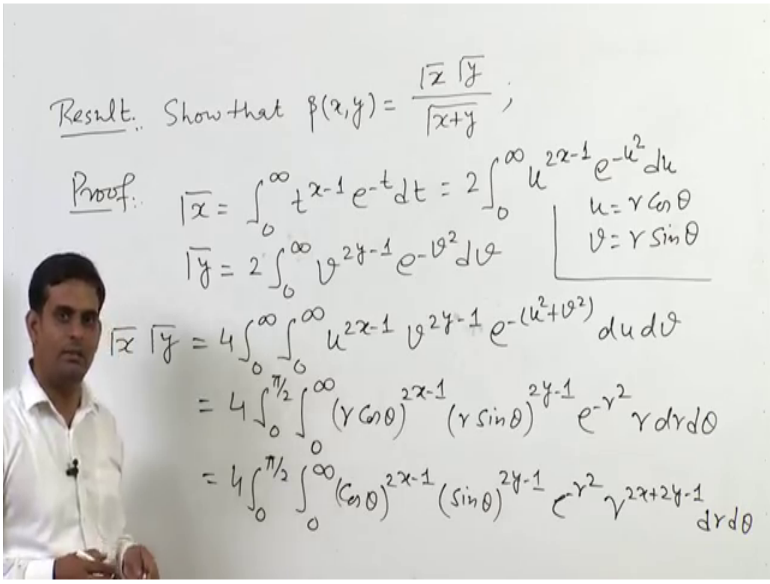

So, result is like that so that beta of x y equals to gamma x gamma y upon gamma x plus y. So, this is the relation between beta and gamma function. So, let us try to get this result. So, we know that gamma x can be defined as 0 to infinity t raise to power x minus 1 e raise to power minus t d t. And if I substitute t equals to u square, then I will get it 2 times 0 to infinity u raise to power 2 x minus 1 because t is u v square. So, u v square raise to power x minus 1 means u raise to power x minus 2 and when you will come from d t, so 2 x minus 1 e raise to power minus u is square d u.

Similarly, I can write gamma y equals to 2 times 0 to infinity u and here I will take a another variable let us say v, v raise to power 2 y minus 1 e raise to power minus v square d v. So, gamma x is that one, gamma y is this one, let us multiply this two relations. So, in the left hand side, I will be having gamma x into gamma y. In the right hand side, I will be having 4 times 0 to infinity 0 to infinity and then I will be having u raise to power 2 x minus 1 v raise to power 2 y minus 1 e raise to power minus u square v square d u d v.

Now, what you do just change it into polar coordinates means make a substitution that is something like this u equals to r cos theta and v equals to r sin theta and d u d v will become r d r d theta. So, this will become 4 times 0 to sin polar coordinates it is first quadrant. So, r will be 0 to infinity and theta will be 0 to pi by 2. U is r into 2 a r r cost theta raise to power 2 x minus 1. So, r cos theta raise to power 2 x minus 1. V will be r sin theta, so r sin theta raise to power 2 y minus 1 and then e raise to power minus r square because u square plus v square will become a r square and then d u d v will become r d r d theta. R is the Jacobian here Jacobian of u v with respect to r theta, so I have to write here.

Now, this can be written as 4 0 to pi by 2 0 to infinity cos theta raise to power 2 x minus 1 sin theta raise to power 2 y minus 1 e raise to power minus r square. And then the total power will be 2 x minus 1 from here 2 y minus 1 from here and one from here. So, 2 x plus 2 y minus 1 d r d theta, please see the limit limits are constant they are not the functions of one another or like that. So, by the Fubini's theorem these double integral can be written as the product of two single integrals.

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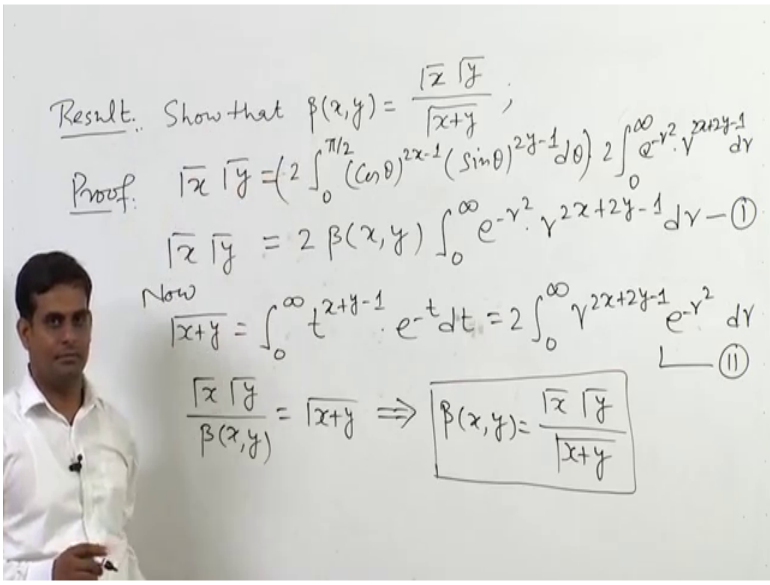

So, let me write it here. So, gamma x gamma y will be I am writing this 2 times 0 to pi by 2 cos theta raise to power 2 x minus 1 sin theta raise to power 2 y minus 1 d theta into 2 times 0 to infinity e raise to power minus r square into r raise to power 2 x plus 2 y minus 1 d r. What is this is the beta of x y just from the second identity 2 beta x y into 0 to infinity e raise to power minus r into r raise to power 2 x plus 2 y minus 1 d r. So, this is gamma x, gamma y. Let us see this our first equation.

Now, see the gamma x plus y. So, from the definition of gamma function it will be 0 to infinity t raise to power x plus y minus 1 into e raise to power minus t d t, put here t equals to r square. So, if I put in this t equals to r square, this will become 2 times 0 to infinity r square raise to power x plus y minus 1. So, it will become r raise to power 2 x plus 2 y minus 2 e raise to power minus t will become e raise to power minus r square and 2 I have taken already, r d r d t will become 2 r d r. So, one r will come r d r. So, I can take out this r, and I can write out here. So, this is the second equation.

And now compare first and second. So, basically what I am having from the I have taken this beta x y in the side in the denominator, then this particular right hand side will be same as this right hand side equals to gamma x plus y. This gives me beta x, y equals to gamma x gamma y upon gamma x plus and this is what we need to prove. So, this is the relation between beta and gamma function.

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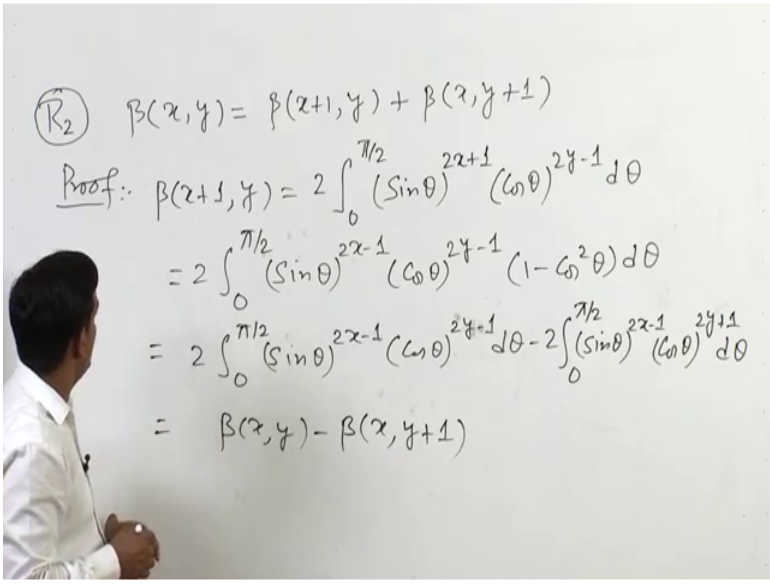

Another important result which I can prove here is, so I will write result two that is beta of x, y can be written as beta of x plus 1 y plus beta x y plus 1. So, let us see how we can obtain this result. So, again I will use the definition not the definition, in fact, I will say the identity two of the beta function. So, beta x plus 1 y can be written as 2 times 0 to pi by 2 that is the integral over 0 to pi by 2 sin theta raise to power in if I am having x here, I used to write 2 x minus 1.

But since I am having x plus 1 here, so it will be 2 x plus 1 into cos theta raise to power 2 y minus 1 d theta; This can be written as 2 times 0 to pi by 2 sin theta raise to power 2 x minus 1. So, I am taking sin square theta out of this, cos theta is already here and then sin square theta which I have taken out from here, I am writing this number as 1 minus cos square theta d theta.

Now, I am breaking this integral into two double two integrals 2 0 to pi by 2 sin theta raise to power 2 x minus 1 cos theta raise to power 2 y minus 1 d theta means from this term and minus 2 times 0 to pi by 2 sin theta raise to power 2 x minus 1 cos theta raise to power 2 y because here 2 y minus 1 into cos square theta, so 2 y plus 1 d theta. What is this, this is my beta of x, y. And another second term will be minus beta of x, and here it is y plus 1 by the second identity. So, beta x plus 1 y equals to beta x, y minus beta x, y plus 1, hence beta x, y equals to beta x plus 1 y plus beta x, y plus 1.

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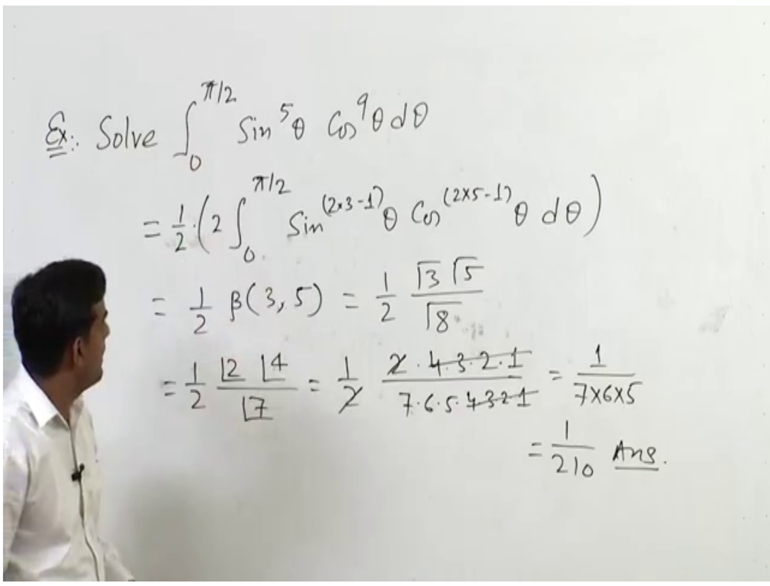

If someone ask you to solve any integral solve 0 to pi by 2 sin five theta into cos 9 theta d theta. So, it is quite lengthy process if we solve it without using beta and gamma functions, but since now we are having the knowledge of beta and gamma function. So, we can applied that here. So, I can write this as one by 2 into 2 0 to pi by 2 sin 5 theta sin 5 theta I can write sine 2 into 3 minus 1. So, 2 into 3 minus 1 theta cos raise to power 2 into 5 minus 1 theta d theta. So, this becomes 1 by 2 and this thing becomes beta of 3 because 2 x minus 1 2 y minus 1 means 5 1 by 2 beta of 3 5 1 by 2 gamma 3 gamma 5 upon gamma 8.

This equals to 1 by 2 gamma 3 will be factorial 2, because gamma x plus 1 equals to factorial explain when x is a positive integer factorial 4 upon factorial 7 factorial 2 is 2. Factorial 4 is 4 into 3 into 2 into 1 upon factorial 7 will be 7 into 6 into 5 into 4 into 3 into 2 into 1. So, this is cancelled out, 2 also cancelled out. So, this is coming out as one upon 7 into 6 into 5 that is 1 upon 210.

So, this makes our life simple if we use beta and gamma function for evaluating such kind of definite integrals. So, in this lecture, I talk about beta function. I have taken few identities of beta function, then I have developed the relation between beta and gamma function which is quite popular and very helpful solving definite integrals. And finally, I have taken one example to show the applicability of beta function for evaluating definite integrals. With this, I will end this lecture.

Thank you.