Multivariable Calculus
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Lecture - 21
Change of order in integration

Hello friends, so welcome to the 21st lecture and the first lecture of the fifth unit of this course. So, in the past couple of lectures, we have learned about multiple integral, especially double and triple integrals. So, in this lecture, I will talk about change of order in integration here integration means double or triple integrals. So, why we do the change of product in multiple integral?

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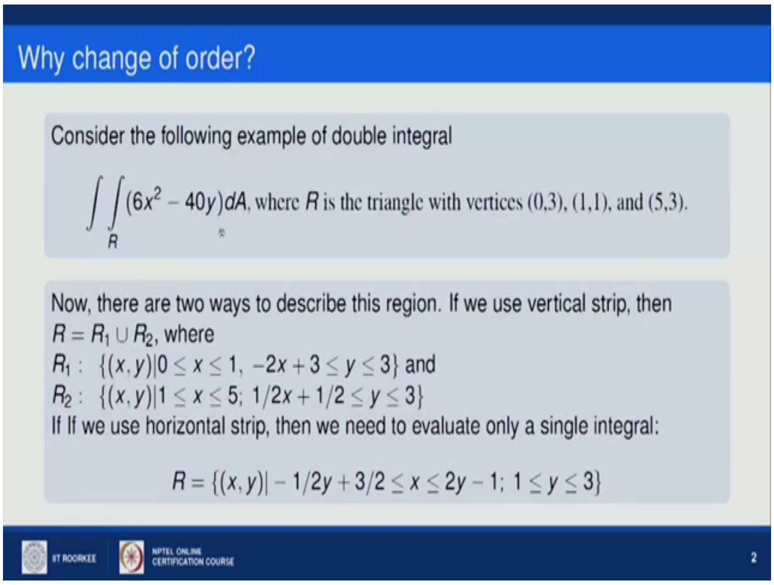

To understand this particular thing let us take this simple example.

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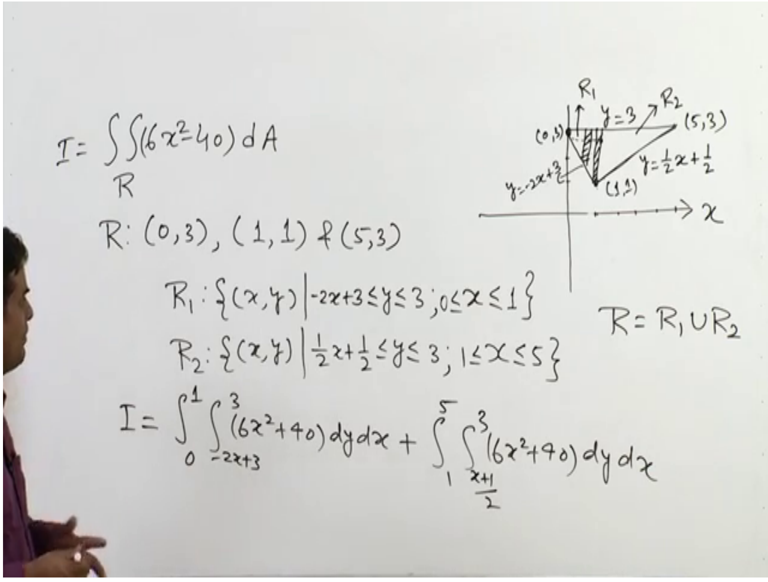

So, I am having a double integral which I have to integrate over a region R and I am having here 6 x square minus 40 dA. And my region R is given by a triangle with vertices 0, 3, 1, 1, and 5, 3. I am having 0, 3, 1, 1, and 5, 3. So, I need to find out this I need to evaluate this double integral over this triangle. So, now, if I takes this particular region, so let us say these are my x and y-axis. First point I am having 0 and 3, so 1, 2, 3. So, this is the vertices 0, 3, then I am having 1, 1. So, 1, 1 will be here. And finally, I am having 5, 3, so 2, 3, 4, 5, so here. So, I am having this triangle at my region of integration. So, this is 0, 3, 1, 1, and 5, 3.

So, this is the region R, now I need to integrate it. And here I can integrate this particular thing over this region in two ways; one is by taking a vertical steep and another one is by taking a horizontally steep. So, basically if I write the equation of these lines, then these equations are given by.

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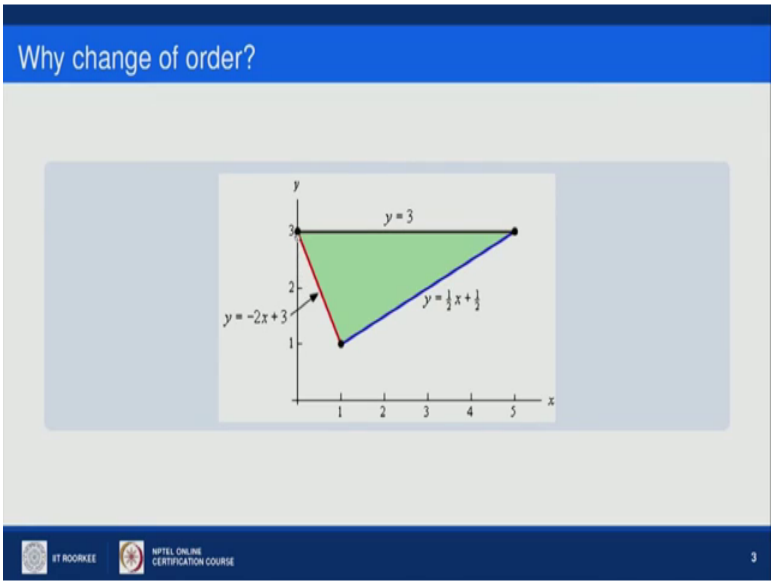

So, by these expressions so this line is given by y equals to minus 2 x plus 3; this line is given by y equals to half x plus half; and this line is given by y equals to 3. So, now, as I told you, if I take a vertically stripe on this region then so this line is y equals to minus 2 x plus 3 this line is y equals to half x plus half and this line is y equals to 3. So, now, if I take a vertical stripe then what I need to do, I need to divide this region into two sub regions; one is R 1 which start from x equals to 0, and comes up to x equals to 1, because here in this region R 1 my vertical stripe will go like this.

So, the lower limit of y will be so region is like this x, y such that the lower limit of y is minus 2 x plus 3. And upper limit is 3, while for x, I am having limits edge at start this vertical stripe I will start from here and it will go up to here. So, it will start from 0 and it will go up to 1. Once the vertical stripe will be here, in the next point the lower limit of this particular stripe will change because here the lower limit will become this line. So, let us define this region as R 2.

Now, what I am having R 2 is x y where the limits of y is half x plus half y and it will go up to again 3. And the limits of x will be from 1 to 5. And the complete area or complete region will be R equals R 1 union R 2. So, here for solving this particular integral using a vertical stripe what I need to do I need to solve two double integrals one for this region and another one for this region. So, integral will becomes 0 to 1 minus 2 x plus 3 to 3 6 x square plus 40 d y d x plus x equals to 1 to 5 x plus 1 upon half to 3 6 x square plus 40 d y d x. And this will be the complete value this integral. This is one of the way of solving this problem.

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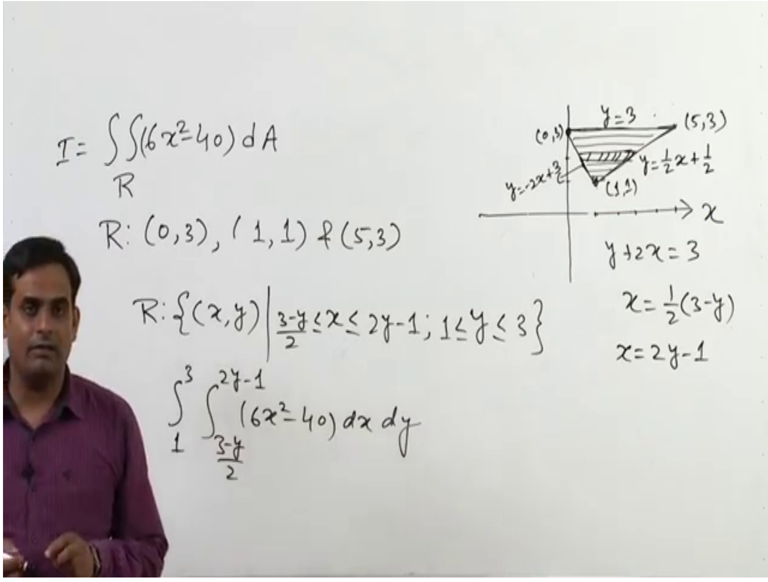

So, the other way of solving this problem is to take a horizontal stripe to cover the whole region R. So, if I take a horizontally stripe to cover this region, so this is step will start from here, and it will cover this region like this. So, here the region will become x, y, where the lower limit of x will be. So, this line so this line is y plus 2 x equals to 3, so x will become 1 by 2 3 minus y. And upper limit will become this 1, so this will become x equals to two y minus 1. And my y will go from y equals to 1,2, y equals to 3. So, here the limits of this double integral will become 3 minus y upon 2 to 2 minus 2 y minus 1. And y will be 1 to 3 6 x square minus 40 d x d y.

So, what I am doing, I am taking two different ways of solving this particular problem in one first I am integrating with respect to y, and then with respect to x means I am taking the variable limits for y and constant limit for x. In this particular thing in the a second where I am taking just opposite, I am taking variable limits for x and a constant limit for y. So, I am integrating it first with respect to x, and then I am doing it with respect to y. In the earlier case, I need to evaluate to double integral, but in this case I need to evaluate only one double integral, and which is quite simple when compared to earlier case. So, hence order matters in evaluation of multiple integral, yes, we have seen in this particular example. So, in one way it becomes easy to calculate; in other way, in other order, it may become difficult. So, this is one of the region for taking or for changing the order of integration.

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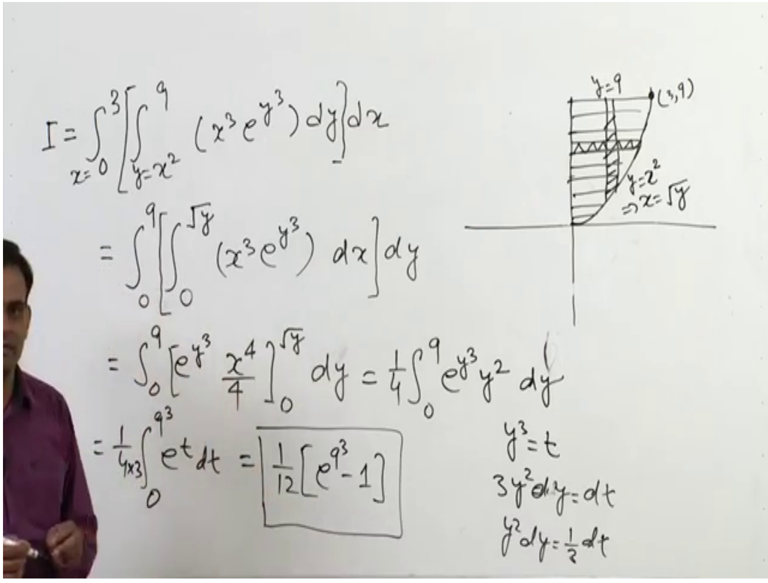

Now, take one more example. And this is example again from the double integral. So, here x equals to 0 to 3, and I am having y equals to x is squared 2 9. In this particular problem or example my function which I need to integrate is x cube into e raise to power y cube and then d y d x. So, what I am having I need to evaluate this particular integral. So, according to given order what I need to do, first I need to calculate this particular thing means I have to integrate it with respect to y. However, I cannot do it. Why, because if I have to integrate it with respect to y, I need a term of y square here due to the cube of y in the exponential term. So, that I do not have and hence I cannot evaluate this integral in this order.

So, what is the solution? Solution is to change the order of integration and the then we will see whether we can solve it or not. But for this as I told you change of order means changing the direction of its stripe. If you are covering you are in a region of integration by horizontally stripe, changing the order means take the vertically stripe and vice versa.

So, let us take a is case the region of this particular example. So, what I am having here y equals to x square. So, y equals to x square will be like this; and it is going up to y equals to 9. So, let us say 3, 6, 9, so this particular things. So, this is the point 3, 9 ; and x is going from 0 to 3. So, my region is this particular area bounded by these three lines.

Now, what I am doing according to the given problem, I am taking a vertically stripe which is having limit edge y equals to x square and going up to y equals to 9, and it is starting from x equals to 0 to 3. Now, what you do change the order of integration, so instead of this vertically stripe let us take this horizontally stripe. So, now, this is equals to the lower limit of x will be 0, upper limit will become this curve and what is this curve y equals to x square. So, this I can write x equals to root y, because now I need to take the variable limit in y.

So, 0 to root y and then what I am having this is stimuli start from here it will cover this region like this. So, it will go from 0 to 9 x cube e raise to power y cube, and then now d y d x will become d x d y now integrate this the integral in square bracket this will become 0 to 9. So, it will become e raise to y cube which is as a as such and x cube will become x 4 upon 4 limit will be 0 to root y and then d y. This thing equals to 0 to 9 1 by 4, when I will put root y, it will become this term as such. So, when I will put root y x raise to power 4 will become y square and then when I will put 0, it will become 0 so into d y ok.

So, now we are having this integral. Now, what I will do I will take y cube as t, so 3 y square will become d t d y equals to d t; and y square d y will become 1 by 3 d t. So, after substituting, here it will be 1 by 4 it will become e raise to power t upon 3. So, I will take 1 by 3 out d t. The only thing I need to change limits when y 0 t 0. So, no change in lower limit; however, when y is nine t will become nine raise to power three. So, this will become 1 upon 12, the integral of e raise to power t will be e raise to power t. So, e raise to power nine raise to power 3 minus 1. This is the final answer of this particular problem.

So, here you have seen the original problem was not solvable in the original order, but once you change the order, we can solve this particular problem. Hence, we are having the necessity of change of order in many double integral or triple integral problems ok. So, basically change of order means to changing the order of integration.

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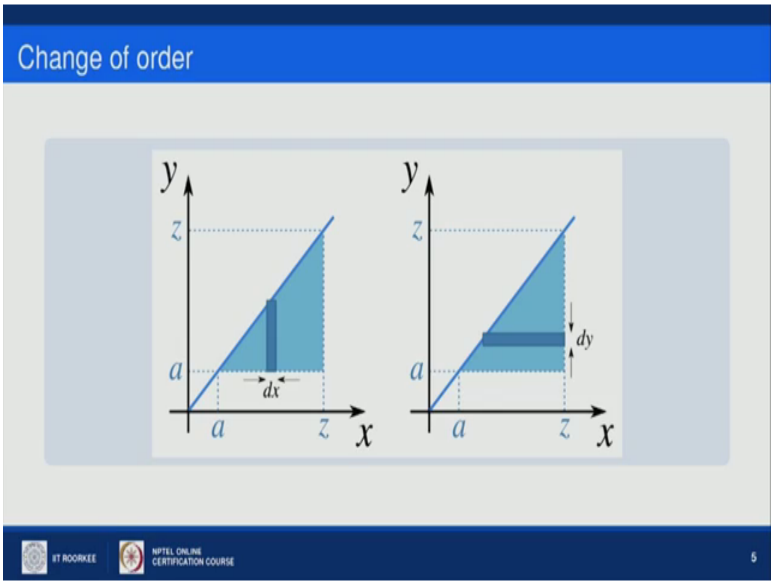

And why we need it that we have seen in these two examples, where sometimes changing the order problem becomes easy; and sometime it is not solvable in the given order. But after changing the order you can integrate it. Some and the idea behind the change of order is just change the direction of by stripe covering the region of the integration. For example, if you are having vertical stripe, make it horizontal; and if you are having horizontally stripe make it vertical. So, basically if it is d x d y, make d y d x and vice versa.

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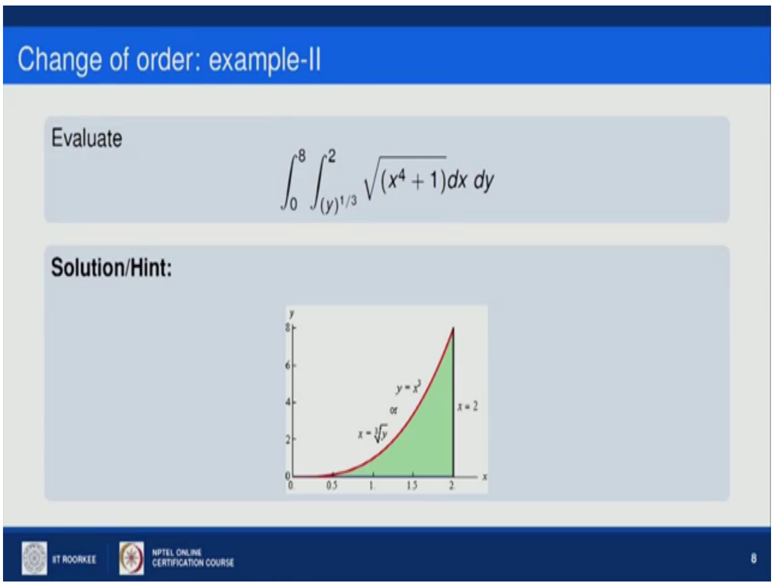

So, let us take one more example of the same type where we needs change of the order.

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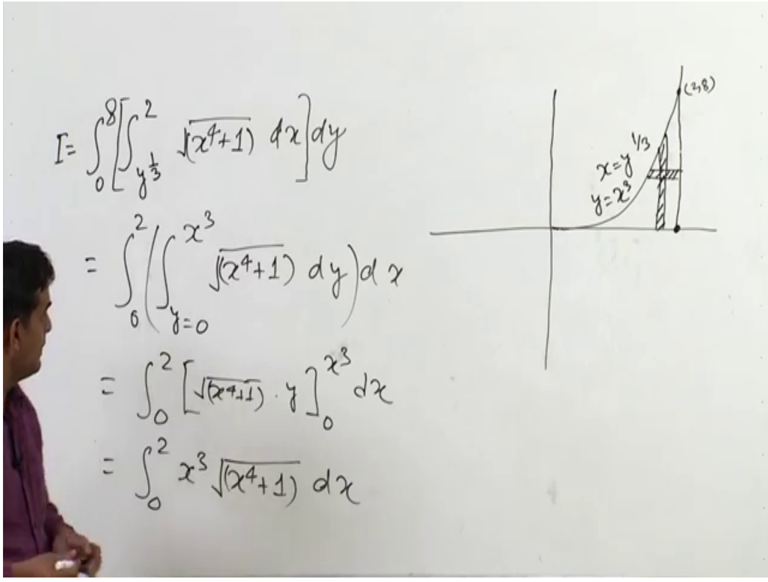

So, my example is double integral, it is coming is the limit y raise to power 1 by 3 to 2 and x a 0 to 8 and the function f of x y is x raise to power four plus 1 a square root and then d x d y. So, find the value of this double integral. Now, in the original form what I need to do first I need to evaluate the definite integral in the square bracket. However, for doing it what I need when I am going to integrate this particular thing, I need a term x raise to power 3 out of this because then only I can assume that x raise to power 4 plus 1 equals to t. I can make some substitution I can eliminate this thing I can integrate this thing but that I am not having here.

So, I cannot integrated it as such means I cannot integrate this with respect to x. So, the solution is change of order. So, changing the order means let us do it first with respect to y. And if you are doing it first with respect to y then you need constant limit for x. So, what I need to do, I need to change the direction of by stripe. So, for that first I need to is case the region and region will be this is the our y equals to x cube. The other one is it will go up to a start from here and it will go up to 2.

So, let us say this is the point 2, 8. So, region is this one. And at this moment what I am having I am having this particular stripe. Now, change the direction of this. So, let me take this kind of the stripe that is the vertically stripe and this problem, so it will become the lower limit of this vertical stripe that is y, y equals to 0; upper limit will become x cube, and then it will start from here 0, it will go up to 2. Now, when I am integrating it 0 to 2 so, this is a such into y 0 to x cube d x once I put the limit. So, this will become x cube a square root x raise to power 4 plus 1 d x and now you can evaluate easily.

So, this is another example where we cannot solve the original problem in it is in the given order, but after changing the order we can solve it very easily. So, far we have talked about change of order in Cartesian coordinates; we did not touch polar coordinates. So, now, let us take some example in polar coordinates where we need change of order.

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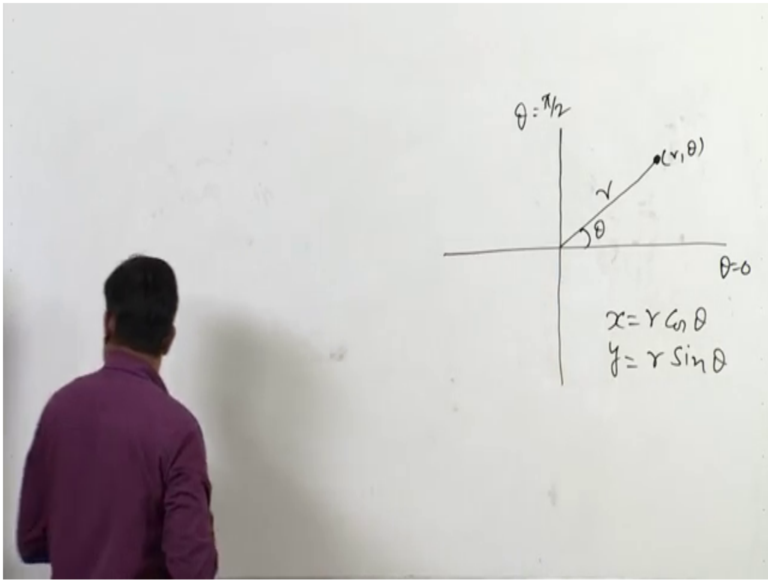

So, before that let me explain how is the double integral or we cover the region of the integration in polar coordinates. So, in polar coordinates what we are having theta equals 0, theta equals to pi by 2 and then we are having some point r theta here. So, r theta means this distance is r and this angle is theta. So, this is the meaning of polar coordinates. So, if this point is x y in Cartesian coordinate then x will be r cos theta and y will be r sin theta that is the projection of this on x-axis and y-axis.

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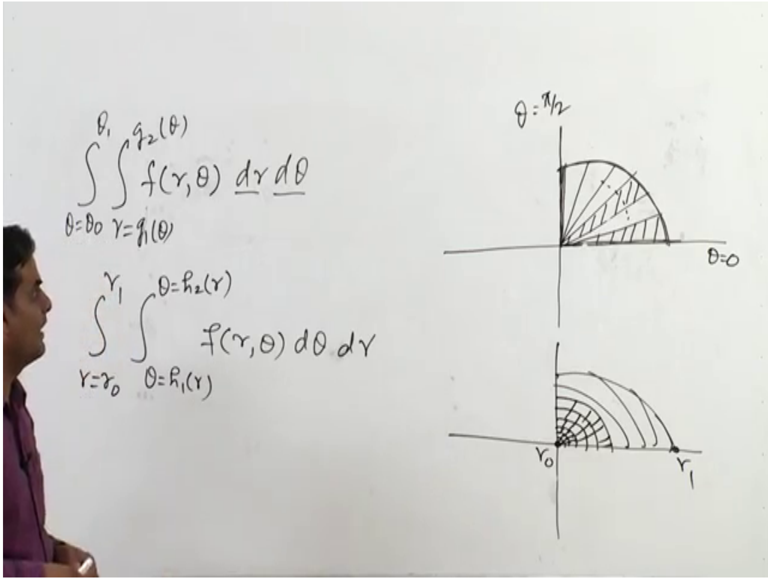

Now, let me have this region in some problem over which I need to integrate. So, in polar I am having these two, then some integrate that is f of r theta and d r d theta. So, r equals to some g 1 theta to g 2 theta and theta is sum theta 0 to theta 1. So, theta 0 and theta 1 are constant limits why for r I am having functions of theta. So, in this way how it will work I will take a radial stripe like this, and it will cover this region like this.

So, means the lower value of r the upper value because this curve will be some curve of r equals to some function of theta, and it will start in this example let us say theta equal to 0 to pi by 2. So, theta 0 will be 0; theta 1 will be pi by 2. So, this is the way when you are using the order first d r and then d theta.

If you are taking the order in another way means you are having variable limits for theta those are functions of r and then some r 0 to r 1 means constant limit for r. So, in this order f r theta will become means d r theta will become d theta d r means first we have to integrate with respect to theta and then with respect to r. So, in the this particular thing how we will cover our region.

So, in this case region will be covered using the circular arc like this. So, here I will be having some this value of theta and this value of theta and then constant limit for r means it will start from here and it will go like this up to here. So, r 0, r 1 like this, it will cover the whole region. So, changing the order means if you are taking this kind of stripe that is the radial one, make it circular; or if you are having circular means you are having this order d theta d r, make it radial. So, this is the idea of change of order of integration in polar coordinates.

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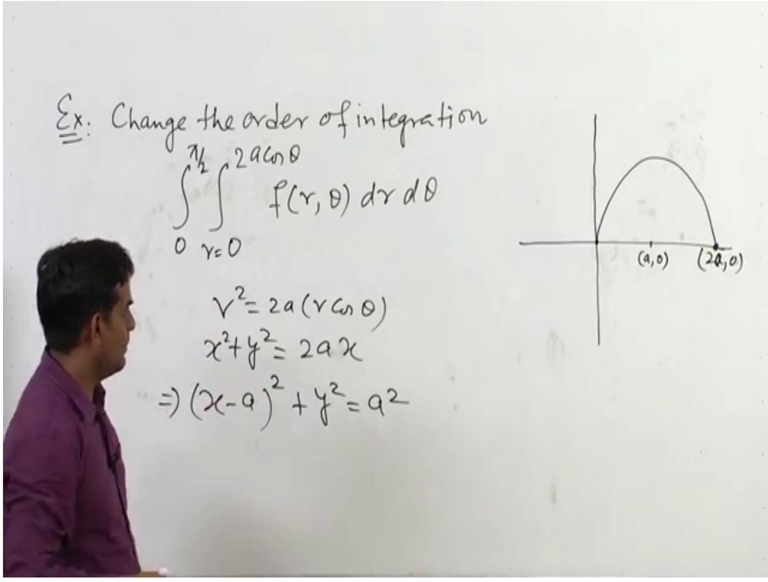

Now let us take one or two example of this case. So, my first example is change the order of integration in the integral theta I 0 to pi by 2 and r is 0 to 2 cos theta f r theta d r d theta. So, here these are the limits for r 0 to 2 a cos theta and I am having the constant limit that is from 0 to pi by 2 means in first quadrant. So, lets us is case this region first. So, r equals to 0 means here and another curve is r equals 2 a cos theta r equals 2 a cos theta means r a square is 2 a into r cos theta as you know r is square is x square plus y square equals 2 a x. So, it means r equals 2 a cos theta is nothing but the circle having centre at a and 0 and having radius a.

So, in first quadrant this will be this circle. So, it will be centre will be at a, 0 and radius is a. So, this point will be 2a, 0. So, r is a, theta is 0, r is 2 a, theta is 0. Theta is going 0 to pi by 2, so we are taking only first quadrant and according to the given problem we are taking the radial stripe.

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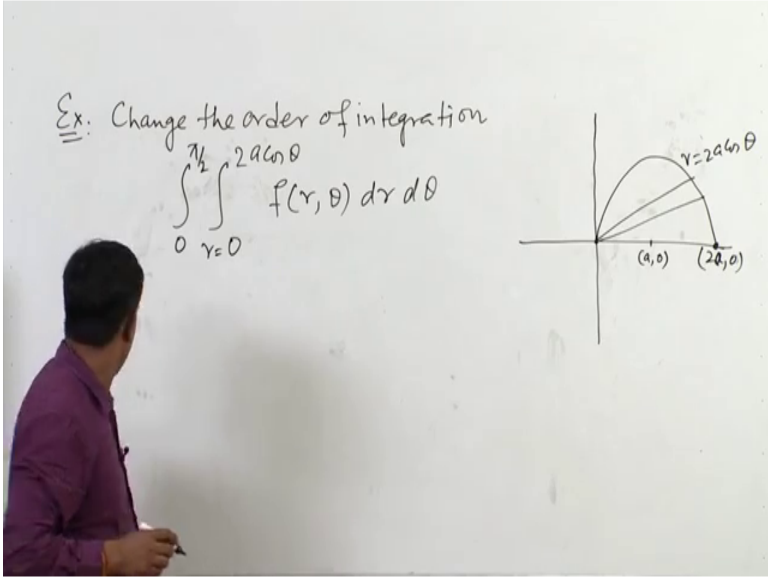

Why, radial stripe because my stripe is starting from this point that is r is equals to 0 and finishing at this curve which is r equals 2 a cos theta. So, 0 to 2 a cos theta and this is radial stably start from here and it will go up to here that is the theta equals to pi by 2. So, now, cover this region by circular the arc.

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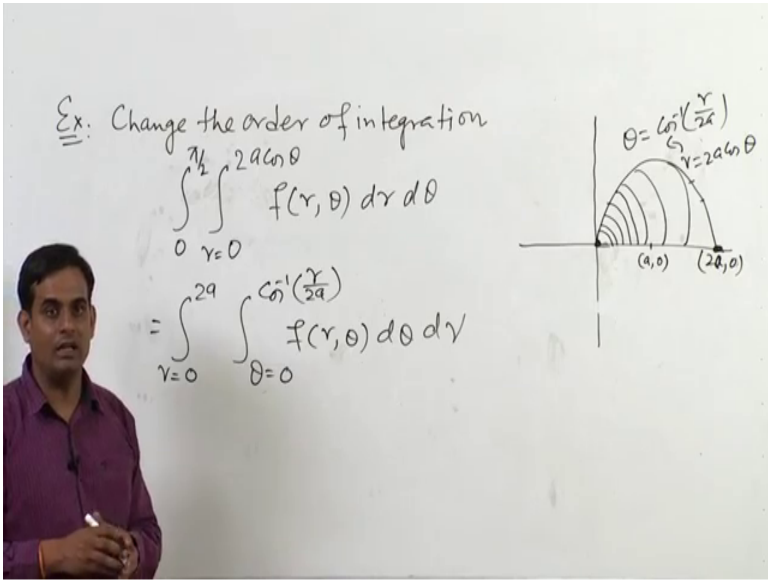

So, circular arc will go like this. So, from it is starting it is starting from theta equals to zero line. So, theta equal to 0. Where it is finishing, it is finishing on this curve. So, what is this curve, this curve is theta equals to cos inverse r upon 2 a. So, upper limit of theta will become cos inverse r upon 2 a and now the stripe will start from here. So, here r is zero and then it will go like this up to here that is 2 a, so 0 to 2 a f r theta d theta d r. So, this is the change of order in polar coordinates for this particular example. Take one more example to make a very clear understanding of this particular concept.

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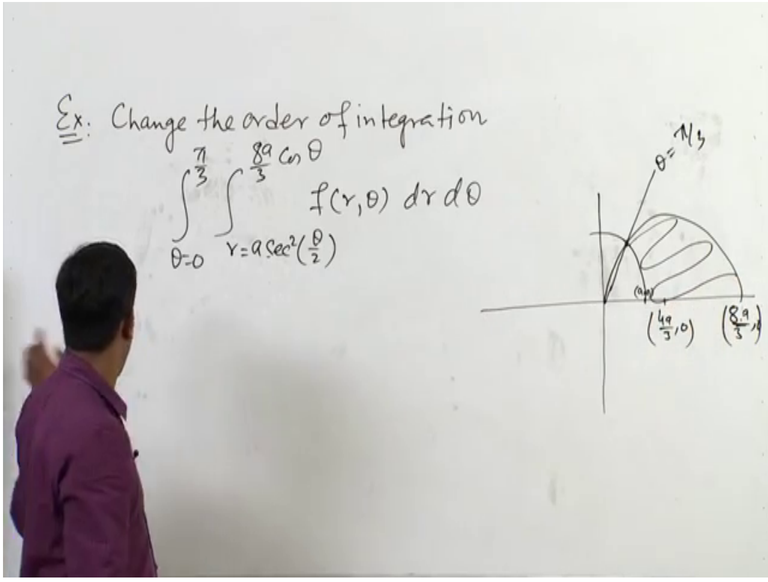

Change the order of integrals theta equals to pi to pi by 3 and r is a sec square theta by 2 to 8 a upon 3 cos theta f r theta d r d theta. So, we need to solve or we need to change the order of this particular integration. Here a limits are looking at bit complicated it compare to the earlier example. So, let us see. So, like earlier case the upper limit is r equals to 8 a by 3 cos theta. So, it is clearly a semicircle in first quadrant having centre at 4 a by 3 0 and having radius 4 a by 3. So, it is the circle 8 a by 3 9.

Now, the other curve is r equals to a sec square theta by 2. So, when theta is 9, r is this term is 1, because a sec zero is. So, a sec 0 is 1. So, r equals to a. So, this curve will be start from r equals a. And then it will intersect this curve at pi by 3. So, it will go like this and then we do not care after this because this is so as I told you it will intersect at pi by 3. So, this line is theta equals to pi by 3.

So, my region of integration is theta equals to 0 to pi by 3, and this curve a region between this curve and this curve. So, this is the region of integration. I am taking the radial stripe according to the original problem. So, these radial stripe is going like this.

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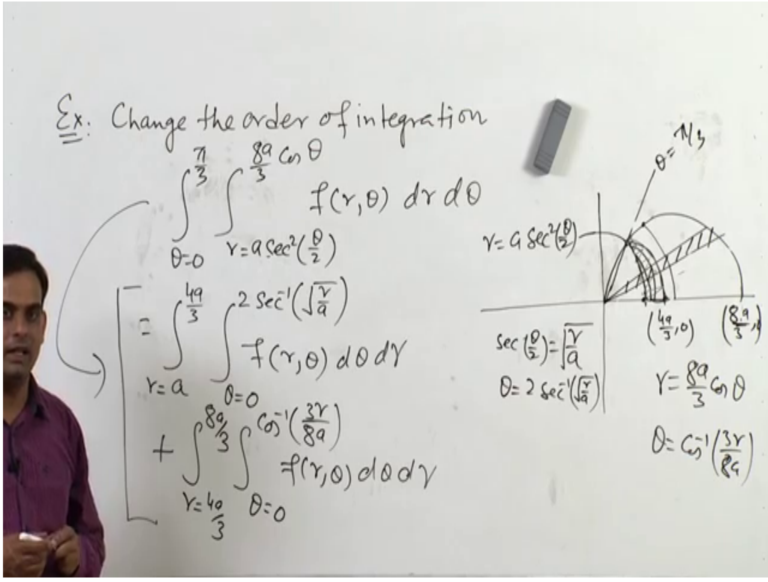

Means the lower limit is this one a sec squared theta by 2, upper limit is 8 a by 3, cos theta and it starting from theta equals to 0 going up to theta equals to pi by 3. Now, take the circular stripe. So, my region is start here. So, this is a. So, circular stripe will be like this. When it is 4 a by 3, this is 4 a by 3, because the radius of this. So, mean circular stripe will start from this line that is the theta equals to 0 an ending on this curve.

So, what is this curve r equal to a sec square theta by 2. So, form here I can write sec square theta by 2 it r upon a. So, sec theta by 2 is square root r upon a. So, theta will be 2 sec inverse a square root r upon a. So, 0 and it is (Refer Time: 35:12) 2 sec inverse is square root r upon a. And r is starting from r equals 2 a and going up to 4 a by 3 f r theta d theta d r, but please look so far we have cover only this much region this region is left. Because why because after that theta will start from 0, but ending up on this curve instead of this curve.

So, for this I have to take one more region. So, plus theta equals to 0 ending on this curve. So, this curve is r equals to 8 a by 3 cos theta. So, theta will be cos inverse 3 r upon 8 a. And then r will go from 4 a by 3 2 8 a by 3 f r theta d theta d r. So, this is the complete solution after changing the order of the integral.

Now, my region is divided into two sub regions, and I need to calculate the double integral separately on each sub region. So, this is the change of order in polar coordinates. So, in this lecture, we talk about change of order, and we have taken some examples from Cartesian coordinates and then from the polar coordinates. We also discuss why we need change of order in multiple integrals.

So, thank you very much.