Multivariable Calculus
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Lecture - 02
Limits for multivariable functions- I

Hello friends. So, welcome to lecture series on multivariable calculus in the last lecture we have seen that what do we mean by a function of several variables, domain and range of two or more than two variable functions. Whether, it is open closed, mounted, un-mounted all those we is all these things we have seen in the last lecture. Now, how can you find limit of multivariable functions.

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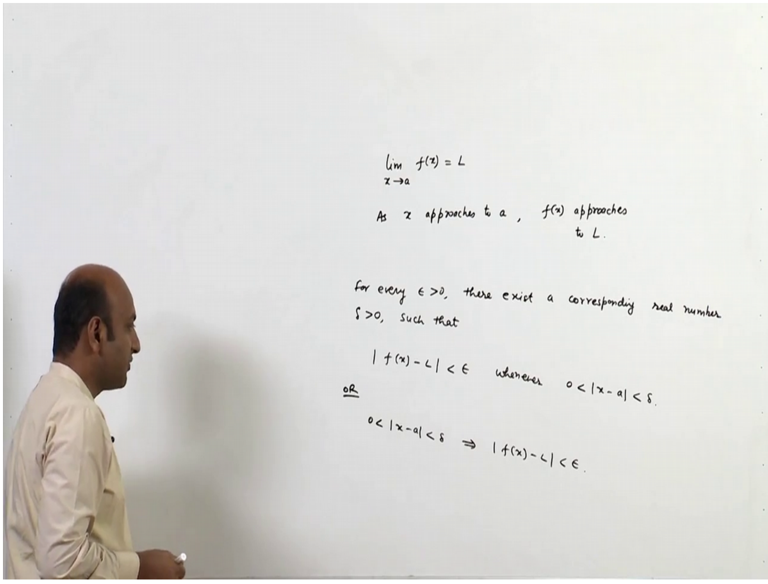

So, we recall it first you recall for limits for one variable functions so, suppose we have limit x tending to a f x is equal to L.

So, what does that mean it means it means that as x approaches to a f x approaches to L so, roughly speaking when we say that limit x extend to a f x equals to L. This means as f tending to a f x tends to L, now how can we defined this mathematically. So, the mathematical definition of limit is for every epsilon greater than 0, there exists a corresponding real number delta greater than 0, such that mod f x minus L less than epsilon whenever 0 less than mod x minus a less than delta.

You take any epsilon greater than 0, no matter how small how large epsilon you are taking. There will always exist a corresponding delta greater than 0 such that f x minus L, the mod of this less than epsilon whenever this quantity less than delta greater than 0 or we can say that this implies mod of f x minus L less than delta. Now what does it mean what this definition mean, now let us see.

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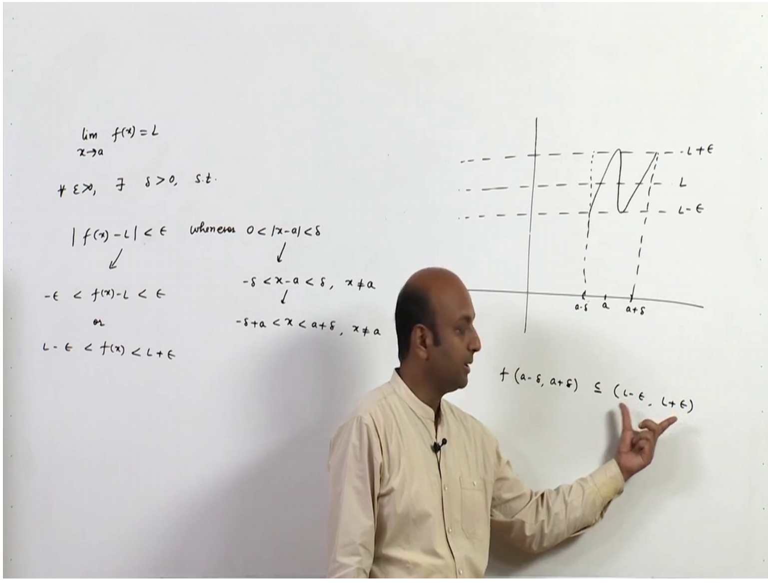

Now here we are having that limit x tending into a f x is equals to L, this means for every epsilon greater than 0 there exist corresponding delta greater than 0 such that mod f x minus L less than epsilon whenever 0 less than mod x minus a there is a delta.

Now, this means this means that x minus a is less than delta greater than minus delta and this means that x is less than a plus delta and greater than delta minus delta plus a and of course, x should not equal to a because if x equal to a then it is equal to 0 and x equal not equal to a. This sometimes we call as deleted neighbourhood this we are also called as a deleted neighbourhood of x at a.

Now, same similarly what this means what this inequality represent this is f x minus L less than epsilon greater than minus epsilon or f x is less than L plus epsilon and greater than L minus epsilon. So, this means now this is this is suppose L, this is f x equal to L now this is suppose L minus epsilon and this is suppose L plus epsilon. Now x is between a minus delta to a plus delta you take suppose this is x equal to a suppose this is a minus delta and this is suppose a plus delta.

Now, this definition means this means you take any epsilon greater than 0, no matter how small how large epsilon you are taking there will always exist a delta such that the image of all those x lying in this interval will always be contained in this band means you take any epsilon greater than 0 no matter how small how large you are taking, there will always exist a corresponding delta such that image of all those x lying in this interval will always be contained in this band.

Now, if you take epsilon very small tending to L tending to 0 then this delta will definitely tending to zero; that means, that as x approaches to a f x will approach to L because as epsilon tend 0 this will tends to L and as delta tend to 0 this will tends to a. So, x extend to a f x will tends to L so, we can say that f of all those a minus delta to a plus delta this all those x lying in this interval will be contained in L minus epsilon to L plus epsilon so; that means, for every epsilon greater than 0 there will exist a deleted neighbourhood of x at a, such that the image of all those x lying in this interval will be contained in totally inside this penned totally inside this interval ok so, this is how we can defined limit of single variable functions.

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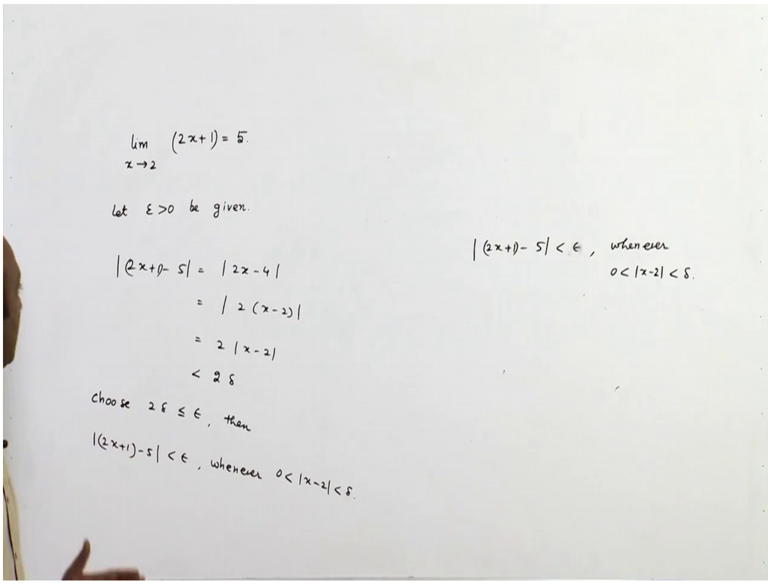

Now, let us first discuss few examples based on this for example, limit extending to 2, 2 x plus 4 plus 1 is equals to 5. This is very simple you one can easily see that limit of this simple function 2 x plus 1, is x into 2 is 5 how can we prove this using delta epsilon definition.

So, let us see how can we prove so, let epsilon greater than 0 be given you take any epsilon greater than 0 you take some epsilon greater than 0. The same process will repeat for any greater than zero so, we can say that for epsilon greater than 0 there will exist some delta. So, basically what we have to show, we have to show that for you take any epsilon greater than 0 there will always exist corresponding delta such that definition holds. What we have to prove basically that for every epsilon greater than 0 there will exist corresponding delta greater than 0 such that mod of 2 x plus 1 minus 5, this is f x minus L less than epsilon yeah less than epsilon whenever 0 less than mod x minus 1 less than delta.

So, basically x minus 2 less than delta so, basically we have to show the correspondence of delta in terms of epsilon. We have to show the existence of delta so, how can we do that. Now, we start with this inequality this is mod 2 x plus 1 minus 5, this is equals to mod 2 x minus 4, this is equals to mod 2 x minus 2, this is equals to 2 mod x minus 2 and this is equals to if you take if this is less than delta. So, this is less than delta. Now this quantity; this is less than 2 delta because this quantity is less than delta. Now, if we choose 2 delta less than equals to epsilon, then mod of 2 x plus 1 minus 5 less than epsilon whenever 0 less than mod x minus 2 less than delta.

Now, this is because wherever you choose whenever you choose delta less than equal to epsilon by 2 then this inequality always holds because this is less than 2 delta which is less than equal to epsilon so; that means, this is less than this is less than epsilon whenever this is less than delta. So, this inequality always holds whenever delta is less than equal to epsilon by 2 you choose any delta greater than 0 you can always you can choose any epsilon greater than 0, you can always find delta which is less than equal to epsilon by 2 for which this inequality holds.

So, we have shown the existence of delta in terms of epsilon for different delta for different epsilon delta will be different, but we have shown the existence of delta in terms of epsilon such that this inequality holds. Hence, we can show we can say that this limit exists and is equal to 5.

So, this is how using delta epsilon definition we can show that the existence of a limit of a function. Now, same concept we extend for 2 variable functions, now how can you do that let us see, now we say that for 2 variable functions we say that f approaches the limit L as x y approaches to x naught y naught.

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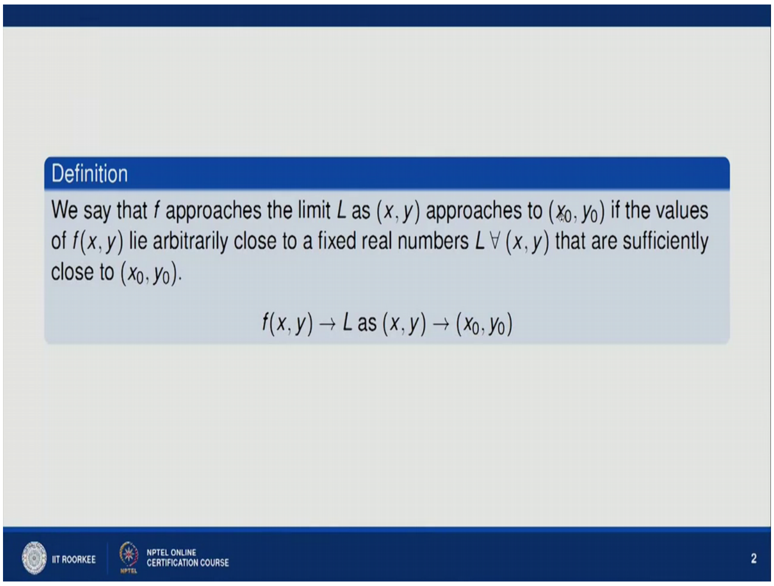

If the values of x y values of f x y lie arbitrarily close to a fixed real numbers L for every x y data sufficiently close to x naught y naught. You take you take any x y that is sufficiently close to x naught y naught, f x will arbitrarily close to l; that means, f x will approach to L as x y is approaches to x naught y naught, x y is arbitrarily close to x naught y naught.

How can you define this in terms of delta epsilon let us see, now here we are having two variable functions instead of a single variable function.

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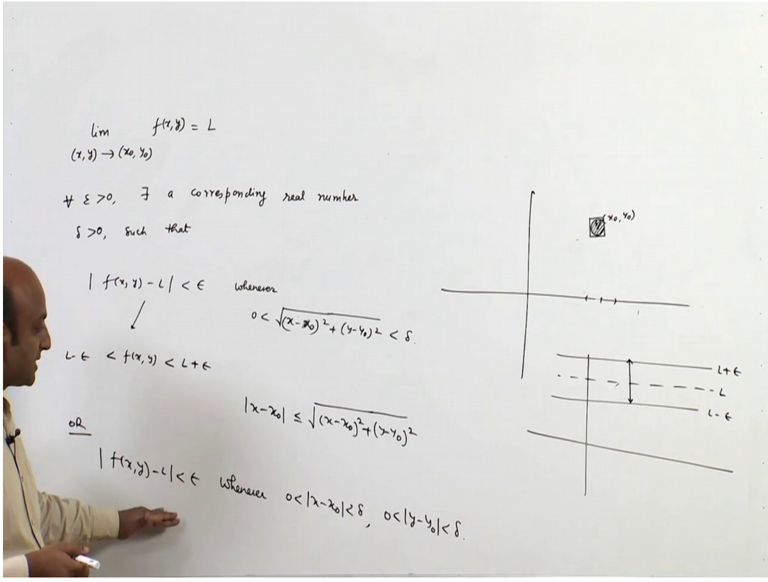

Now, a single in single variable function in single variable functions say we have x equal to a so, when we take disk at this point at x equal to a. So, this will be interval it is a minus delta to a plus delta this is an interval. Now, when we take point in two variable a function of two variable say here x naught y naught now instead of a interval it will be a disk centred at this point, it will be a disk centred at x naught y naught. So, how can I define limit now, limit x y tending to x naught y naught f x y is equals to L. So, how can we prove whether limit exists and is equal to L.

So, for that again we will repeat the same definition for a two variable functions for all epsilon greater than 0, there exist corresponding the real number delta greater than 0 such that mod f x minus f x y minus L less than epsilon whenever 0 less than under root x minus x naught whole square plus y minus y naught whole square less than delta. So, you take you take any epsilon greater than 0 there will always exist a corresponding real number data greater than 0 such that this inequality hold whenever this is less than delta or greater than zero. So, this is a deleted neighbourhood of x naught y naught of radius delta so, this is a basically a circle radius delta.

Now, again what does it mean it means that now this inequality means that f x y is less than L plus epsilon to L minus epsilon and this means all x y data lies inside the region inside a disk of radius delta and centre x naught y naught; that means, this region centre x naught y naught and radius delta this region and this means this is L minus epsilon and this is L plus epsilon and this is some L.

Now, you choose any epsilon greater than 0 there will always exist a corresponding delta such that all those x y which lie in this disk, now image of all those x y is lying in this disk will lies totally in this band which will lie totally inside this band. No matter how small epsilon or how large epsilon you are taking they will always exist corresponding delta such that the image of all those x y lying in this disk will totally contain in this band.

So, as epsilon tending to 0 this will tends to L and this will tends to x naught y naught so, this is how we can defined we can defined limit for two variable functions. Now, since mod of x minus x naught is always less than equals to under root x minus x naught whole square plus y minus y naught whole square because this is always true.

So, this definition can also be written as mod f x y minus L less than epsilon whenever 0 less than mod x minus x naught less than delta and 0 less than mod y minus y naught less than delta; that means, instead of disk it may be a rectangle instead this it may be a rectangle, then also we can apply the same definition. Now, let us discuss few examples based on this so, that concept of limit will be more clear.

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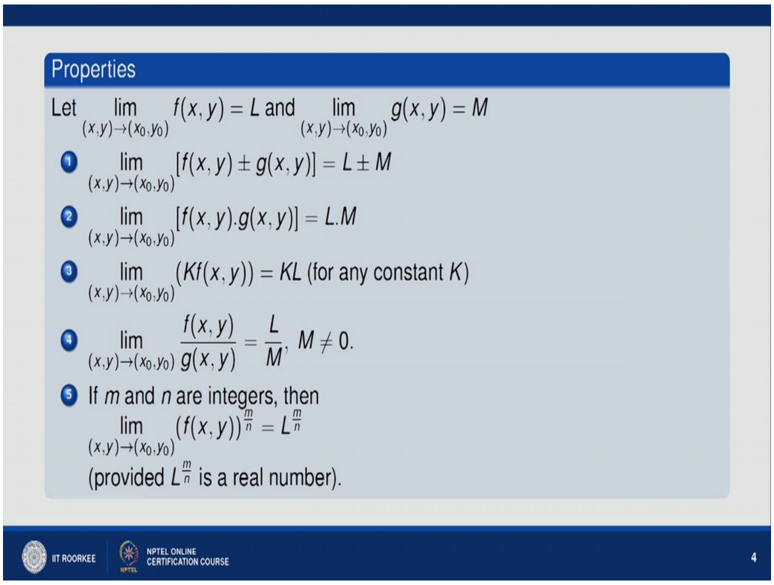

First we have some properties of limit, now limit suppose limit x y naught interest naught y naught of x y is L and limit x naught into x naught y naught g x y is M, then the addition of addition and subtraction of f x y and g x y limit x y tend to x naught y naught is L plus minus M and similarly we have the product of two functions then the limit will also be product. Then scalar multiplication will be K into L division by up f upon g will be simply L upon M. M should not equal to 0 and similarly we have the next property these are very straightforward.

Now, come to problems based on delta epsilon definition using delta epsilon definition show that this is equal to this.

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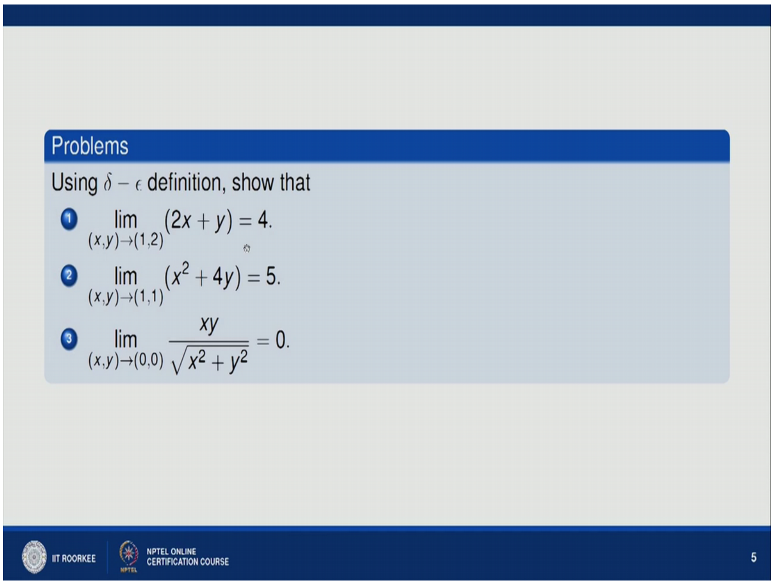

The first problem it is limit x y tending to 1 comma 2, 2 x plus y is equal to 4.

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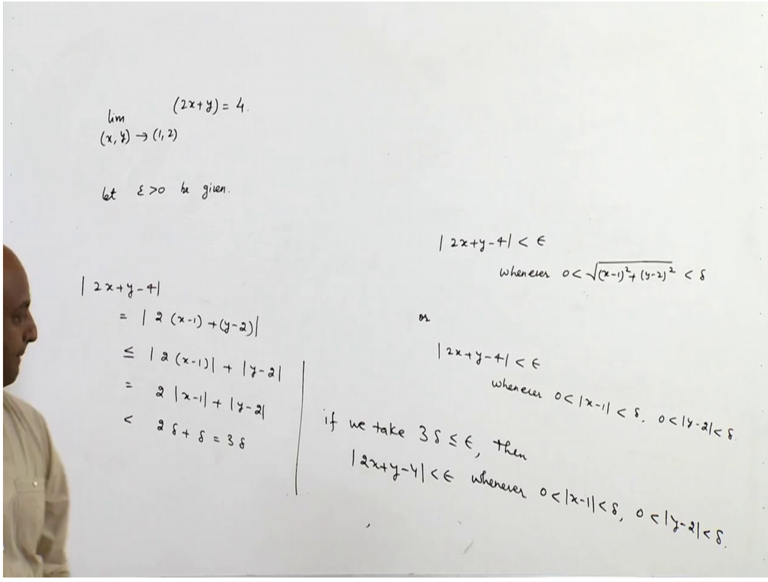

So, how can we how can you prove it where other way it is very simple you see when you substitute x equal to 1 and y equal to 2, the value is 2 plus 2 which is 4. Now, if somebody asked how can we prove mathematically that this limit exists and is equal to 4. So, we have the only option is delta epsilon definition we can use delta epsilon definition to show that this limit exists and equal to 4, how can you proceed for that we let epsilon greater than 0 be given.

Now, again we have to show the correspondence of a delta corresponding delta greater than 0 such that we have to show that there exists a corresponding delta greater than 0, such that mod 2 x plus y less than 4 less than epsilon whenever 0 less than under root x minus 1 whole square plus y minus 2 whole square less than delta or mod 2 x plus y minus 4 less than epsilon, whenever 0 less than mod x minus 1 less than delta and 0 less than mod y minus 2 less than delta.

So, we can use any definition either this or this to prove this result now you take mod 2 x plus y minus 4 this is equals to mod of 2 into x minus 1 plus y minus 2. You can easily see that it is 2 x plus y which is 2 x plus y minus 2 minus 2 is minus 4. Now, this is less than or equals to mod of 2 x minus 2 x minus 1 plus mod of y minus 2 because mod of a plus b is less than equals to mod a plus mod b.

Now, this is equal to 2 times mod x minus 1 plus y minus 2 now mod x minus 1 so, it will be better if you apply this definition because this is direct for this particular problem.

Now, mod x minus 1 is less than delta if you take so, this is less than 2 delta and this is again delta which is 3 delta. So, if we choose or if you take 3 delta less than equal to epsilon then mod of 2 x plus 5 minus 4 less than epsilon whenever 0 less than mod x minus 1 less than delta and 0 less than mod y minus 2 less than delta. If we choose 3 delta less than equal to epsilon, then this quantity will be less than epsilon and whenever this is less than delta and this is less than delta.

So, this inequality holds if this delta is less than equal to epsilon by 3 so, for any epsilon we have shown the corresponding delta existence of correspond delta such that this inequality holds hence we can say that this limit exists and is equal to 4.

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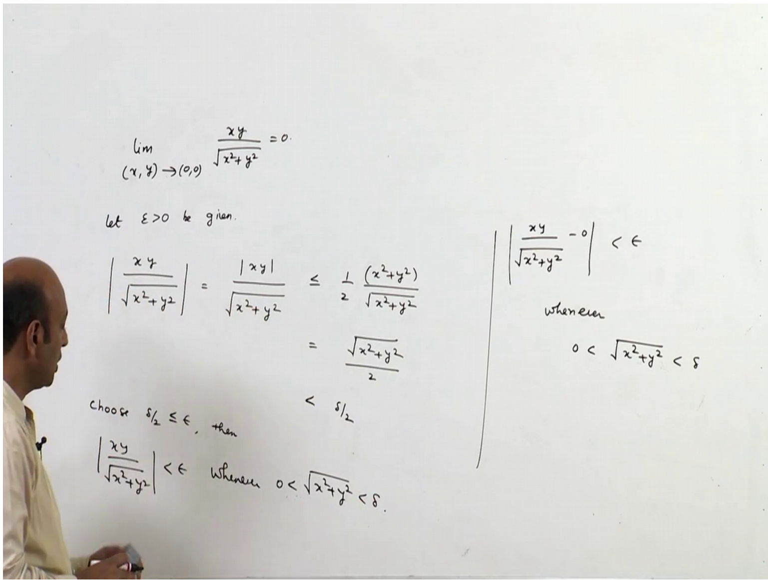

So, the next example now limit now let us discuss this is simple now here it is basically a 0 by 0 form so, it is very difficult to say whether this limit exists or not. Again if this limit exists so, we have to show this by delta epsilon definition; that means, we have to show the existence of delta corresponding to every epsilon greater than 0. So, how can you proceed we have supposes limit exist and we have to show the existence of this limit

So, let epsilon greater than 0 be given now you take x y upon under root x square plus y square so, this is equals to mod of x y upon under root x square plus y square. Now, we know that we know that x minus y whole square is always greater than equal to 0 so, x square plus y square minus 2 x y is also greater than equal 0. So, x y will always be less than or equals to x square plus y square by 2 so, mod of this quantity will also be less than equals to x square plus y square by 0.

So, this is less than or equals to 1 by 2 x square plus y square upon under root x square by y square so, this is equals to under root x square plus y square upon 2. Now, what we have to show here we have to show that x y upon under root x square minus y square plus y square minus 0 will be less than epsilon whenever 0 less than under root x square plus y square less than delta.

So, this quantity will be less than delta by 2 so, if we choose delta by 2 less than equal to epsilon then mod of x y upon under root x square plus y square will be less than epsilon whenever 0 less than under root x square plus y square less than delta. So, if you take any delta satisfying this inequality so, this inequality will be satisfied. So, we have shown that this term to delta corresponding to any epsilon such that this inequality holds. So, hence we can say that this limit exists and is equal to 0 so, in this way we can prove that using delta epsilon definition that that the limit of two or more than two variable function exists and is equal to L. Now, we will see some more properties of limit of several variable functions in the next lecture.

Thank you very much.