**Multivariable Calculus**

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**Lecture - 20**
**Multiple Integral**

Hello friends. So, welcome to a lecture series on multivariable calculus. So, today we will dealing with multiple integrals. Now, we already know what integral y dx.

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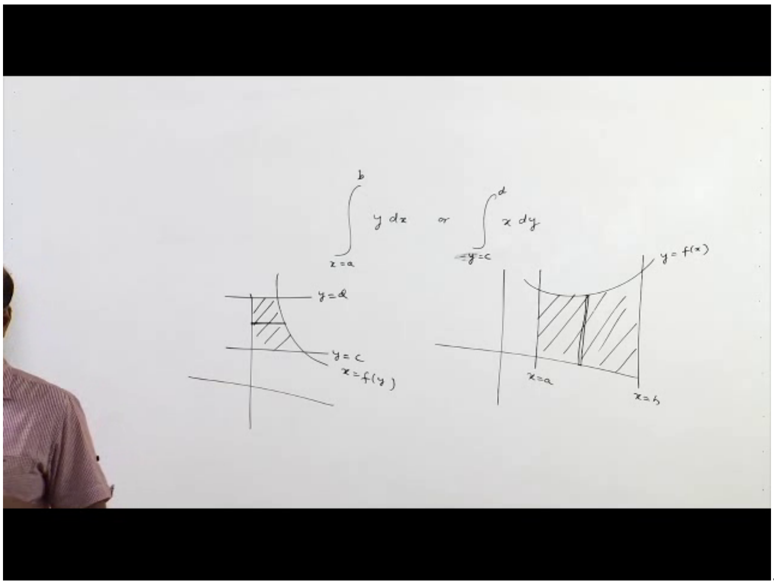


Or integral x dy gives, you see when we talk about integral y dx or integral x dy, what does it give, it gives area, you see, when you have a curve, say y equal to fx and say x is varying from a to b and you are interested to find out, suppose this area. So, how you find this area, you take a strip along y axis. So, this is y, this is dx. So, the area is y into dx and you sum up it over all their strips, you move this strip over the entire region, you sum up it, sum up, all these, small strips and this will give integral y dx from x 2 goes to a 2 x 2 goes to b.

So, it basically gives, area below the curve below the curve y equals to fx above the x axis and between line x equal to a and x equal to b similarly, if you talk about this integral; so, this integral, for this integral curve is something like this. This is x 2, goes to f y, where y is varying. Suppose, from c to d and we are talking about this area.

So, we take a strip parallel to x axis, this is x and this is dy. So, this is x dy and then we sum up this strip over the entire region, this will give the area of the shaded portion which is integral, which is in single integral x dy and x is varying from and y is varying from c to d. This is the single integral. Now, how can we define double integral, the double integral? So, double integral over tingles consider a function.

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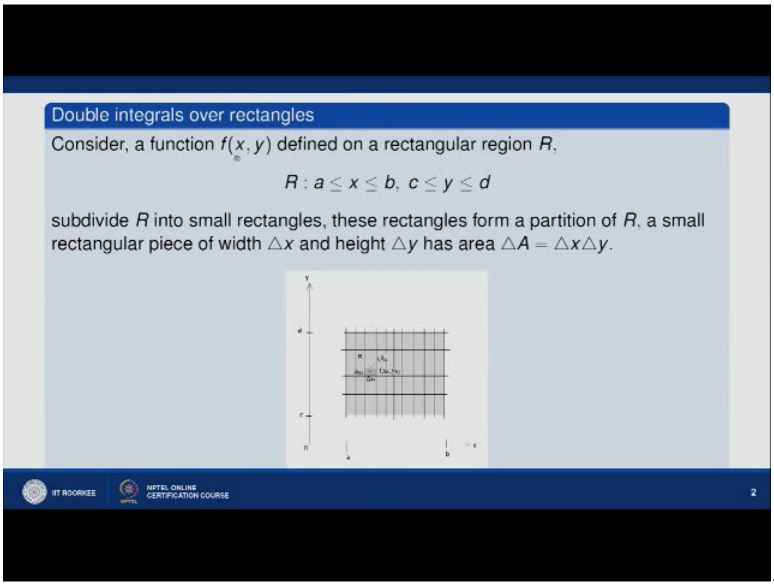


F xy defined on a rectangular region r, which is x varying from a to b and is y, is varying from c to d. So, you have a rectangle.

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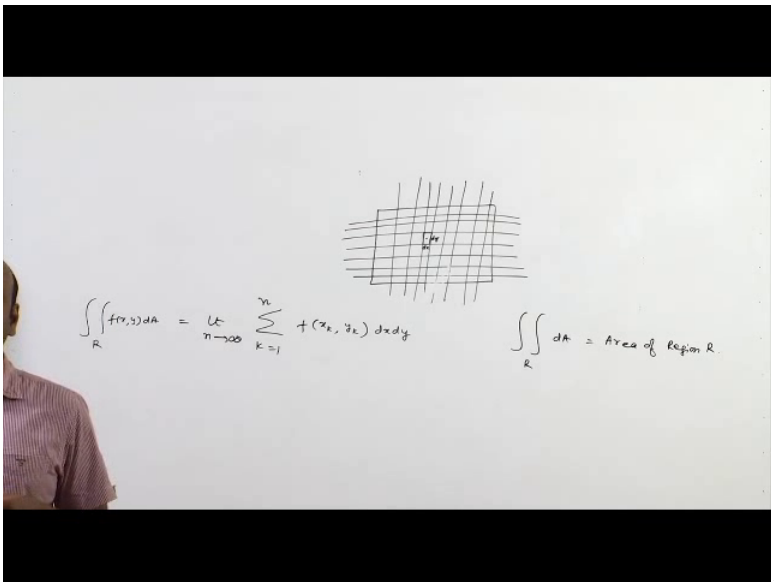

Now, suppose, you have a rectangle, now, to find out, the, to define double integral of f over this rectangle, we first divide this rectangle into number of vertical and horizontal strips. Now, we take a small portion, say this portion, say this is dx and this is dy and in this small portion take a point say x K y K. The functional value at this point is f of x Ky x K comma yK into dx dy. Now, what we are having basically?

We have a rectangle, where we have a rectangular. Suppose, this is that a tingle, we divide this rectangle into a number of vertical and horizontal stripes, take a small portion of area dx dy, take a point xk yk on that the small portion. Now, at that portion, at that point f xk yk gives height at that point, f xk yk gives height and dx dy is that small portion. So, f into dx dy will give volume of that small strip, this volume of that ah, small portion and you sum up it over entire, rectangle. You sum it up, a from 1 to n say n number of strips and then tend and then take limit n tends to infinity; that means, limit tend n tends to infinity means you are taking that stress smaller and smaller.

So, this will converge to this will gave double integral over rectangle r of f xy into da da may be dx dy or dy dx. So, this will be the double integral f xy dx dy or dy dx. So, what does it give basically ? It gives volume of the solid over the region R and height is governed by x, a z is equals to f xy the region is R here, we are defined region as rectangle, region may be any ah, anything.

Region may be square, region may be some other portion ok, on xy plane, region may be anything. Now, this gives volume of a solid, which is obtained over the region R. And height is governed by a z equal to f xy. Now, if f xy is 1, now. Now, this double integral over R da; this basically, gives area of the region R. So, area for our region can be computed by a single integral y dx or x dy or can computed by double integral, also that is double integral over r dx dy or dy dx.

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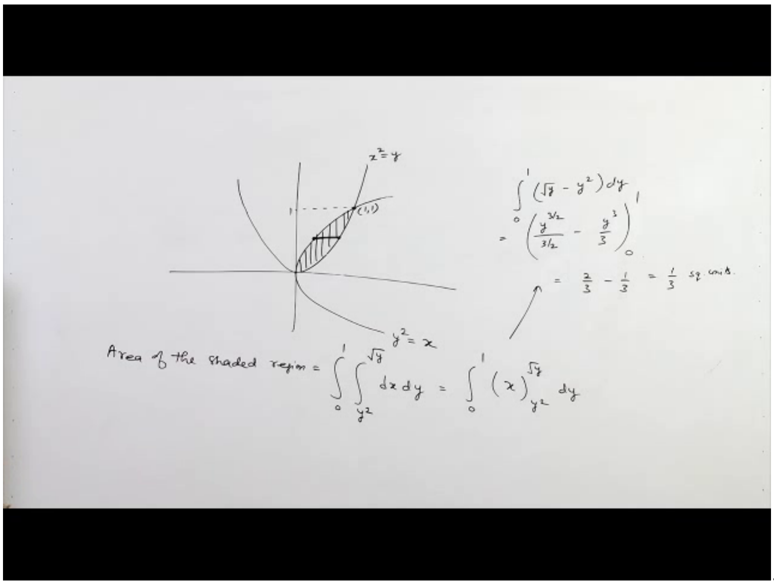

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Suppose, these are two curves say, it is y square equal to x and say it is x square equal to y and you want to find out the area enclosed by these two curves. What is that area enclosed by these two curves is this area.

Point of intersection is clearly 1 comma 1 1 2 satisfy this 1 1, satisfy this. Now, we want to find out the shaded region, shaded region R. So, we can compute this shaded region. So, area of the shaded region will be double integral. We can take dx dy or dy dx. Suppose, we are taking dx dy. Now, in dx dy, if you are taking dx dy. So, take a strip parallel to x axis first.

So, take a strip parallel to x axis, take a strip parallel to x axis. Now, what is x here, lower bound and this is upper bound of x, what is x here? X here is this curve, that is y square and what is the x here ? x here is under root y. And y is reading from which point to which point? You see y is varying from 0 to 1 in the entire region, y is only between 0 to 1. So, y is varying from 0 to 1.

So, this is how we can obtained the area of shaded region by double integral. Now, you can simply solve this integral. How can you solve this? It is 0 to 1. Now, first you, first you integral with respect to x, keeping y constant, integral dx will be x, this, it is y square. This is under root y dy. Now, upper limit minus lower limit, this give integral 0 to 1.

This is under root y minus y square, upper limit minus lower limit into d y. Now, you integrate over y. So, integration of y raise to power 1 by 2 is y raised to power 3 by 2 open 3 by 2 minus y cube upon 3. And it is 0 to 1. So, when y is 1, it is 2 by 3 minus 1 by 3, at 0 to 0. So, it is 1 by 3, the square units. So, this will be the required area of this shaded portion. Now, this area can also, we find out if in instead taking dx dy, we take dy dx.

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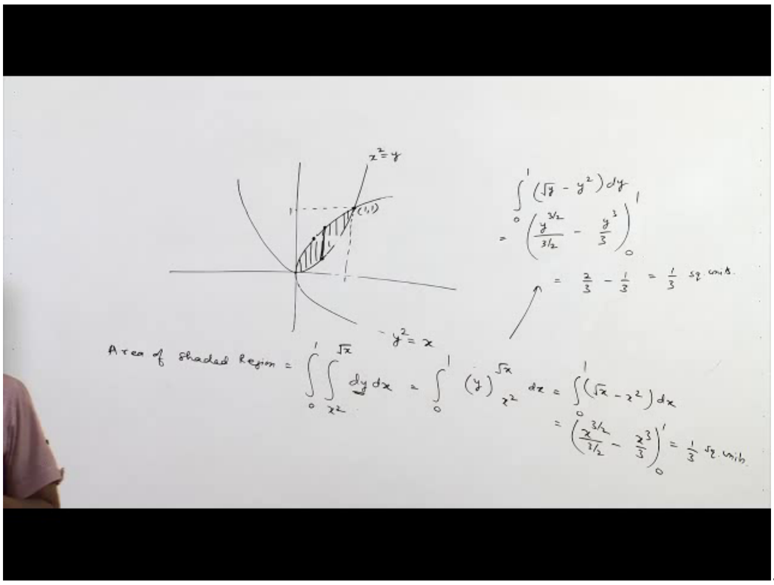


Suppose, you take dy dx. So, area of shaded region is equal to double integral. Suppose, you take d by dx. Now, if you are taking dy first. So, take a strip parallel to y axis. So, when you should taking a strip parallel to x axis, take strip parallel to y axis. Now, if you take a strip parallel to y axis, what is y here? y here is x square, and what is y here ? y here is under root x.

And x and x is varying from this point to this point. The minimum value of x is 0, maximum value is x is 1, in this region x is always varying between 0 and 1 only. So, 0 to 1. So, this will be equal to 0 to 1 and the integration first. You integrate with respect to y, keeping x constant. This is equal to 0 to 1, upper limit minus lower limit. So, this is s raise to power 3 by 2, a whole is 2 upon 3 by 2 minus x 2 by 3 from 0 to 1. So, we get the same answer, which is 1 by 3, the square units. So, that is how we can find out the value of the area of the shaded region, using double integral. Now, suppose we have this type of problem.

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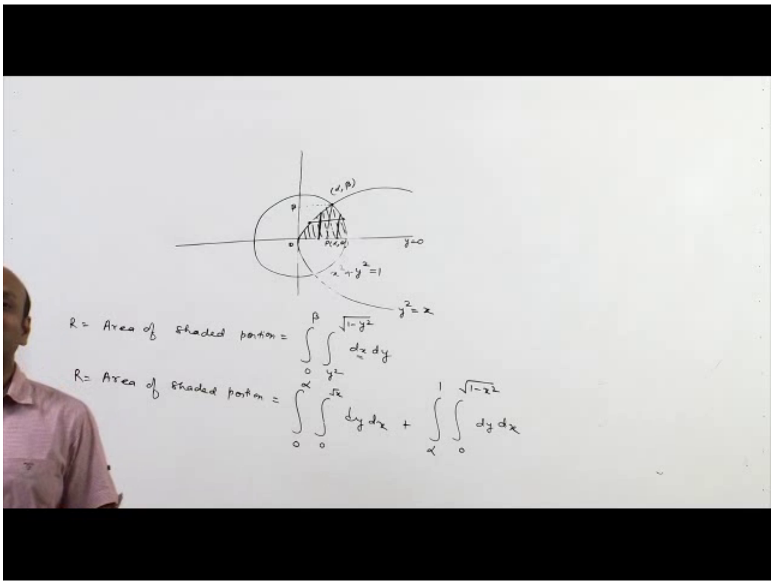


We have a circle say x square plus y square equal to 1. We have a parabola say y square equal to x. Now, you are interested to find out, this shaded area. So, how can you find this? Now, suppose this point is alpha beta, this can, this point can easily find out by finding the point of intersection of these two curves, say at this point of intersection is alpha comma beta. Now, we are interested to find out the area of the shaded portion. So, how can you find out? So, area of shaded portion or shaded region will be equals to double integral. Suppose, we take dx dy.

So, dx is first. So, take a strip parallel to x axis. So, you take a strip parallel to x axis. What is x here? X here is y square and what is x here ? X here is under root 1 minus y square and y is reading from which point to which point ? Here, y is 0 and here y is beta. So, y is varying from 0 to beta. So, this will be the presentation of area of the shaded region. Now, this is, this area can also, you find note, if you initially taking dx dy, we take dy dx. So, this suppose, this is R. So, same R, which is the area of this region can also, I find like this, you see dy dx. So, you have to take a strip parallel to y axis, if you take a strip parallel to y axis here.

Now, if you take a strip parallel to y axis is here. So, here the lower bound is, lower bound is ah, y equal to 0 and upper bound is this; however, if you take the same, the strip over here, the lower bound is y equal to 0 and the upper bound is on the circle; that means,, when we move this strip on the, on the shaded region, the lower and the upper limits are changing. So, we have to split the region from this intersection point. If you draw a line perpendicular to x axis from there to here, from o to P, you have; so, you have to divide this region into 2, the first region is this, where y is varying from 0 to this point. Here, y is for this curve, y is under root x and x is varying from this point to this point.

This point is alpha 0 from 0 to alpha and in this region when you take a strip parallel to a y axis here, y is 0 and here, y is on the circle, which is given by under root 1 minus x square and x is varying from alpha to 1, because here, here is alpha x is alpha. Here, x is 1. So, sum of these 2 will give area of the shaded region. So, whenever you plot test, whenever you take a strip, either parallel to x axis or parallel to y axis, you moved at a strip in the entire region. If the lower and the upper bound of the region is not changing. So, we do not ah, need to split the region, if it is changing then, we have to split the region, in the, in the same.

If we take a strip parallel to x axis for this portion, if we move this strip in the region, the lower and upper curve are not changing. So, there is no need to split the region; however, if you take a strip parallel to y axis and move this strip in the region, move this strip in the region. So, the here, lower and upper curve, lower is an x, x axis. Upper curve is the parabola. Here, it is an x axis and upper curve is a circle. So, it is changing. So, we have to split the region. So, that is how we can find out area of a region, using double integral.

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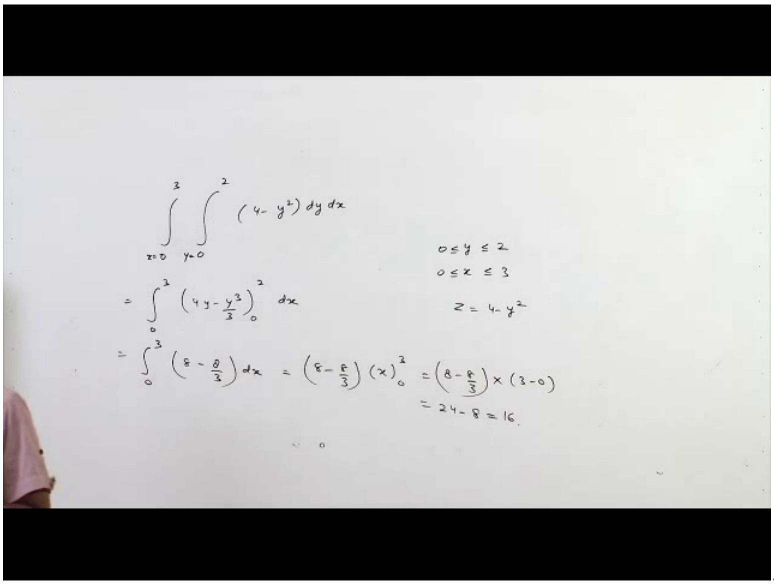


Now, because all these problems also say the first problem, the first problem is double integral 0 to 3 0 to 2, it is 4 minus, y square dy dx. Now, what basically, this is, this is volume of a solid enacted, on a rectangle enacted on a rectangle, where y is varying from 0 to 2 and x is varying from 0 to 3 and height is governed by a 4 minus y square. You have a rectangle on xy plane, where x is varying from 0 to 3 and y is varying from 0 to 2.

We have a rectangle and height is governed by a, for every xy in that rectangle. We have a corresponding height, which is governed by 4 minus y square and 4 minus y square into dx dy or dy dx will give the volume of the solid, which is erected on that rectangle. So, this is basically, volume of a solid volume of a solid on the region this. And the height is governed by ah, this expression. So, how to evaluate this? So, this is, this is limits for y, this is limits for x.

So, this is 0 to 3, you simply integrate this keeping x constant. So, it is 4 y minus y cube 0 to 2 dx. It is 0 to 3, it is 8 and it is 4 y minus y cube by 3, then it is 8 minus 8 by 3 dx and it is equals to 8 minus 8 by 3 integral of dx is x from 0 to 3, which is 8 minus 8 by 3 into 3 minus 0. Which is 24 minus 8 equals to 60. So, that should be the value of this expression. Now, similarly if you want to solve the second problem or the next problem, that also can you find out?

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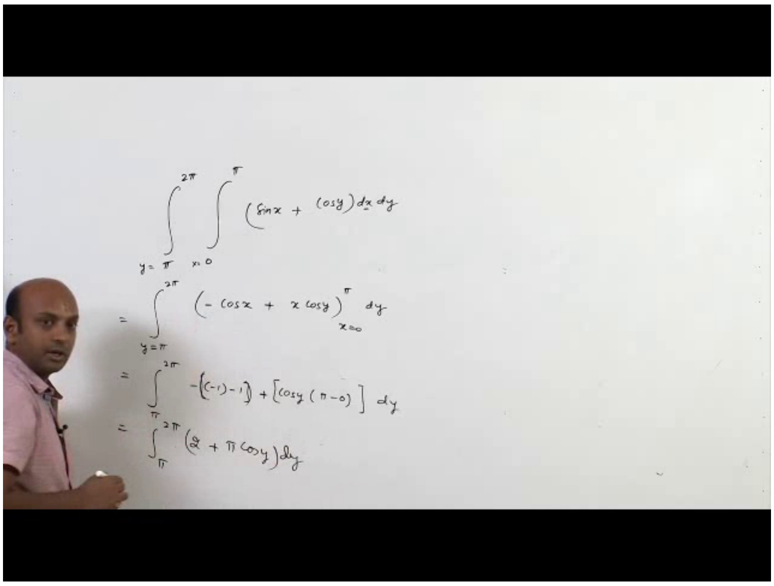

It is integral pi to 2 pi integral. It is 0 to Pi sin x plus, it is cos y and dx dy. So, these are the limits for x, because dx is coming first and these are the limits for y. So, y is from Pi to 2 Pi. Now, integral respect to x keeping y, a constant. So, integral of sin x is minus cos x plus integral of this is x cos y and x is varying from 0 to Pi dy. So, this is equal to Pi to 2 Pi. Now, you substitute, x equal to Pi. It is minus minus 1.

Upper limit minus lower limit minus 1, then cos y comes out, because we are taking y as constant. These are limits for x, you take x equal to pi and then x equal to 0 and this is dy. Now, this is integral Pi to 2 Pi. This is 2 plus Pi cos y into dy.

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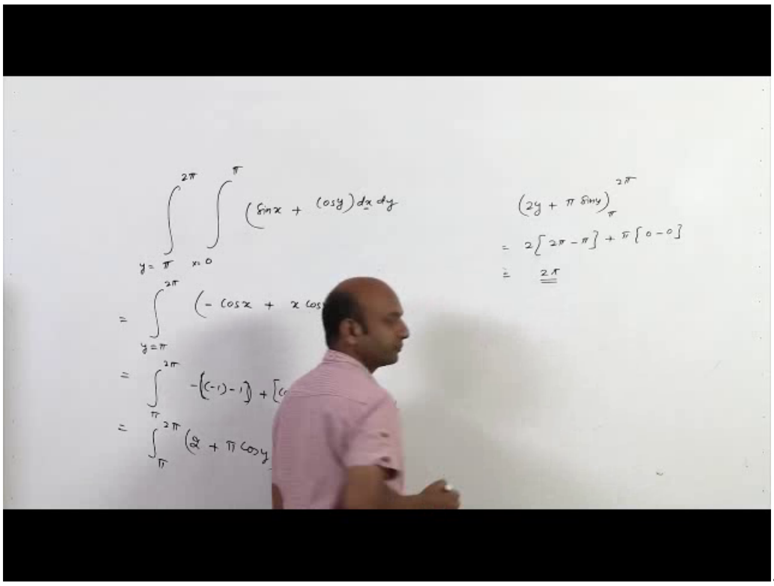

Now, this will be 2 y plus pi sin y from Pi to 2 Pi. This is equals to Pi 2 Pi minus Pi, upper limit minus lower limit and Pi 0 minus 0. So, this is simply 2 Pi. So, that is how we can, find out the values of these double integrals.

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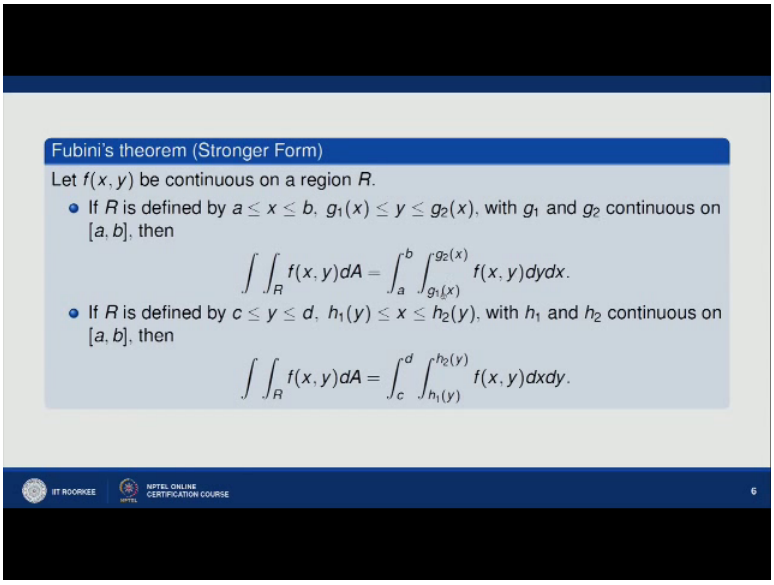


Now, a Fubini's first form, what is that, if f x y is continuous throughout the rectangular region R, which is a, where x is varying from a to b and y is varying from c to d, then if you interchange the limits, then the value will remain the same. So, if limits are constant, if limits are constant. So, and we interchange the limits. So, the value of the, a double integral will not change, will not affect it. This is Fubini's theorem or the first form. Now, the stronger form of Fubini is. Let f xy, we continuous on the region R, if R is defined by a x, varying from a to b and y is varying from g 1 x to g 2 x with g 1 and g 2 are continuous on closed interval a to b, then double integral over R f xy dA will be given by. Now, here y is varying from g 1 to g 2.

So, you take the limit of y g 1 to g 2 and x is varying from a to b f xy and then dy dx, because, first you are putting the limits for y and then for x. So, here will be dy dx. Similarly, the another form of this is, if we defined R else, y is varying from c to d and x is varying from h 1 y h 2 y with h 1 and s 2 continuous on closed interval a to b, then double integral over R f xy dA will be given as, now, your first integrate respect to x taking, the limits of x from h 1 to h 2 and then y from c 1 to c 2 d and the and take dx dy, because you are, we are putting the values of limits for x first. Now, suppose you want to solve these problems.

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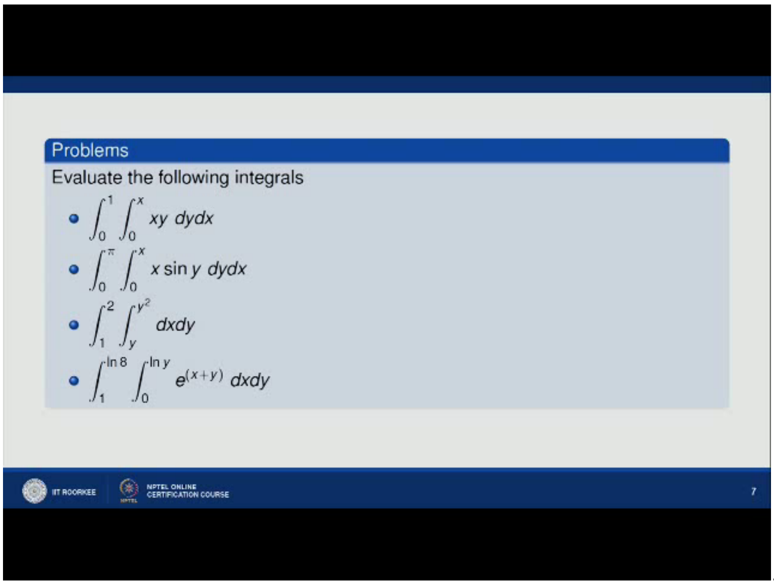


So, we can take the first problem.

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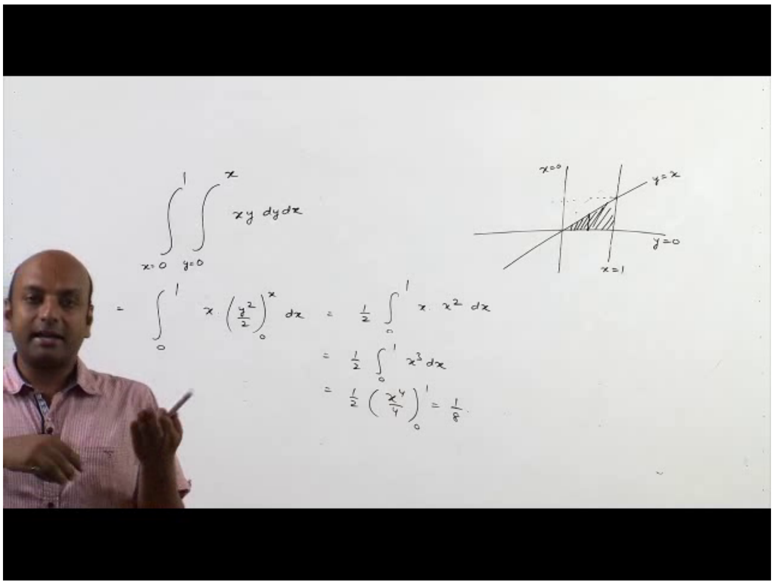

The first problem is double integral. Now, here it is 0 to x and then 0 to 1 dx dy dx. So, these are limits for y and these are limits for x of course, because y is, first to their limits for y and these for limits for x. So, how can you find this integral? So, this is integral 0 to 1. You first integrate this respect to y, keeping x constant. So, it is x into y square by 2 and y is varying from 0 to x and dx. So, 1 by 2 can come out integral 0 to 1. It is x into x square dx, because for 0 to 0 is equals to 1 by 2 integral 0 to 1, x cube dx, which is equal to 1 by 2 x raise to power 4 by 4 0 to 1 which is 1 by h.

Now, what this represent basically, let us see. Now, this here, here y is 0 here, y is equal to x here, x equal to 0 and x equal to 1 x equal to 1 means this line. So, which region you are having. So, you take a strip parallel to y axis dy is there, you take a strip parallel to y axis, here y is 0 and here y is x and x is varying from 0 to 1. So, this is the region, because in order to check, whether this is, this triangle or the above triangle you first, you take a strip parallel to y axis, because dy is there first. You take a strip parallel to y axis. Now, here y is 0 and here y is x. So, here y is 0 to x.

And x is varying from 0 to 1. So, this shaded region is, this region R. So, what we are having basically, we are having a triangle, which is given by this shaded region on the xy plane and a, and a height is governed by a z equal to x y and the solid directed over this, where, and the volume of solid directed over this will be is, 1 by 8. Now, similarly, you can solve these problems. Suppose, you want to solve the last problem. Similarly, you can solve second and three, the last problem let us.

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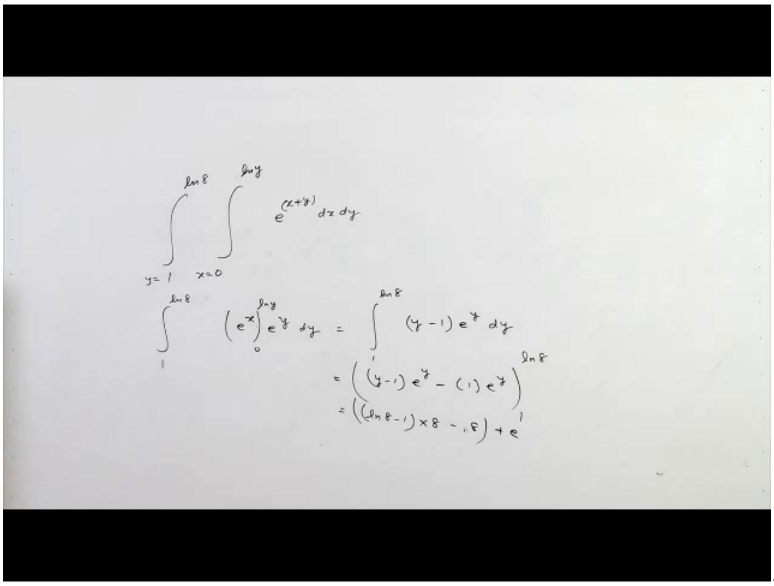

So, let us suppose, it is integral 0 to 1 to, ln 8 integral 0 to ln y e raise to power minus x square minus e, e raise to power x plus y. Sorry and it is dx dy. So, these for x these for y. So, again you can integral in the same way, it e raised power x into e raised to power y, where you integral respect to y, it will be e raise to power x will remain as it is and e raise to power x i will remain as it is integral of e raised to power x will be e raise power x and it is 0 to ln y dy. So, this is equals to 1 to ln 8. Now, when x is ln e, ln y.

So, it is y and at 0, it is 1 and it is e raise to power y dy. Now, I simply integrate this using product tool. So, it is, first as it is, then it is e raise to power y minus derivative of, this is one integral, this is y. So, this is 1 to e raise to ln 8 and then, you can simply substitute the upper limit minus lower limit at y equal to 1, it is 0 at y equal to 1, it is e. So, this will be the value of this double integral. So, these are some of the property of double integral.

Now, triple integral. Now, suppose, we have a solid. Now, we if, we have a solid say, we have a cuboid. So, we divide this along x axis, along y axis along z axis, we divide that solid along x axis, y axis, and z axis and take a small solid of volume delta x, delta y, delta z, and pick a point say xk yk zk in that point and the value of the function that point will be f xk yk zk, then we take, we take the sum of this.

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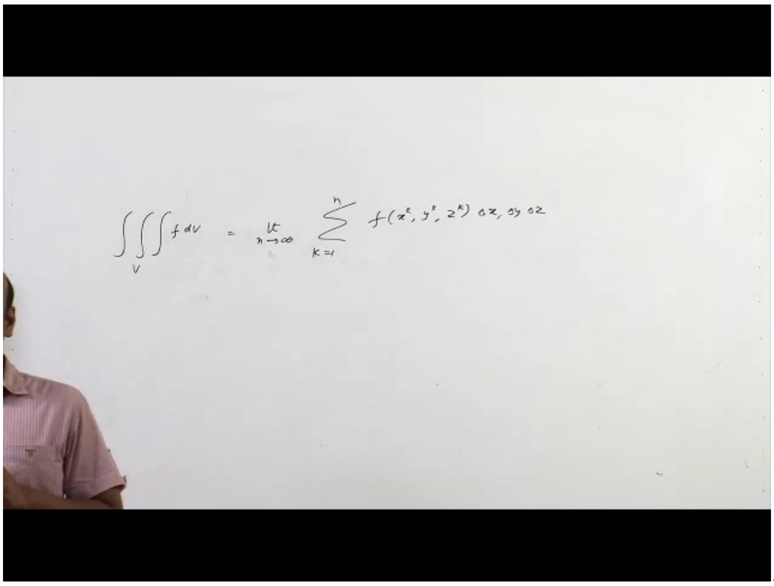

Which is f at xK yK zK delta x, delta y, delta z and K varying from 1 to n and limit n tending to infinity means, we are making, we are taking the solid small and the smaller very close to 0. I mean very, the small solid, then this will be triple integral over volume V of f dV. So, this is how again, you find triple integrals.

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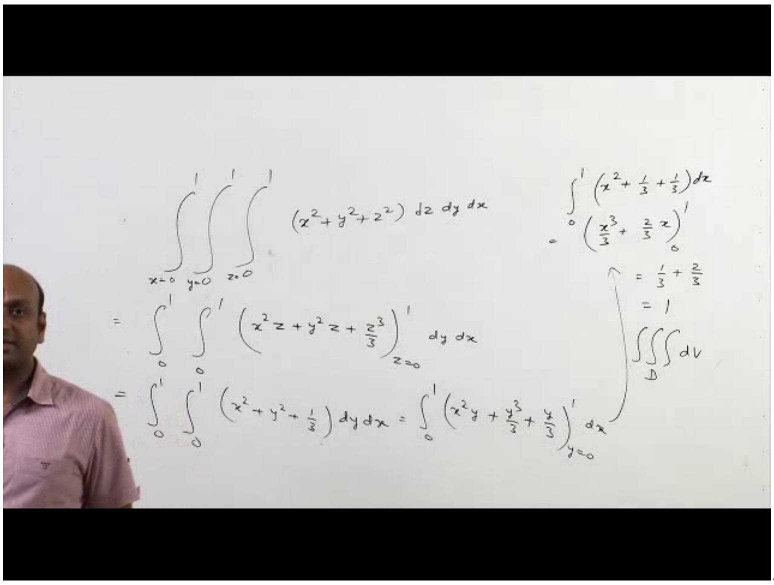


Now suppose, you want to solve the first problem, this is triple integral 0 to 1, 0 to 1, 0 to 1 x square plus y square plus z square. First, we have dz then dy, then dx this means, this is for z, because z is dz is the first one, then it is y and there it is x. So, first two integral respect to z, keeping all our variables constant. So, this will be double integral 0 to 1, 0 to 1, integral respect to z, keeping other variable constant.

So, x of square z plus y square z plus that cube by 3 and z is varying from 0 to 1 dy dx. So, this is equal to 0 to 1, 0 to 1. So, it is x square plus y square plus 1 by 3. When you take z equal to 0, all are 0. So, it is dy dx. Now, you integral respect to y keeping, other variable constant, other variable is x keeping that constant. So, it is 0 to 1.

It is x square y plus, it is y cube by 3 plus and 1 by 3 is y by 3 from y equal to 0 to 1 and dx. So, this value will be, it is 0 to 1, you put y equal to 1, then it is x square plus 1 by 3 plus 1 by 3 and for y equal to 0, all are 0 and 2 dx. So, this is again equal to.

Now, it is x cube by 3 plus 2 by 3 x from 0 to 1. So, it is 1 by 3 plus 2 by 3, which is 1. So, the, final answer is 1. So, in this way, you can solve such type of problems and similarly, you can also solve the next problem of this slide. These are some of the properties of triple integral like you can take constant out.

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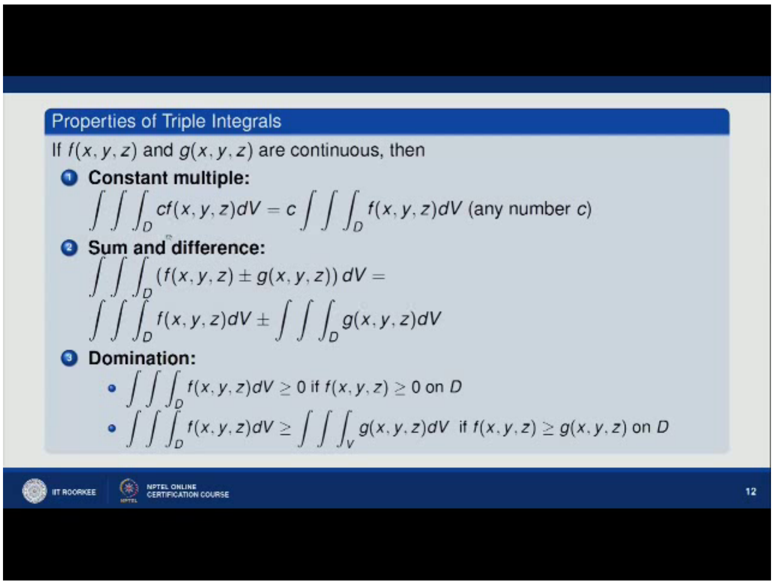


If we have a addition or subtraction of 2 tipple integrals, it can be write separately and if fxyz is greater than or equal to 0 on d then the triple integral of fxyz into dV is also greater than equal to 0 and similarly, if fxyz is greater than equal to gx yz, then the corresponding triple integral of f will be greater than equal to triple integral of g. Now, if, if f is 1, you see, if we are, if we are talking about triple integral over d of dV; so, what is, what this give? This give volume of the solid d, this gives volume of the solid d.

Thank you very much.