Multivariable Calculus
Dr. Sanjeev Kumar
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 22
Change of Variables in Multiple Integral

Hello friends, so welcome to the second lecture of the fifth unit of this course. You know in the last lecture, we have discuss about the change of order in the multiple integral. In this lecture, I will talk about another important concept related to the multiple integral and that is change of variables in multiple integrals. So, let us understand what is; Change of Variable.

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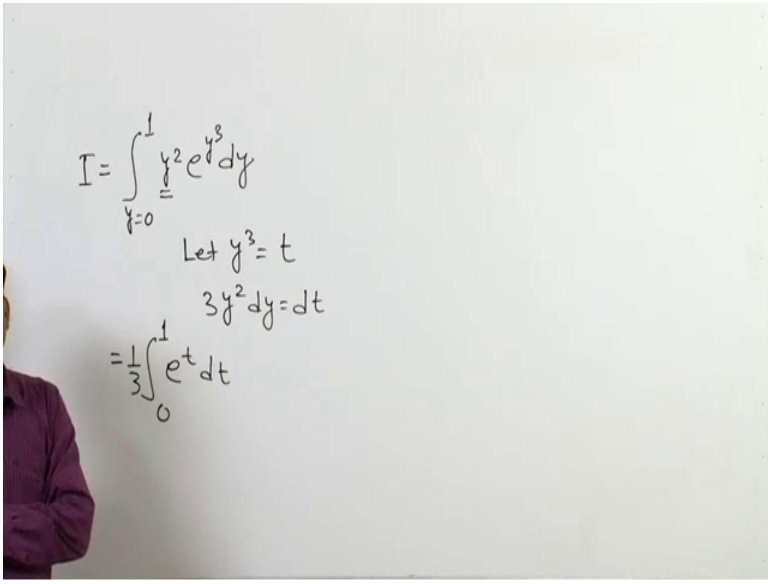

So, first take a simple example of definite integral. So, suppose, I need to solve this particular integral, so for solving this integral what I need to do I need to make some substitution. Substitution means I need to change the variable y. So, for doing this what I need to do let y cube equals to t; and from here I will get 3 y square d y equals to d t. Since, I am having this y square term here. So, my this integral will become 1 by 3 e raise to power t d t. When y is 0, t will be 0; when y is 1, t will be 1.

So, my original problem was in y, but now my variable is t. And if you look at both of these definite integrals, this is looking quite easy when compared to this one. So, we have made the change of variable from y to t by putting y cube equals to t, and then we have convert our integral in a simple form. So, we have done change of variable in definite integral also.

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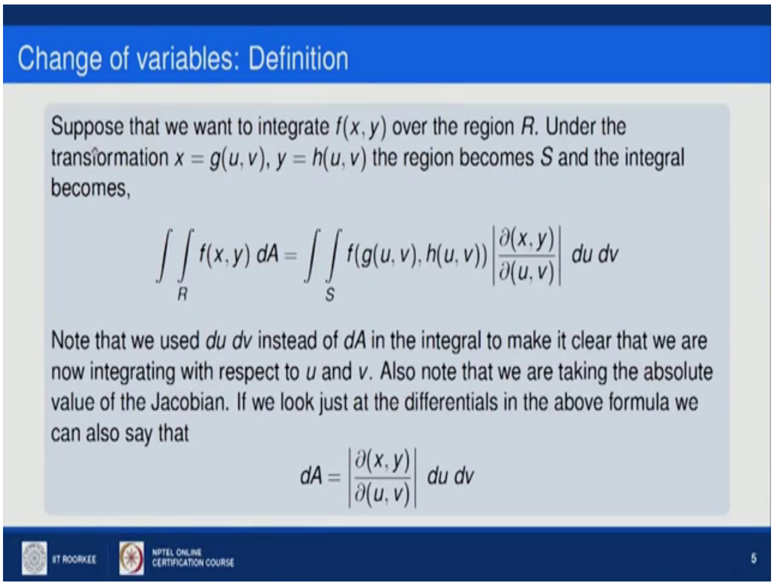

Now, what we need we need to extend this concept in case of multiple integrals. in multiple integrals, what is change of variables. Suppose, we want to integrate a function f of x y over the region R; under the transformation x equals to g of u v and y equals to h of u v the region becomes S. Like in case of definite integral our limit change. Here the region of integration will also change when you make the transformations or change of variables.

So, region becomes S and the integral becomes now means our original integral was like this. A double integral over the region R f of x y dA this will convert into the region is now S x will be g of u v y will be h of u v. Absolute value of the Jacobian of x y with respect to u v into d u d v.

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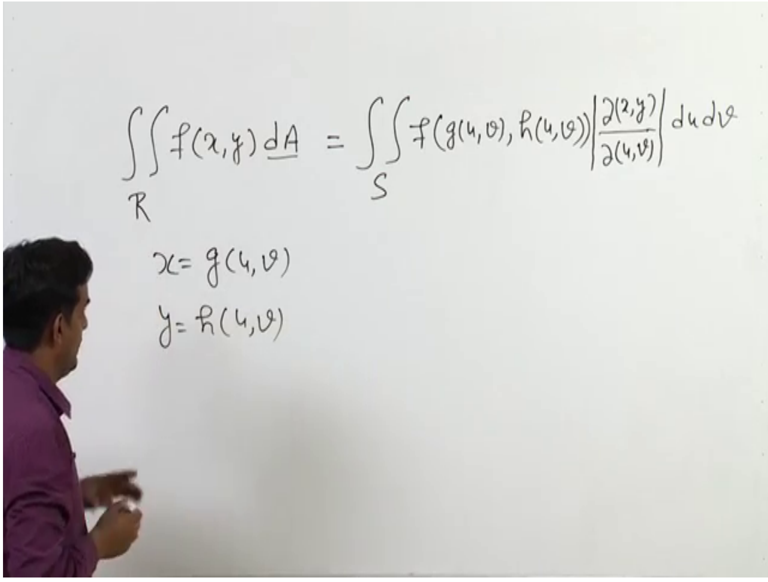

So, basically what we are having, we are having R f of x, y dA. This dA is where be d x d y or d y d x, we are not taking the order here mean exact to the write a d x d y or d y d x. So, it may be anything which ever will be the simple one. So, now what I am doing I am making a transformation x equals to g of u, v y equals to h of u, v. So, after making these two transformations means my original problem is in variable x and y, I am changing it into the variable u and v. So, after doing this as I told you region will change. And we will see by the examples how region will change.

So, earlier region what R now it will become S f of x will become g of u, v, y will become h of u, v and then this dA; dA will become absolute value of Jacobian x y up or with respect to u, v, and then d u d v. So, this is a change of variables in case of double integral similar concept can be extended to the triple integral.

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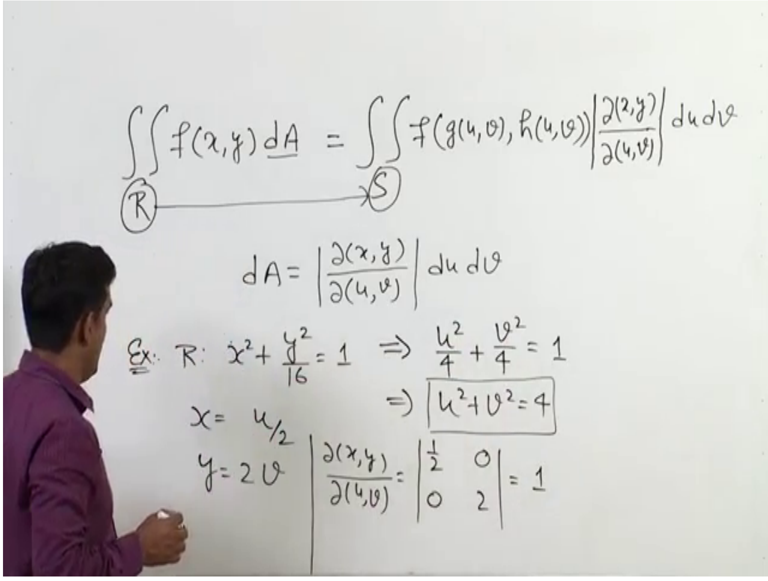

Here you have to notice one is earlier region what R, now it become S. The other thing is this dA is now Jacobian of x y with respect to u v d u d v. And these two things are coming from the region. For example, suppose I am having a region let us say initially I am having region R is an ellipse given by x's square y square upon 16 equals to 1. So, this is my original region ok.

Let us take I made the transformation x equals to u by 2, and y equals to 2 v 1. So, if I apply these two transformations on this region, what will happen, this region will convert x square means u square upon 4 plus y square up on 16, so 1 by 16 into for v square so v square upon 4 equals to 1. And this comes out is u v square plus v square equals to 4. So, my original region was an ellipse, now it becomes a circle. And you know that integration over a circle is quite easy when compared to an ellipse, because there you can convert it into polar coordinates, and you can have constant limit for R as well as for theta.

The other thing which is important to notice here what will be dA in this case so, here first of all to we need to calculate the Jacobian. So, x, y upon del of u, v, so del x over del u del x over del u will become 1 by 2, del x over del v del x over del v 0 then del y over del u which is 0 and del y over del v that is 2. So, here Jacobian is coming out to be 1. So, hence dA will become simply d u d v.

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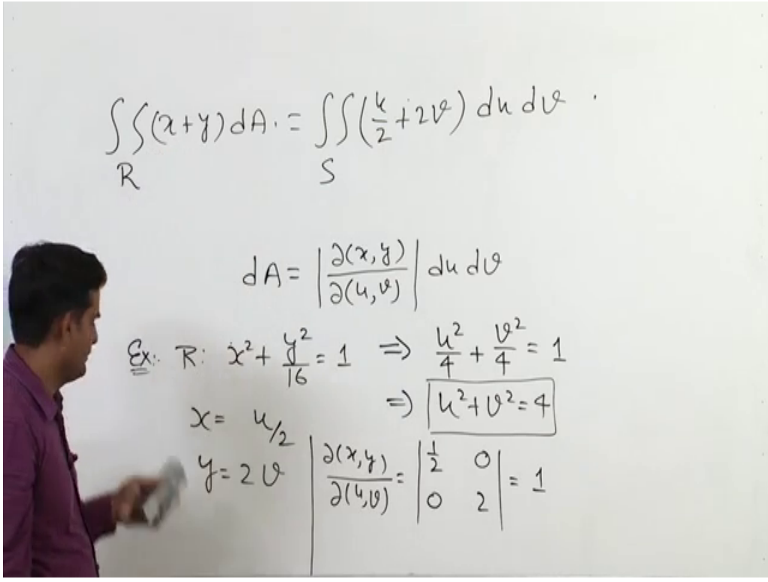

So, if I am having a problem on this particular region, for example if I am having an a an integral R which is let us say x plus y dA, now it will become S. So, R was in ellipse earlier, now this is a circle of radius two having centre at the origin, x plus y will become u by 2 plus 3 v my Jacobian is 1 so d u d v. So, this is the change of variables which I have explain with the help of this simple example.

Now, there are two regions for changing the variables. The first region is suppose we are having the region of integration it is complicated. So, can we apply some transformation, so that the region will become a simple region where the integration will become easy. The other regions is if we are having complicated function here under the integration on which we have to perform integration. So, by applying the some transformations can we change this particular function into a simple form. So, these are the two regions behind the concept of change of variables.

Now, we will take some example, and I will explain I will take an example where region earlier region will be the complicated, but later on it will become simple and the other one also.

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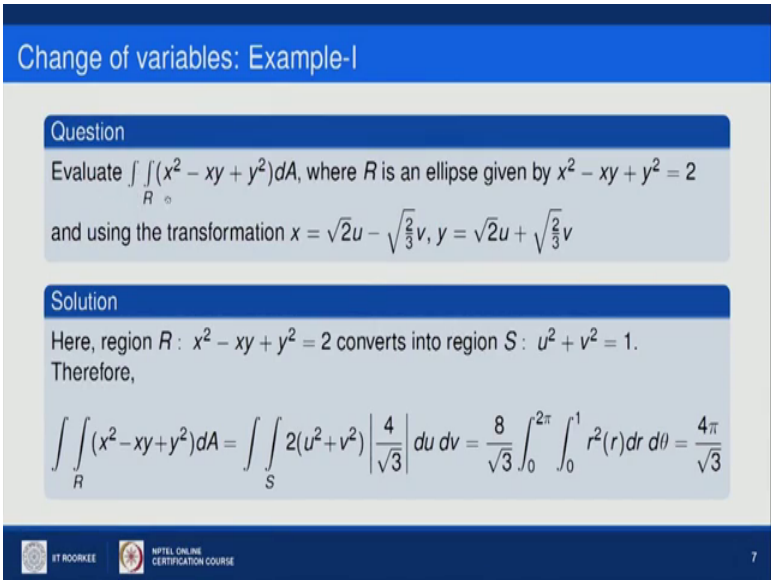

So, let us take this particular example here we need to evaluate x square minus x y plus y square over the region R, where R is given by the ellipse x square minus x y plus y square is equals 2. So, region under inside this ellipse and on the boundary of this ellipse, and using the transformation x equals to square root 2 into u minus square root 2 by 3 into v, and y equals to square root 2 u plus square root 2 by 3 in to v.

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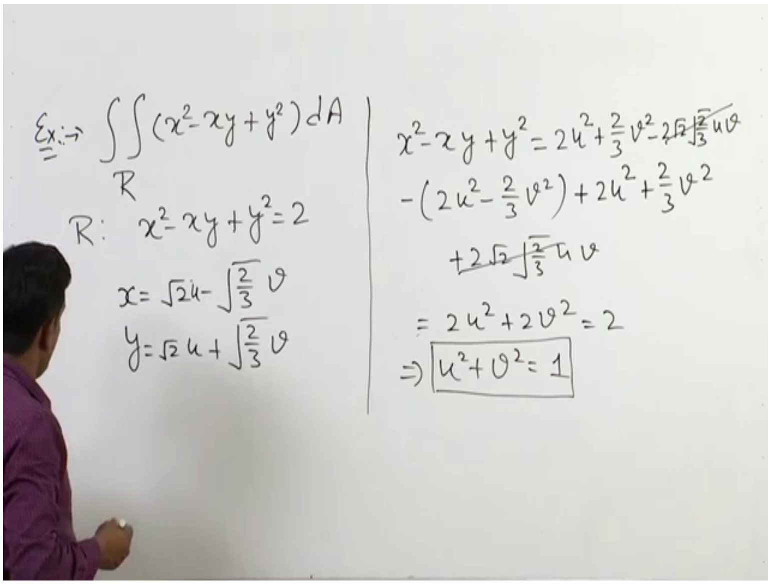

So, I am having x square minus x y plus y square over the region R, where R is given by the ellipse x square minus x y plus y square equals to two. The transformation is which I need to apply x equals to root 2 u minus square root 2 by 3 v and y equals to square root 2 u plus square root 2 by 3 v. So, here we are having something some easy thing that this is dA the integral and the region are having same function.

So, let us see after applying these two transformations what will be my new function. So, earlier I am having this one. So, x square minus x y plus y square x square will become 2 u square plus 2 by 3 v square minus 2 root 2 root 2 by 3 into u into v; Then minus x, y, so minus x y will become 2 u square minus 2 by 3 v square because a minus b into a plus b, so a square minus b square. And finally, y square so plus 2 u square plus 2 by 3 v square minus and now it will become plus 2 times root 2 root 2 by 3 u into v.

So, these two terms are cancel out 2 u square minus 2 u square plus 2 u square. So, it will become that I should it equals to 2 u square 2 by 3 v square plus 2 by 3 v square plus 2 by 3 v square so, 6 by 3 v square, so plus 2 v square. So, we have taken all the terms, it was equals to 2. So, this equals to 2. So, from here after change of variables, the region becomes a circle of unit radius and having centre at the origin.

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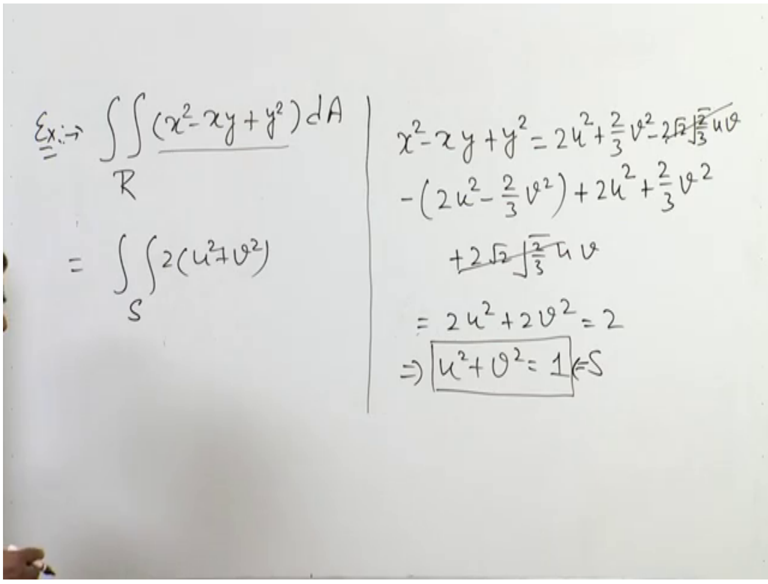

So, now, I can write this integral. So, this will become 2 u square plus v square because this term equals to twice u square plus twice v square; only thing I need to calculate Jacobian. The limit will become the new region S. Here S is given by the circle of unit radius.

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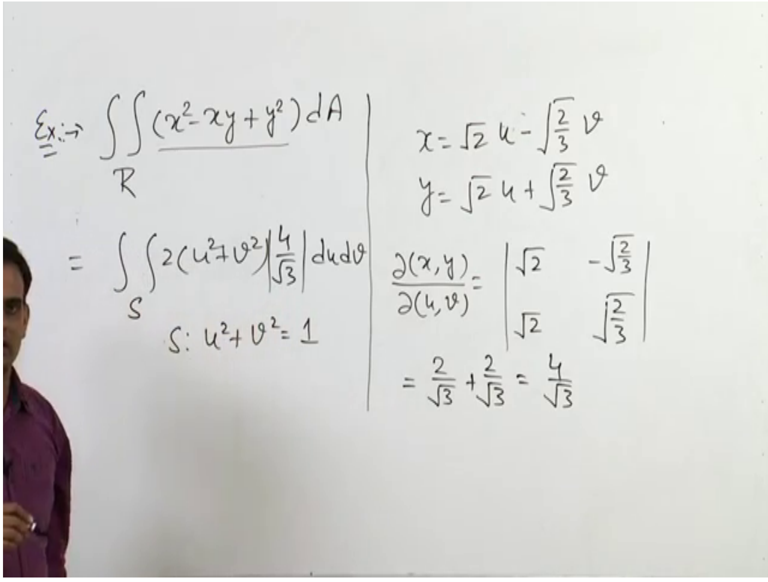

So, now, let me calculate Jacobian. So, x is root 2 u minus root 2 by 3 v and y is root 2 u plus root 2 by 3 v. So, del x y with respect 2 u v del x over del u. So, del x over del u will be root 2 del x over del v del y over del u del y over del v. So, this will become 2 by root 3 because root 2 into root 2 will become 2 upon root 3 plus 2 by root 3 minus minus will become plus so 4 upon root 3. So, 4 upon root 3 absolute value of this into d u d v; and S is given by this particular circle.

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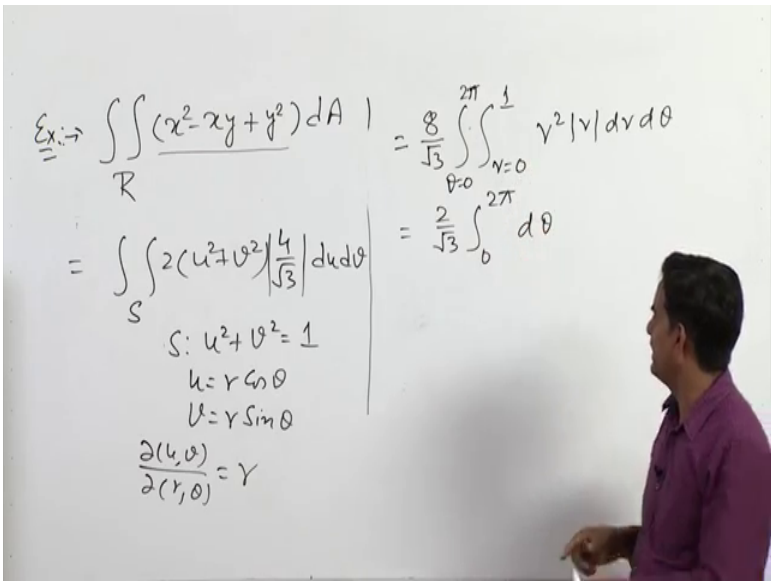

Now, this integral can be written as 2 into 4 8 upon root 3 and if I change it into polar coordinates because it is a circle. So, means again change of variable u equals to r cos theta v equals to r sin theta. So, del u v del r theta will become here r. So, it means d u d v will be r d r d theta it is a circle of unit radius r will go from 0 to 1 and theta will go 0 to 2 pi u v square plus v square will become r square into r d r d theta, because it will become r cube r cube will become r 4 up on 4. So, I have adjusted for here earlier it was 8, now it is 2; r for when you substitute limit when you will put one it will become one when zero it will become 0, so simply theta.

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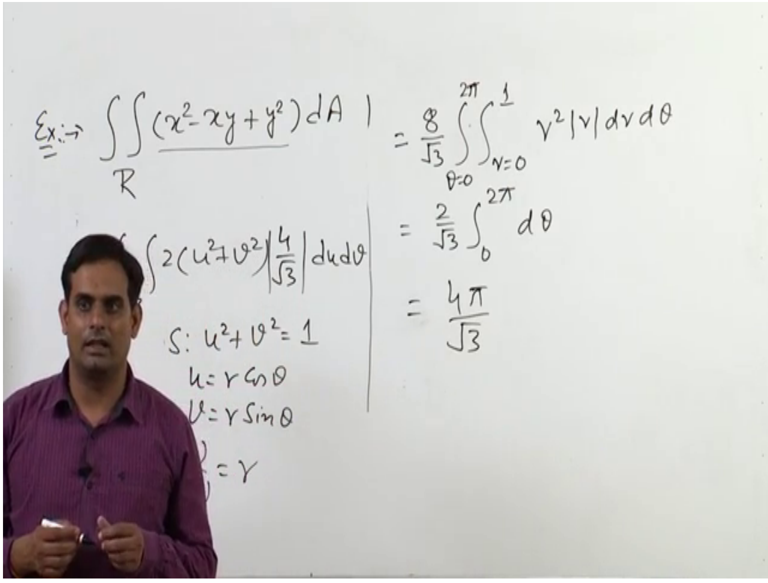

So, basically it will be 4 pi over root 3 now take another example.

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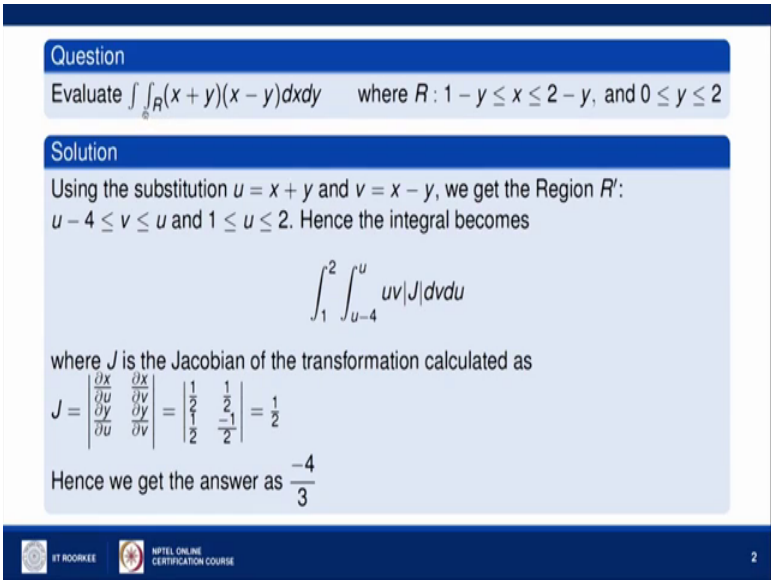

So, in this example, we have to evaluate this particular double integral over a region R. So, here double integral x plus y x minus y d x d y, and the region R is given in such a way that x is bounded by 1 minus y 2 2 minus y and y is between 0 to 2. So, first of all we cannot evaluate this integral in an easy manner directly in this particular region. So, what we need to do we need make some transformations or change of variables.

So, let us make change of variables using these two transformations that is u equals to x plus y, and v equals to x minus y. If we used these two transformations my region R transform into R dash, which is given by v is between u minus 4 to u and u is between 1 to 2. Hence, the double integral converts into this particular double integral that is since my x plus y is u x minus y is v. So, u into v then I need to find out the Jacobian and then d v d u. Here I am writing d v d u as you know that I am having constant limits for u. So, I am taking d u at the end, I will evaluate it later.

So, when I calculate Jacobian, Jacobian is given in this way del x over del u del x over del v del y over del u del y over del v. So, here del x over del u becomes 1 by 2 because x will be simply become u plus v by 2, and y will become u minus v by 2 that we are getting from these two transformations. So, del u by del x sorry del x over del u is 1 by 2 del x over del u is 1 by 2, del y over del u is over 1 by 2, and del y over del v is minus 1 by 2. So, when I evaluate these particular integral this will become 1 by 2 that is the absolute value of this; otherwise it will come out to be minus 1 by 2. And since I have to take the absolute value, so it is 1 by 2. If I substitute it here it will become 1 by 2 double integral u v d v d u. So, after evaluating this particular integral, I got the value of this integral h minus 4 by 3.

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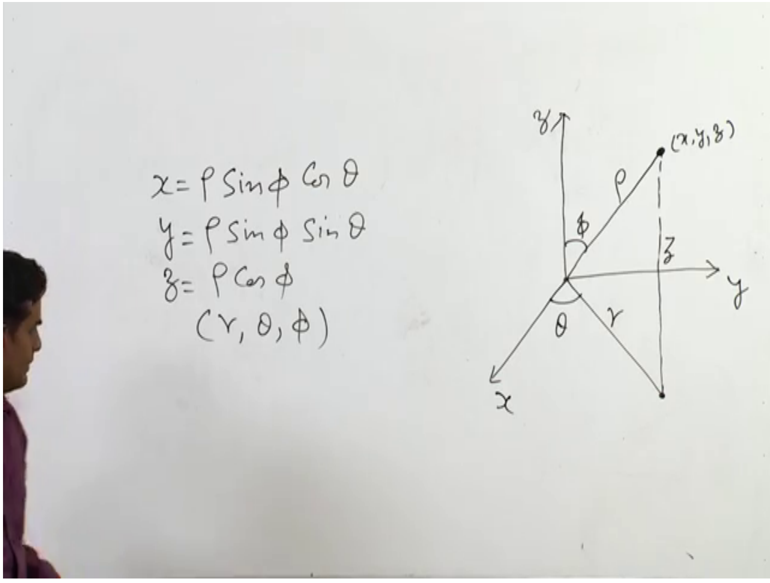

Now take one more example and that is we take in spherical coordinate. So, as you know that if I am having x, y, z coordinate system; and I am having a point here x, y, z ok. And I want to represent this point. So, this is let us say my x-axis, y-axis and z-axis. So, let me take this angle is phi, this distance as rho. So, what will happen this distance will be z, this angle let us take theta. So, let us take at the said r.

So, what I will be having x equals to rho, so rho sin phi cos theta then y will become rho sin phi sin theta. So, basically R is rho sin here higher project this on to this line and then z will become rho cos phi. So, this transformation is called converting from Cartesian coordinates to spherical coordinates. And now my new coordinates is r, theta and phi, where phi will be between 0 to phi.

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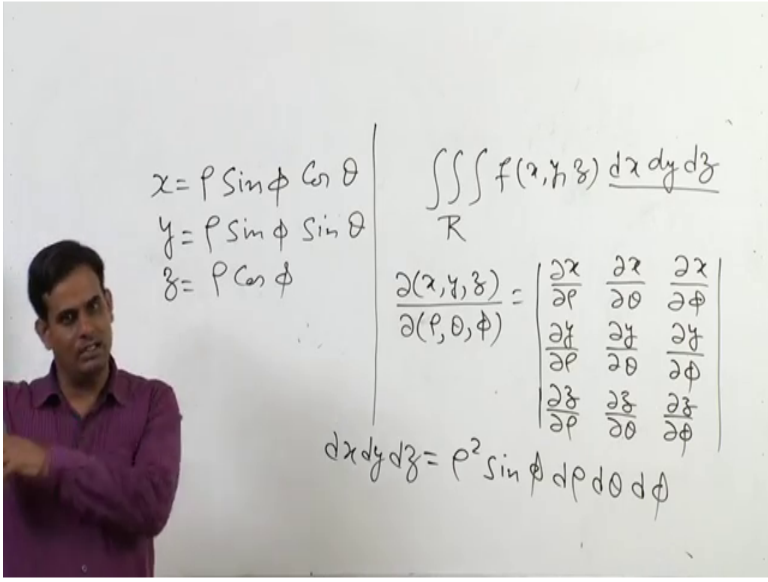

Now, if I convert my problem into this that is let us say I am having a triple integral over a region three dimensional region R f x, y, z d x d y d z. So, what d x, d y, d z will be come in this is spherical coordinates after applying these transformation. So, here you I need to calculate del x y z over del rho theta phi. So, it will be a Jacobian that is del x over del rho del x over del theta del x over del phi del y over del rho del y over del theta del y over del phi, and then del z over del rho del z over del theta del z over del phi.

So, if I simplify by this, the Jacobian comes out be rho square sin phi d rho d theta d phi and which will be equals to d x d y d z. So, this is the change your variables when you are changing your variables from Cartesian to spherical coordinates that is the extension of polar coordinates into theta d. Similarly in cylindrical d x d y d z will become r, d r, d theta, d z because their coordinates are r, theta and z.

So, in this lecture what we have done we have taken the definition of change of variables. Then we have seen few examples for the region y we do it. After that we have seen how we can convert our triple integral which is given in Cartesian coordinates to the spherical coordinates by using the concept of change of variables.

Thank you.