Multivariable Calculus
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Lecture – 19
Polar-curves

Hello friends welcome to lecture series on Multivariable Calculus. So, today our topic is polar curves, what polar curves are and how can we trace it. Actually this is required in double or triple integrals, when we do; when we convert a Cartesian coordinate to polar coordinate system, there the limits of rn theta is required and for that for that knowledge of polar curves is must.

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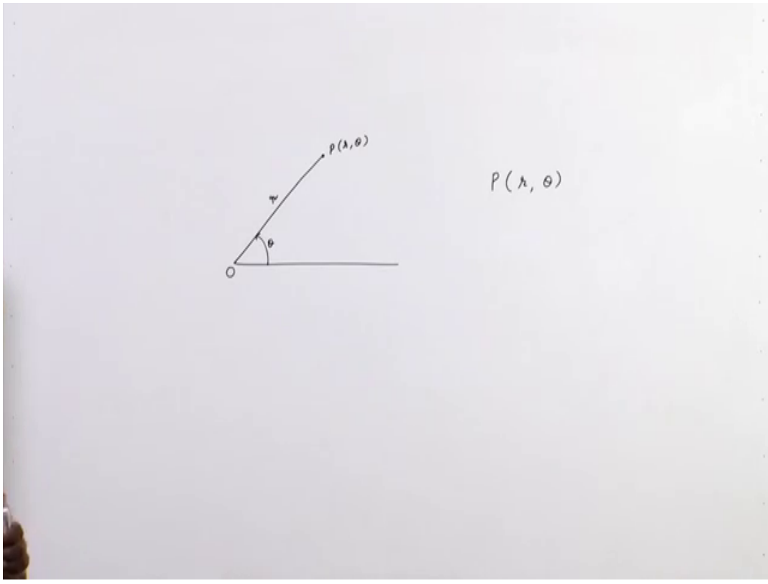

So, what polar curve is let us see you see in polar curve we denote a point by r comma theta in Cartesian curve we take it x comma y. In polar curve we take it as r comma theta what it represent you first fix a point O, which we call as pole then take the initial line this line we call as a initial line.

And r is basically if this point is say r comma theta then this length OP this length OP is basically r and theta is angle. Which this line segment OP makes with initial line this is theta. So, basically r is a distance OP this length OP and theta is angle which this OP makes with the initial line. So, this is basically r comma theta.

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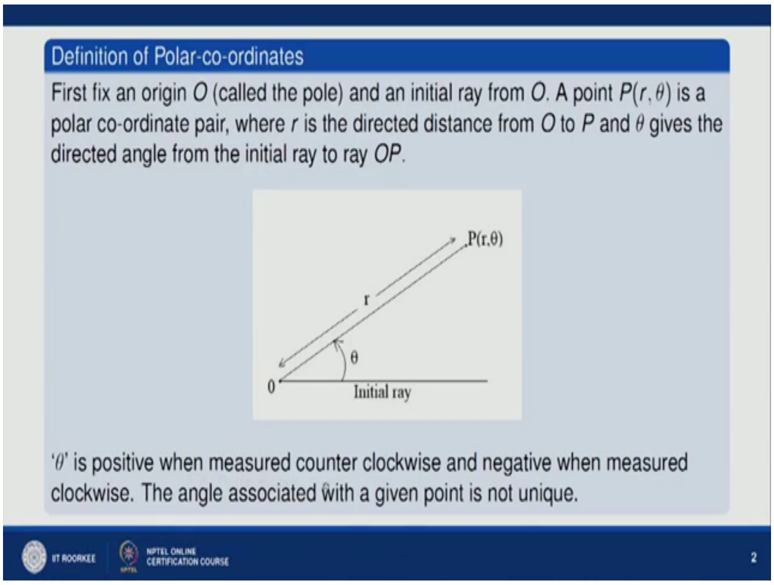

So, theta is positive when measured counter clockwise, and we take it as negative when measured clockwise, the angle associated with a given point is not unique. So, what does it mean we will discuss it afterwards, now r may take negative values also how it can take negative values.

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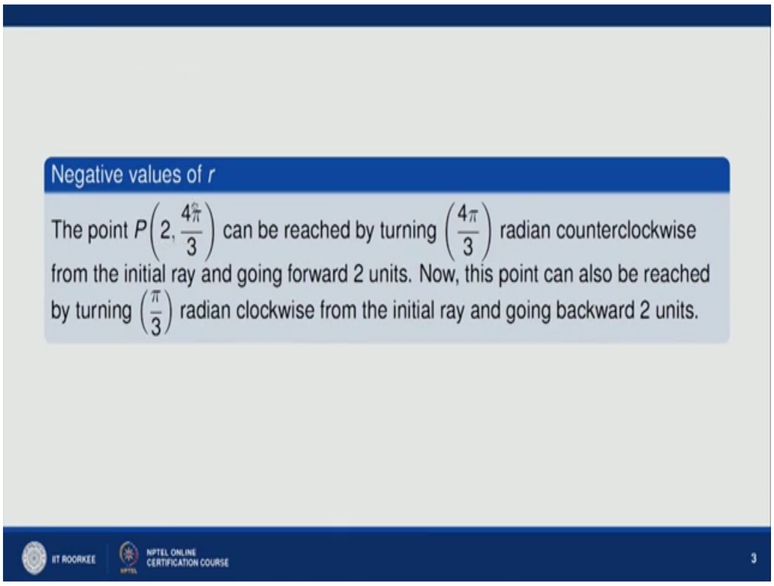

For example, you take a point P which is 2 comma 4 pi by 3, now what this point is now this is pole.

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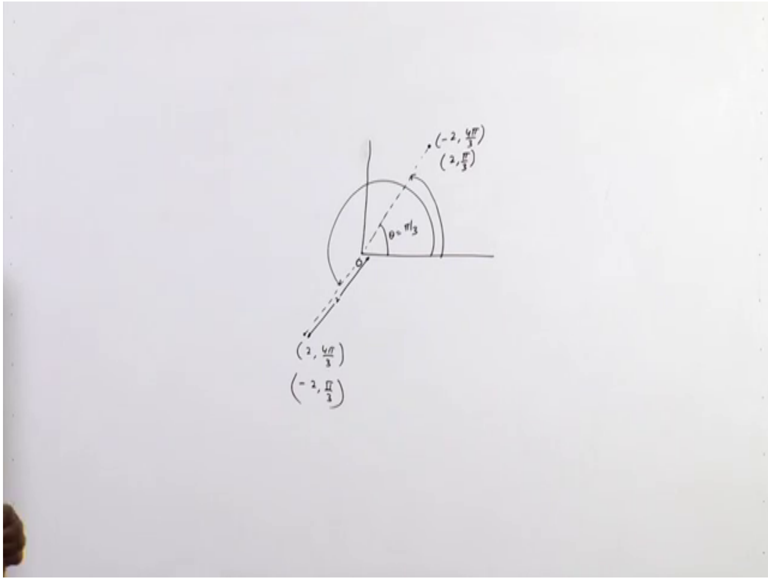

This is initial line, now you have to move this is this is theta equal to pi by 2. Now, you have to move now theta is 4 pi by 3 that is pi plus pi by 3 this is this is pi by 3 60 degree, this is pi by 3 ok, this theta is pi by 3 and pi plus pi by 3 somewhat here this is pi plus pi by 3.

So, say this point is 2 comma 4 pi by 3 theta is positive. So, have to we have to move anti clockwise direction from the initial line. So, this point is 2 comma 4 pi by 3 now if we take the representation of this point on this ray that is on the negative side of r, this length is 2. Now if you take this length on this side, then we may take this point as minus 2 comma 4 pi by 3. I mean on the other side of the ray if we take the same length 2 then this may be represented as minus 2 comma 4 pi by 3 this may also be represented it as this is pi by 3.

So, this may be also be represented as 2 comma pi by 3 or this point may also be represent as minus 2 comma pi by 3, because this length is 2 and theta is pi by 3 if you take the mirror image of this point from about the origin then this will be minus 2 comma pi by 3. So, that is how we can take negative values of all r. So, this is the, negative values of r.

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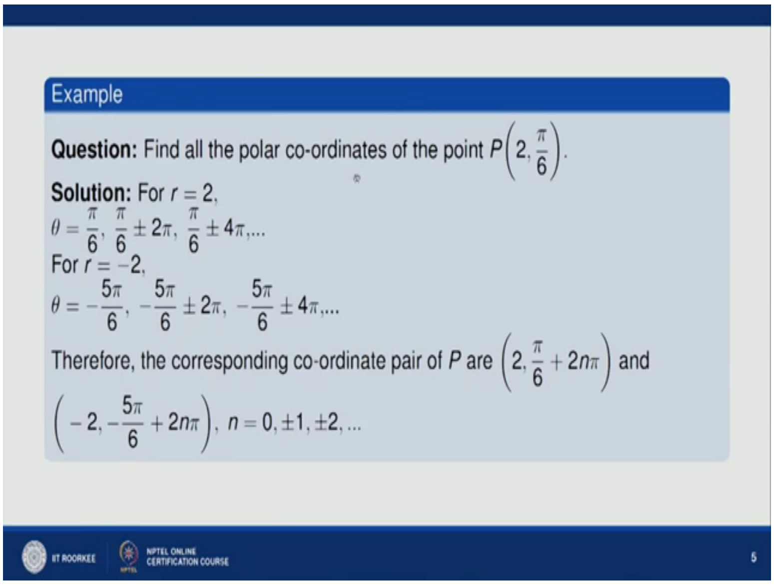

Now let us find all the polar coordinates of the point P 2 comma pi by 6 let us discuss this thing. So, what is 2 and 2 comma pi by 6.

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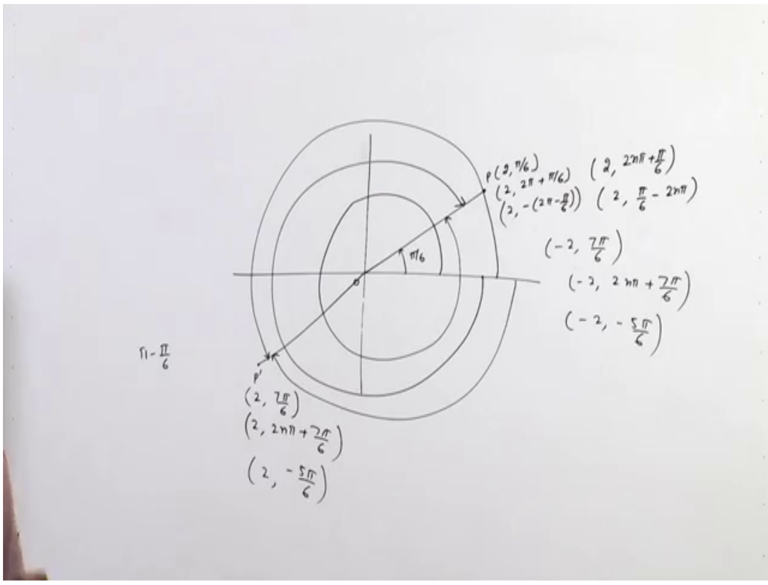

You see this is origin this is the initial line. So, it is 30 degree say at this point is P which is 2 comma pi by 6. So, this theta is anti clockwise direction theta is pi by 6 and this length is and OP is 2. Now there are ways, ways to reach this point see if we move, if we start, from this ray which is the initial ray and comes outer after revolve your entire circle come again to this point.

So, this will be 2 pi plus pi by 6, so this point may be represented as 2 comma 2 pi plus pi by 6. Now this point may also be reached when we take 2 circular round and comes again to this line. So, this will be 2 comma 4 pi plus pi by 6.

So, similarly if we take n number of rounds and come to the ray OP then this representation will be this representation will be 2 comma 2 n pi plus pi by 6, now this point may also be reached.

If we move anti clockwise direction I mean clockwise direction sorry, this is anti-clockwise. If we move start from a initial line initial ray and comes from this side to this side that is, that is clockwise direction then theta will be negative r will remain the same.

So, this will be 2 pi minus pi by 6 2 pi minus pi by 6. So, this is 2 comma minus 2 pi minus pi by 6 because it is anti-clockwise, I mean it is a clockwise direction. So, theta will be negative it is negative of 2 pi minus pi by 6. Similarly, if you take a full round and then come here to this to this line then it will be 2 comma pi by 6 minus 2 n pi.

Where n maybe 1, 2, 3, and so on, if you take n number of rounds now if we, if we take a point here same distance to say P dash which is 2 comma. Now we can move to this point start of the initial line anti clockwise direction to this here.

So, this would be pi plus pi by 6 that is 7 pi by 6. So, this will be 2 comma 7 pi by 6. Now representation of our this point here will be minus 2 comma 7 pi by 6, because if r here is 2 and if we move from the initial line to a backward direction.

That will be minus 2 angle will remain the same that is 7 pi by 6, now similarly if we if for this point we first complete 1 circle and then come again to this point. So, this will be we can say this will be 2 n pi plus 7 pi by 6 after moving n number of circles.

So, representation of this point will be minus 2 comma 2 n pi plus 7 pi by 6 now if we come to this point clockwise direction. So, this will be simply 2 comma this is pi minus pi by 6. So, pi minus pi by 6 will be 5 pi by 6.

So, it will be minus 5 pi by 6 and the representation of this point over here, will be at this point will be simply minus 2 comma minus 5 pi by 6. So, these are all representation of the same point P, the same point P when r is 2 and same point P when r is minus 2, when r is 2 it may be this or this when r is minus 2 it may be this or this.

So, in this way we can find out all the polar coordinate or the point 2 comma pi by 6. So, for r equal to 2 we have this representation when n may be 0 plus minus 1 plus minus 2 and for r equal to minus 2, we are having this representation where n is 0 plus minus 1 plus minus 2 and so on. Now let us graph the set of point whose polar coordinates satisfy the following condition, now the first problem is r is varying from 2 to 3.

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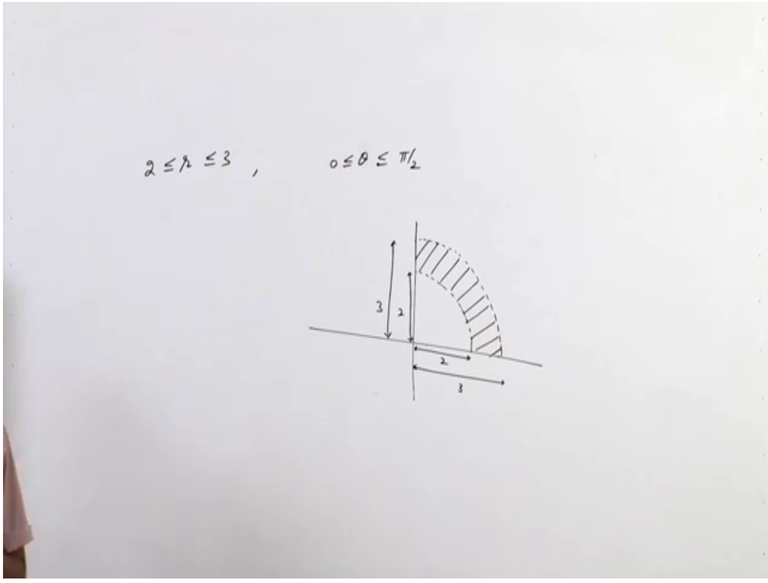

And theta is varying from 0 to pi by 2, now r is first take r equal to 2 what r equal to 2 represent you see r equal to 2 means r is always remain 2. So, it will be a circular arc, when r is always 2 theta may be anything.

So, it will be a circle r equal to 2 ok, theta may be anything similarly r equal to 3 will be a circle, theta may be anything. So, when you take r equal to 2, so r equal to 2 will be something like this is r theta may be anything you see you are taking theta from 0 to pi by 2. So, theta from theta is from 0 to pi by 2 that is only this portion.

This is r equal to 3 and for r equal to 4 it will be something for r equal to 2 sorry, this is 2 for r equal to 3 it will be this thing. Now r is varying from 2 to 3 and theta from 0 to pi by 2 only in the first coordinate, that is why I have plotted this portion in the first quadrant only where theta is varying from 0 to pi by 2.

Now r is varying from 0 to I mean 2 to 3 and theta from this to this. So, this is the region this is a graph of the portion which is covered under this region. So, this is the graph of the first problem now second problem is r less than equal to 0, r is less than equal to 0 and theta is pi by 4.

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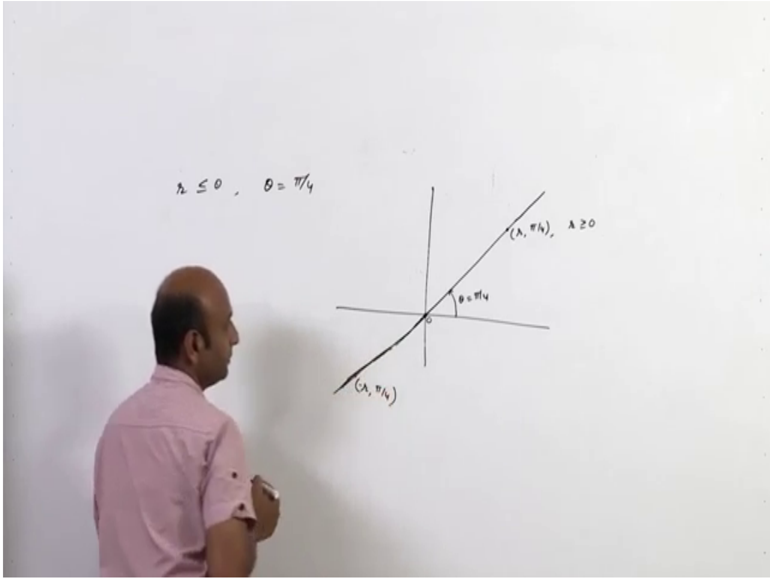

Now first plot, theta equal to pi by 4 when r is positive, and the image of that on the other side we will give r less than equal to 0.

So, theta equal to pi by 4 is this line and over here r is positive over this on this ray they here r is 1 here r is 2 here that is 3 here r is 4. So, on this ray r may be anything any number which is greater than equal to 0, here r is 0 and theta is 5 before.

So, on this ray r is positive and theta is pi by 4 now on the other side of this here on this side, on this side. We want r negative theta is equal to pi by 4 on this side r will be negative theta will be pi by 4 because suppose this point is r comma pi by 4 where r is greater or equal to 0 ok.

So, here the representation of this point will be minus r comma pi by 4. So, what, what is the graph of this the graph of this will be this line, this line segment a starting from this point O towards this side. So, this will be the graph, now how can we convert a polar into Cartesian or Cartesian into polar we can convert Cartesian to polar or polar into partition using this relation x equal to r cos theta y equal to r sin theta.

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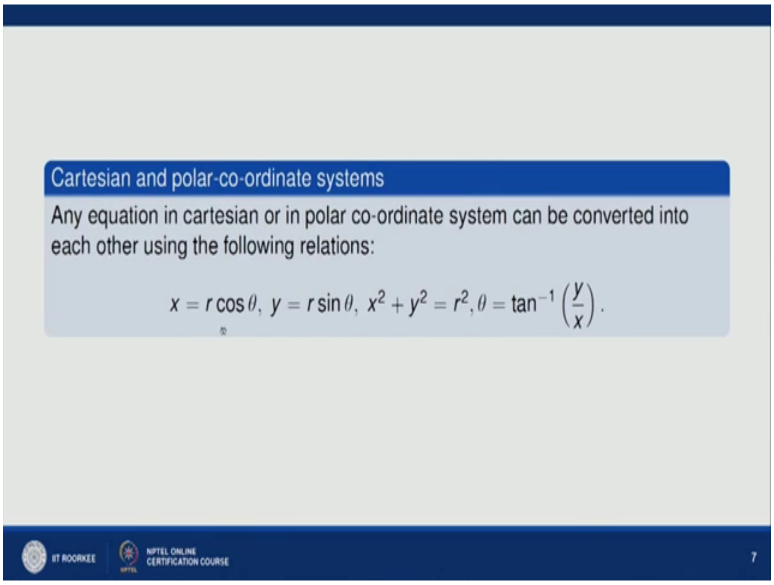

X square plus y square equals to r square or, and theta equal to tan inverse y by x,.

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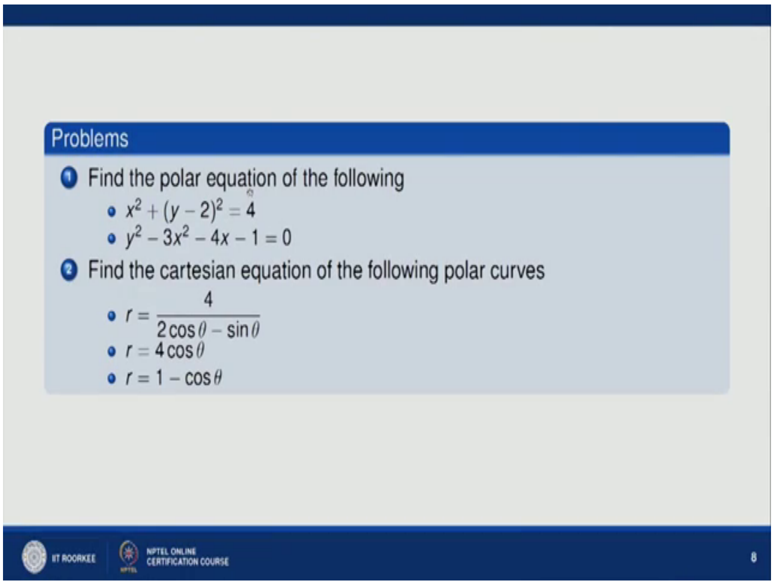

So there are some problems find the polar equation of the following say we have first equation.

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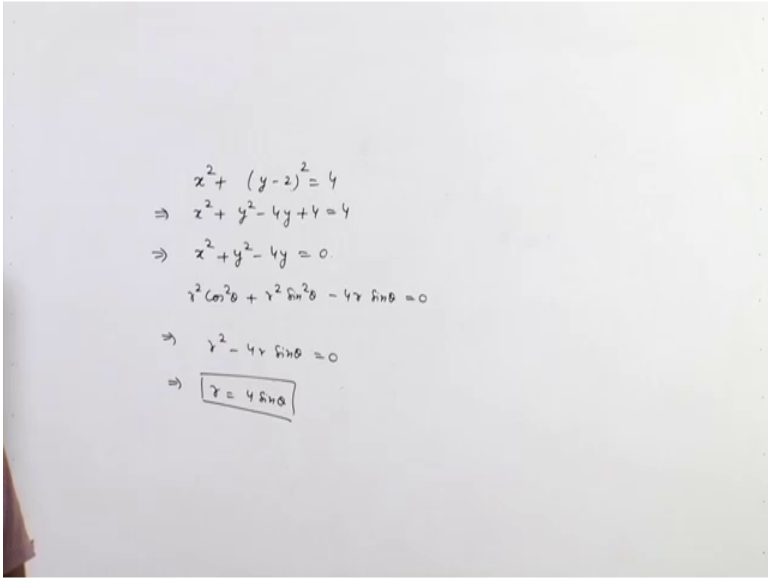

So, the first equation is x square plus y minus 2 whole square equal to 4. So, this is a Cartesian equation of a circle with center 0 comma 2 and radius 2 yes we can clearly see, that this represent a circle with center 0 2 and radius 2.

Now what is an equivalent equation of this in polar coordinate, how can you find that you simply simplify this is x square plus y square minus 4 y plus 4 equals to 4. So, this is equals to x square plus y square minus 4 y equal to 0.

Now for polar coordinate x is r cos theta and y is r sin theta you simply replace x by r cos theta and y by r sin theta, to find out the equivalent polar coordinate polar equation of this curve. So, it is r square cos square theta plus r square sin square theta, minus 4 r sin theta.

Which is equal to 0, so this implies r is square minus 4 r sin theta equal to 0 and this implies r is equal to 4 sin theta. So, this is an equivalent equation of this circle in polar coordinate system. We can cancel r because r cannot be 0, now for a second equation the second equation is the second problem is y square minus.

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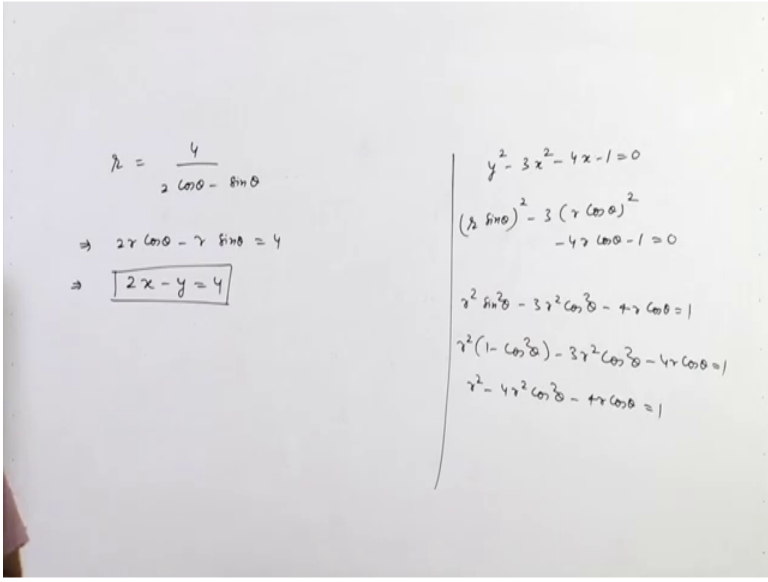

3 x square then minus 4 x minus 1 equal to 0. So, how can you find out the polar equation of this curve you simply replace x by r cos theta and y by r sin theta again.

So, it is r sin theta whole square minus 3 r cos theta whole square minus 4 r cos theta minus 1 equal to 0. So, this will be r square sin square theta minus 3 r square cos square theta minus 4 r cos theta is equal to 1. So, you can replace this sin square by 1 minus cos square for simplification. So, this will be equal to 1.

So, this will be r square minus 4 r square cos square theta minus 4 r cos theta is equals to 1. So, this will be the corresponding equation in polar system for this curve, now second problem find the Cartesian equation of following polar curves now the polar curve is given to us and we have to find out the equivalent.

Equation in Cartesian coordinates, so how can you find that the first problem is r is equals to 4 2 cos theta minus sin theta. So, it is this implies 2 r cos theta minus r sin theta is equal to 4, now r cos theta is x and r sin theta is y.

So, it is 2 x minus y equal to 4. So, it is the equation of a line in Cartesian now the second problem is r equal to 4 cos theta.

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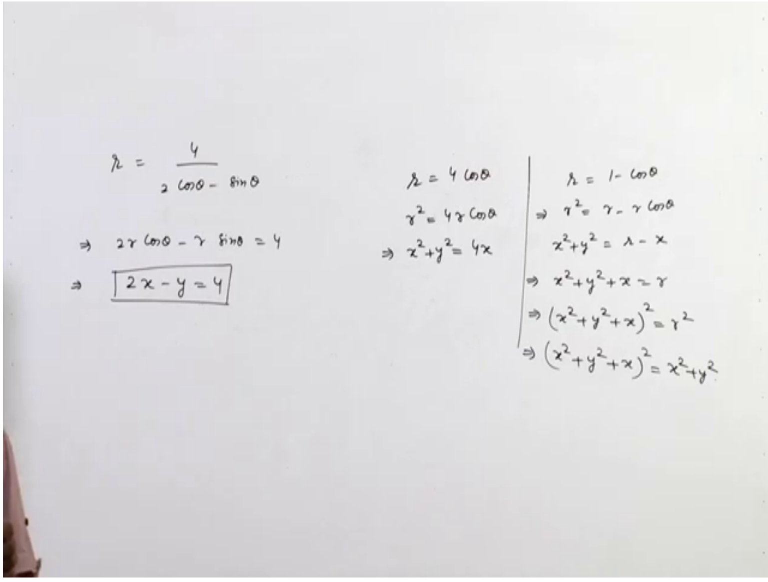

So r equal to 4 cos theta is r square equals to 4 r cos theta and this implies r square a of y square and r cos theta is x. So, this is the equation of a circle the next equation is r equals to 1 minus cos theta.

Now, you multiply both sides by r, so it is r square equals to r minus r cos theta this r square is s square plus y square is equals to r minus x. Now this implies x square plus y square plus x is equals to 1, and this implies x square plus y square plus x whole square is equals to r square and r square is x square plus y square.

So, this implies x square by y square plus x whole square is equals to x square by y square. So, that is how we have convert this polar equation into Cartesian equation. So, that is how we can convert Cartesian coordinate into polar or polar into Cartesian, now how can we graph of a polar coordinate if you want to plot a polar curve how can you do that.

How can you find the range of rn theta. So, we have some properties first is symmetry what do you mean by symmetry ?

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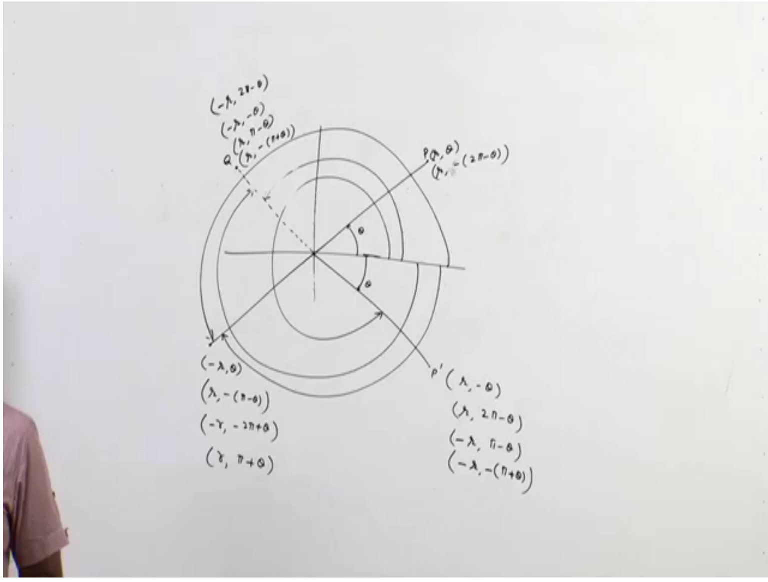

It is suppose this point is, r theta ok, let us suppose the polar curve which is given to was a symmetrical about x axis. If it is symmetrical about x axis this means about x axis if you take the mirror image of this point that will also satisfy the polar curve, because it is symmetrical about x axis.

So, let us suppose a symmetry symmetrical point of this will be P dash if it is theta then this will also be theta, because it a mirror image of this. So, this point can be written as r comma minus theta because we are moving clockwise direction,.

So, we can say that if in the polar curve given polar curve if we replace r by r if we replace r by r and theta by minus theta and the resulting equation will not be affected by this there is no change in the equation; that means, the polar curve the given polar curve is symmetrical about x axis or we can say or at this point you can come from here also.

So, this will be r comma 2 pi minus theta. So, we can also say that if we replace r by r and theta by 2 pi minus theta and there is no change in the equation; that means, symmetrical about x axis or we can say you can you can come here at this from this point also you see.

You see if you take a point here say q and then q will be here it is here it is pi minus theta because this is theta. So, this is r comma pi minus theta and the presentation of this point over here will be minus r comma pi minus theta.

So, we can also say if we replace r by minus r and theta by pi minus theta and there is naught the equation; that means, symmetrical out x axis or we can move at this point when we move from this side also now at from this point it will be r comma it is pi plus theta. So, it is minus pi plus theta and the representation of this point over here will be minus r comma minus pi plus theta.

So, you so if any 1 of the representations satisfy the given equation satisfy means if you replace if you replace r by r or minus r and there is no change the equation ah. Polar equation this means symmetrical about x axis, now if you want to see symmetry about y axis.

Now this is the point suppose this polar curve is symmetrical about y axis this means its representation along y axis that is this point must satisfy the polar curve; that means, if we replace r by r and theta by pi minus theta that no change; that means, symmetrical about y axis.

Similarly, here if you change r by r or theta by minus pi minus theta and there is no change the equation; that means, symmetrical about y axis and the representation of these 2 points representation of these 2 point over here it is minus r comma minus theta representation of this point over here will be minus r comma 2 pi minus theta.

if this if we replace r minus r and theta by 2 pi minus theta and no change; that means, the symmetrical about y axis. So, if any 1 of the 4 coordinate satisfy a given polar equation; that means, symmetrical about y axis, now symmetrical about origin about religion means this point.

So, that the presentation of this point is minus r comma theta the first representation and very obvious representation is minus r comma theta. If we replace r by minus r and theta by theta and no change; that means, symmetrical about origin, and representation of this point. Now you can come to this point from this side also and this is pi minus theta.

So, it is r comma minus pi minus theta because you are coming clockwise direction. So, theta will be negative, so if you replace r by r and theta by minus pi plus theta and there will be no change means symmetrical about origin. Similarly, you can find other representation of this point also you see if you if at this point you come from this side then it is 2 pi minus theta.

So, it is r comma 2 minus r and minus 2 pi minus theta and the representation at this point will be minus r comma minus 2 pi plus theta. So, if any 1 of the points satisfy give an equation; that means, symmetrical about origin. So, so these are the points so 1 more point is there for origin it is r comma pi plus theta when you come from this side when you come from this side it is pi plus theta. So, it is r comma pi plus theta. So, that is how we can check the symmetry.

So, before plotting any polar curve first check the symmetry say whether its symmetrical (Refer Time: 25:21) x axis or y axis or origin let me move to the slope of the polar curve, now what you mean by slope of a polar curve you have a curve r.

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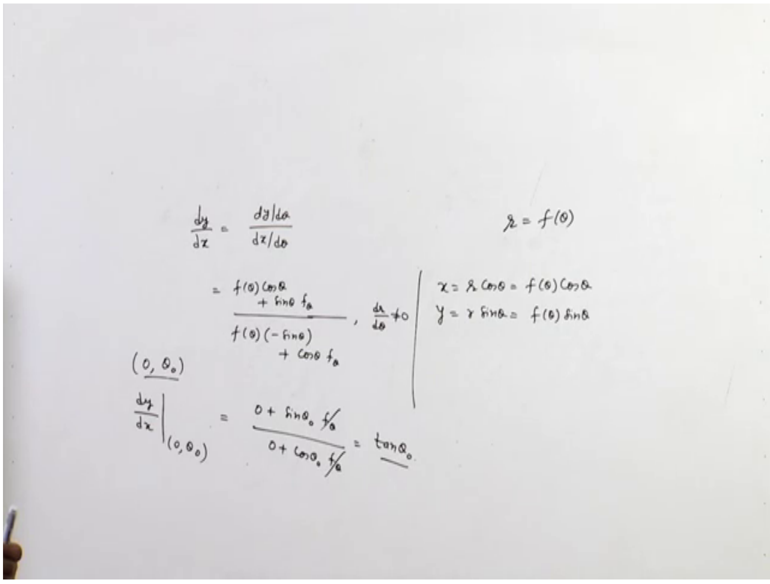

Equals to f theta now slope means dy by dx dy by d is nothing, but dy by d theta upon dx by d theta, now x is r cos theta r is f theta. So, it is f theta cos theta y is r sin theta and r is f theta. So, it is f theta sin theta, now what is dy by d theta dy by d theta will be.

First as it is derivative of second plus second as it is derivative first upon. Now d r upon d theta is again f theta into minus sin theta plus cos theta into f theta provided dr upon d theta is not equal to 0.

So, this is the slope of the polar curve at a point r comma theta, now suppose you are interested to find out the slope at 0 comma theta naught at origin I mean 0 comma theta naught.

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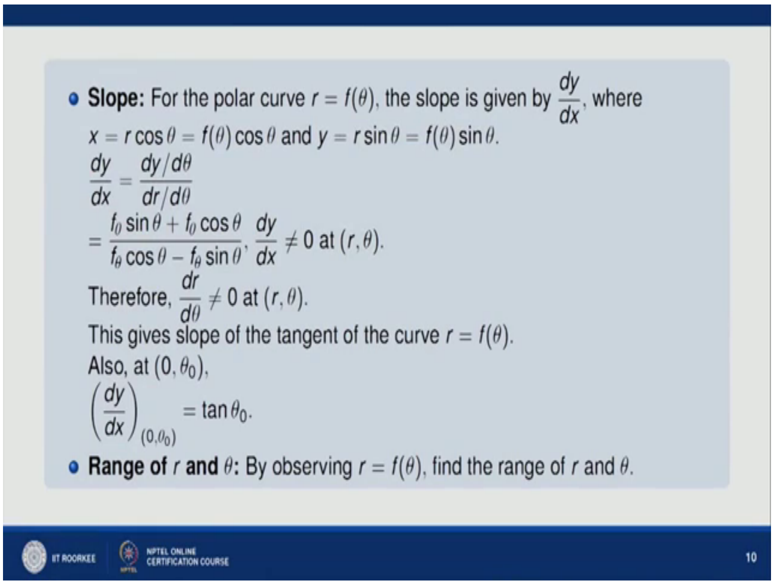

At 0 comma theta naught dy by dx will be, when r is 0 that is that is r 0 theta naught satisfy this curve. So, 0 and is equal to f theta naught. So, f data naught is 0, So, it is 0 it is 0 it is sin theta naught. So, it is equal to 0 plus sin theta naught into f theta upon it is again 0 plus cos theta naught into f theta it cancels out.

So, it is simply 10 theta naught. So, that will be the slope of a polar curve at 0 comma theta naught, now the last point is we find a range of rn theta by observing the polar curve r equal to f theta you find the maximum and minimum value of r by varying the values of theta this also we can observe now let us discuss this thing.

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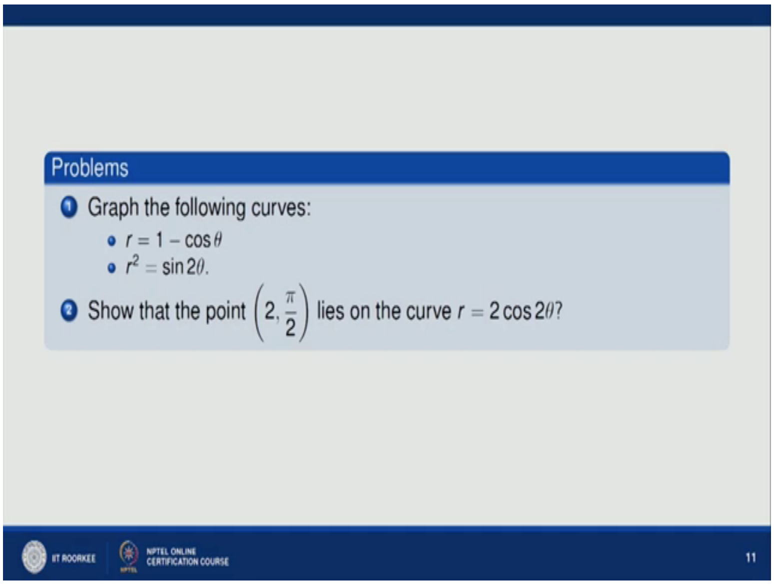

Suppose we want to graph r is equals to 1 minus cos theta.

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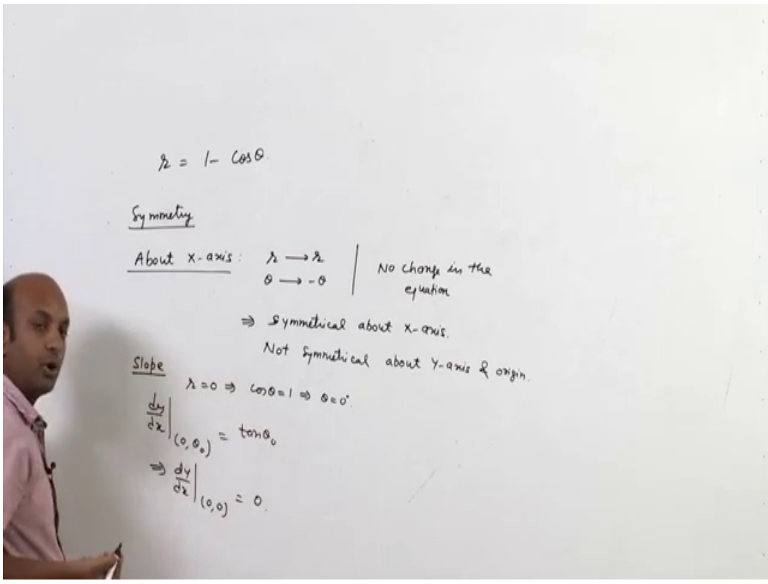

So first is r is equals to 1 minus cos theta. Now first we check for symmetry about x axis. So, you replace r by r and theta by minus theta, if you replace r by r and theta by a minus theta and cos minus theta is cos theta. So, there is no change in this equation.

So, no change, in the equation so; that means, this implies symmetrical about x axis symmetrical about x axis means you plot the curve only from 0 to pi on the on the upper side and since it is symmetrical about x axis.

So, you can take the take the mirror image of the curve which you have plotted on the above side of x axis because it is symmetrical about x axis, now is it symmetrical about y axis.

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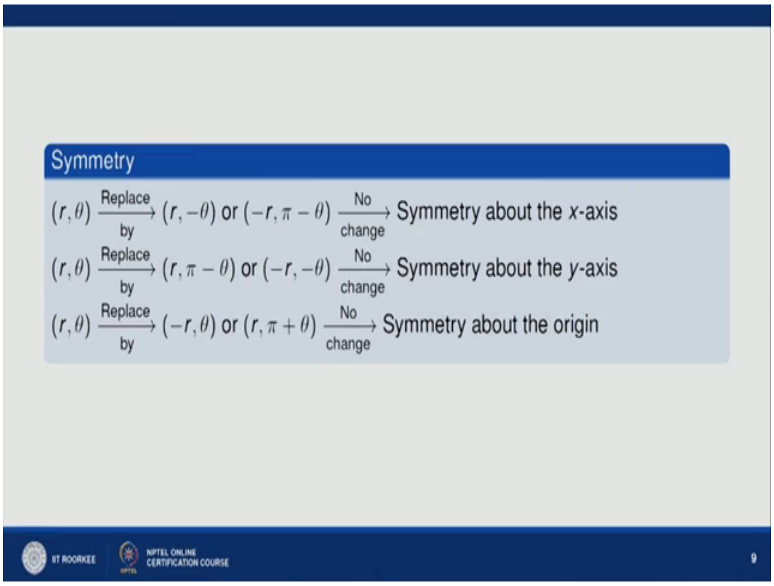

So, we can check for y axis, if you replace r by r and theta by pi minus theta, now when you replace theta by pi minus theta it is minus cos theta, so equation is changing.

So, this is not satisfying now second point is minus r minus theta if we replace r by minus r and theta 1 minus theta again equation is changing and the other 2 points will also not satisfy. So, we can say that this is not symmetrical about y axis or about origin.

So, not symmetrical about y axis and origin now the next is next is slope. Now when r is 0 cos theta is 1 r is 0 implies cos theta is equals to 1 this implies theta is equals to 0 so; that means, 0 is the point which lying on this polar curve and what your slope at 0 0, because we know that dy by dx at 0 comma theta naught is 10 theta naught.

And here theta naught is 0 because the point we satisfying this curve is 0 0. So, this will be this implies dy by dx at 0 0 will be 0 because theta naught is 0 so; that means, at origin at pole the curve is touching x axis because slope is 0 slope is 0; that means, it is touching x axis at origin.

Now you can find out the maximum and minimum value of r you see this cos theta may take the many maximum value is 1 and minimum value is minus 1. So, r will r will be less than equals to r will be greater than equals to 0 and less than equals to 2 the maximum value of r is 2 and the minimum value of r is 0, now having all these things in our mind let us try to plot this graph of this curve.

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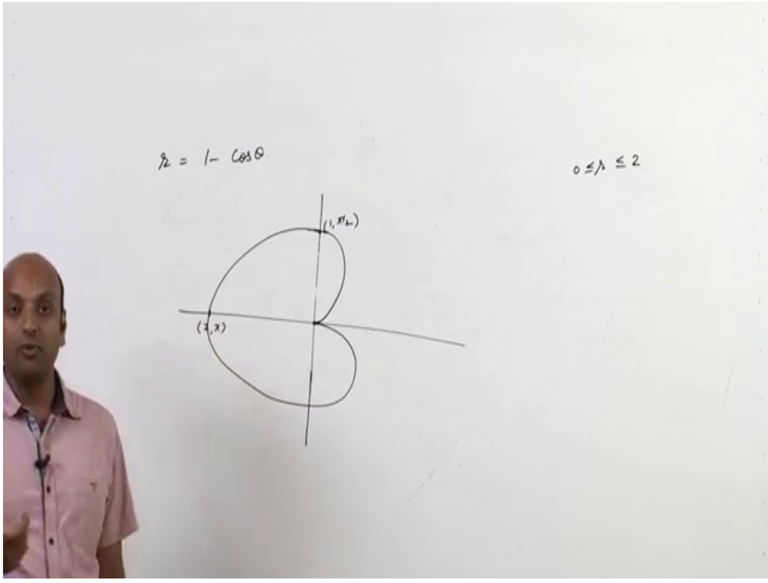

Now this is the origin this is pole at pole at pole curve is touching at pole carve is stretching x axis because slope is 0 slope is 0; that means, it is touching x axis. And at theta equal to this is theta equal to pi by 2 at theta equal to pi by 2 r is 1.

So, this is something suppose this is 1 comma pi by 2 and at r equals 2 pi theta equal to pi r is 2. So, suppose this is 2 comma pi because here theta is 0 and here it is pi now and it is also symmetrical out x axis. So, first we will plot the curve lying, on the upper side of x axis and then we will similarly plot the curve lying on the below side of the x axis.

Now here it is touching x axis and then it is touching this point and then it is coming to this point. So, this is a rough shape of this curve and by symmetry we can simply say that this is also touching this. So, this would be the rough shape of this polar curve is it.

So, r is r minimum is 0 and maximum r is 2 and theta equal to pi we can draw some more points when theta is say when theta I say pi minus pi by 3 or pi minus pi by 6. We can take some more points find out the values of r and then we can simply plot the curve.

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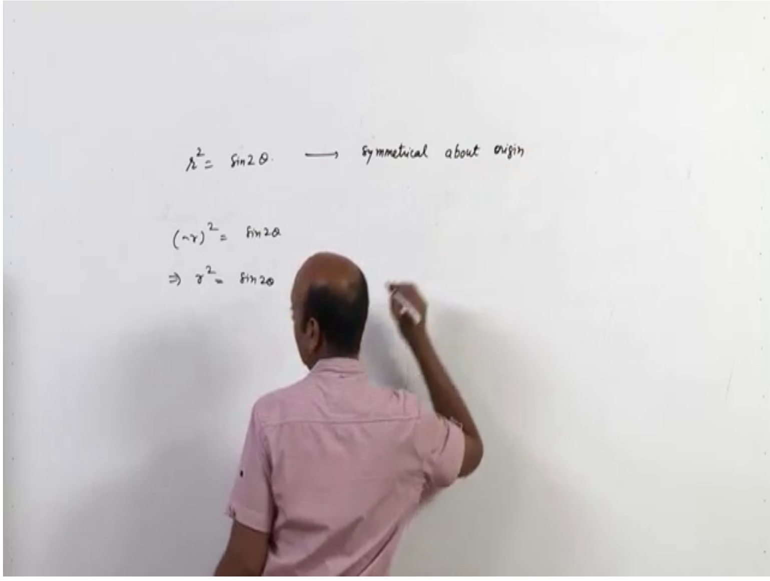

Now say we have a second curve r square equal to sin 2 theta, for sure we will check for the symmetry. Now when you take symmetry about origin you see when you replace r by minus r and theta by theta, when you replace r by minus r and theta remain theta.

So, this is simply r square equal to sin 2 theta; that means, there is no change in this equation. So, we can say that it is symmetrical about origin; however, when you take symmetry about x axis try to check symmetry about x axis will replace r by r theta by minus theta. So, there is a change in the equation because sin minus theta is minus sin theta, similarly if you replace r by minus rn theta by pi minus theta again the equations will change, so it is not symmetrical about x axis.

Similarly, we can check for y axis also. So, from here we conclude that it is symmetrical about origin, now next is symmetrical about the origin now next is slope when r 0.

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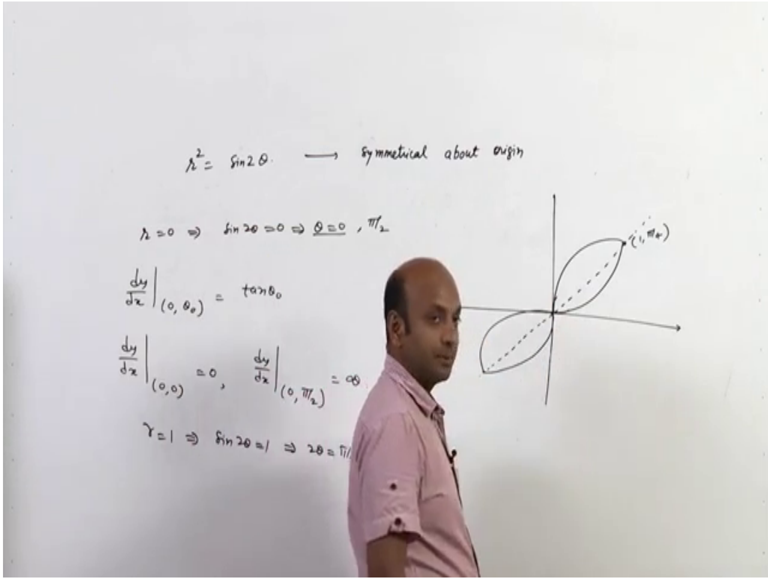

Implies sin 2 theta 0 so; that means, theta equal to 0 or theta equal to pi by 2 also, now slope at 0 comma theta naught is 10 theta naught.

So, we can say that dy by dx at 0 comma 0 is 0 and dy by dx at 0 comma pi by 2 is infinity. So, at this point at this point it is touching x axis and at this point it is touching y axis because slope is infinite and how can you send.

The maximum and minimum value of r the maximum value of sin 2 theta is 1. So, r will be 1 and minimum value is minus 1 when sin 2 theta is 1 again, now this is origin. So, when, when r is 1 r equals to 1 implies sin 2 theta is 1 at that means 2 theta is pi by 2 implies theta is pi be 4.

So, you plot theta equal to pi by 4 theta equal to pi by 4 r is 1 which is the maximum value to say say it is 1 comma pi by 4 at theta equal to at r equal to 0 theta is 0 or pi by 2. So, here if we are moving along this side it is only taking it is really touching this side now at this it is touching.

It is touching x axis at pole if you are coming from this side it is also touching y axis and then we have to come to this point. So, this point is here and this point is here and since it is a symmetrical origin. So, here also we have this thing or something like this.

So, they are rough shape of this polar curve. So, we can easily verify that this polar curve does not exist in this and this coordinate, because otherwise our u square will be negative and r square will be negative means no r no real r this we can easily verify. Now the next problem is show that the point 2 pi by 2 lies on this curve.

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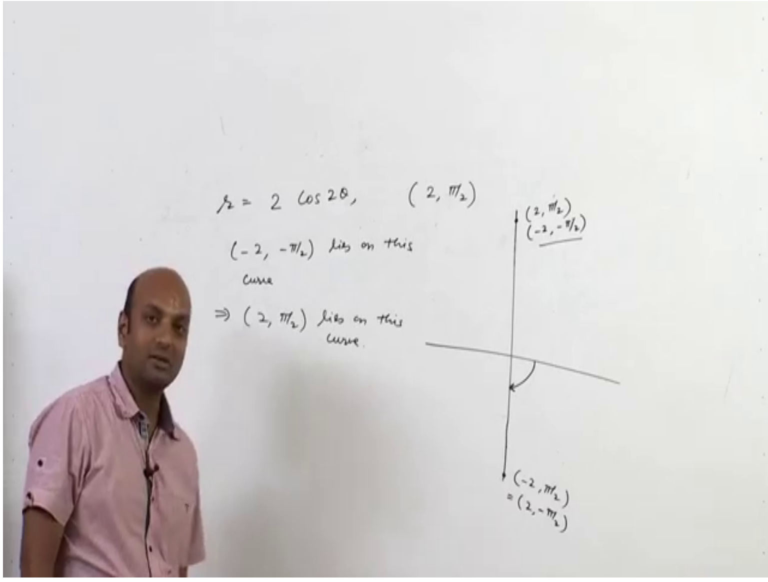

So, this is very simple problem this is r is equal to 2 cos theta and 2 cos 2 theta and point is 2 pi by 2, now when you substitute theta equals to pi by 2. So, it is cos phi and cos phi is minus 1.

So, r is minus 2, so from here you can you can simply say that this point does not lie on this curve because this point is not satisfying this equation; however, its another representation is satisfying this curve you see you have a point 2 pi by 2 this point the representation of this point on this side is minus 2 pi by 2 now if you come to this point from this side clockwise direction.

So, it will be, so it will be 2 comma minus pi by 2 and the representation of this point on this side will be minus 2 will be 2 comma minus pi by 2 oh sorry minus 2 comma minus pi by 2, I what I want to say that this point has infinite representations and if any 1 of the representations satisfy the given polar equation this means this also this point lies on this curve. It does not mean that this is this point lies another representation same point lies on the curve.

Now, this representation which is the representation of same point when you substitute theta equal to minus pi by 2 it is cos minus pi which is minus 1, and hence r is minus 2 which is r equal to minus 2.

Hence, this point minus 2 minus pi by 2 lies on this polar curve and this implies 2 comma pi by 2 lies on this curve, because this is 1 of the representation this is 1 of the representation of this point or this is 1 of the representation of this point. So, hence we can say that this point lies on this curve, so that is all about polar curves so.

Thank you.