Multivariable Calculus
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Lecture - 17
Taylor's theorem

Hello friends. Welcome to lecture series on Multivariable Calculus. So, today we deal with Taylor's theorem, what Taylor's theorem is, and how can we expend a two variable or more than two variable function in terms of the polynomial ok, how can I do that. So, let us see. So, let us start with a Taylor's theorem for a single variable function first.

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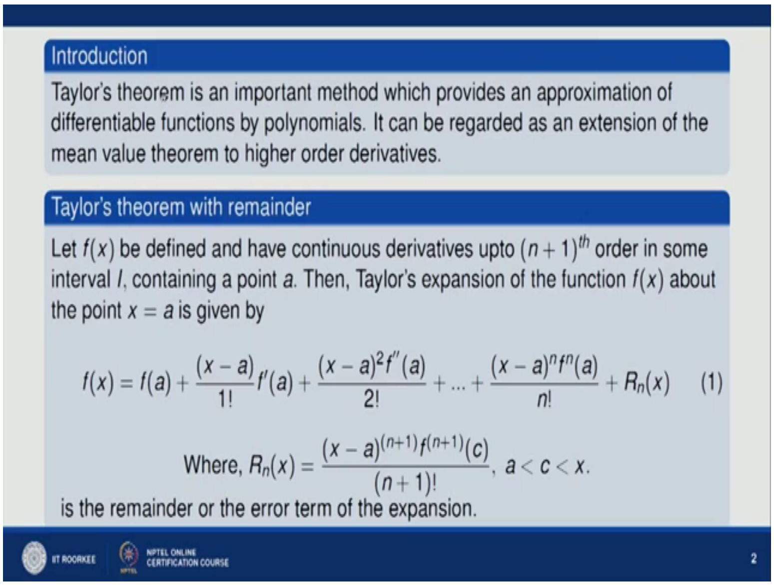

So, Taylor's theorem is an important method which provides an approximation of a differentiable functions by polynomials. So, if you have a differentiable function and you want to approximate it by polynomials we use Taylor's theorem. It can be regarded as an extension of mean value theorem to higher order derivatives. Now what Taylor's theorem with remainder state? It is states that let f x be defined and have continuous derivatives up to n plus 1th order in some interval I containing point a.

So, let us suppose you have a function f which is defined and have continuous derivative up to n plus 1th order in some interval say I which contain a point a, then Taylor's expansion of the function f about a point x equal to a is given by this expansion. So, what is this expansion?

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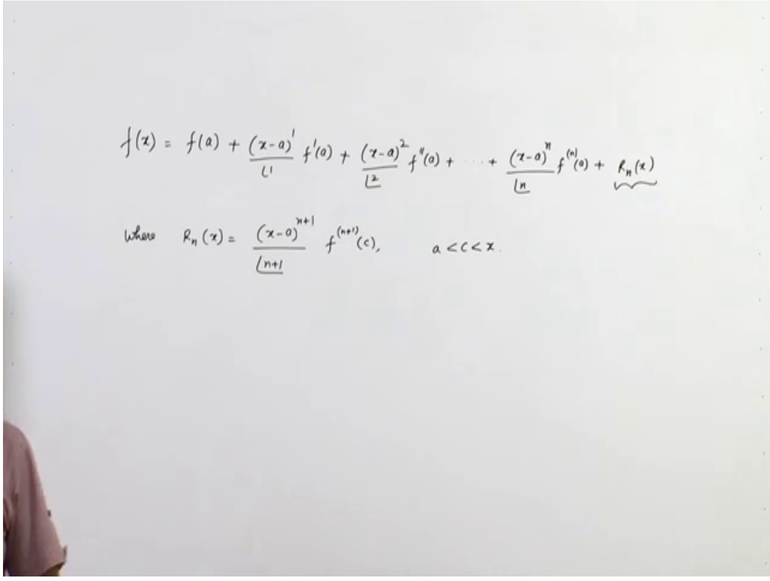

This expansion is f x is equals to f a plus x minus a raised to power 1 upon factorial 1 f dash at a plus x minus a whole square upon factorial 2 double derivative at a and so on plus x minus a whole dash to power n upon factorial n nf derivative at a plus R n x. Now this R n x is called remainder term or error term. How we defined R n x? So, where R n x is given as x minus a whole raise to power n plus 1 upon factorial n plus 1 plus n-th n plus 1th derivative at some c where this c is between a to x; is between a to x.

So, this term plus remainder term we call as Taylor theorem with remainder. Now how can we prove it? What is the proof of this theorem? So, let us discuss the proof. Now for the proof of this theorem we first find a polynomial say P n x.

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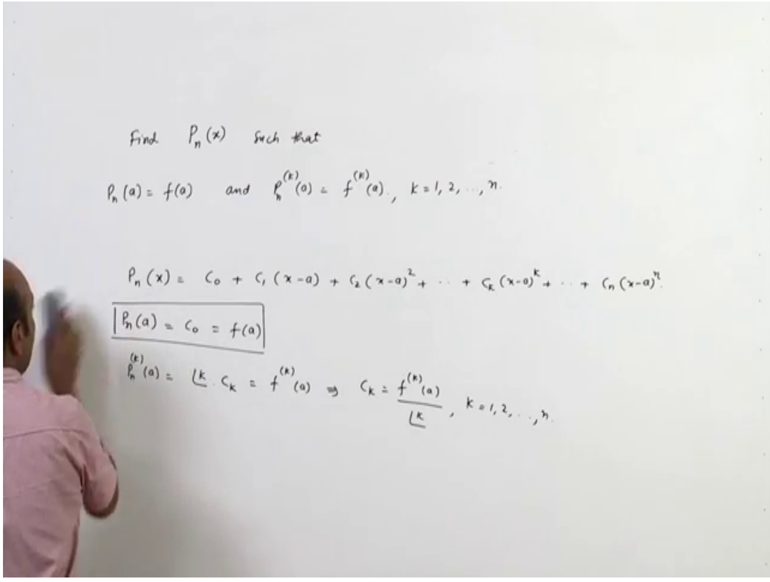

We first find a polynomial say P n x such that find polynomial P n x; P n x such that P n a is f a and P n of k-th derivative at x equal to a its k-th derivative of f at x equals to a.

So, you first try to find out a polynomial P n of degree n such that it satisfies these two properties, this is for all k; k from 1 to n. Now you first write any polynomial of degree n. Say P n x is c naught plus c 1 x minus a plus c 2 x minus a whole square and so on plus c k x minus a raise to power k plus and so on plus c n x minus a holders to power n.

Now what is P n a? P n a is simply c naught, because when you substitute x equal to a all these terms vanishes we are only left with c naught and c naught is f a. So, this is equals to f a this is the first term. Now when we take when we take a k-th derivative of P n x at x equals to a so what we will obtain; you see when you take k-th derivative of P n x all those terms in this polynomial whose degree is less than k will vanish, where left with this term and all other terms.

And when you substitute x equal to all the terms after this term will contain x minus a, so will be 0, where left is only this term. And what is the k-th derivative of this? K-th derivative of this will be factorial k into c k this you can easily check. You find out k-th derivative for k-th derivative all the terms which are up before this will vanish and after this will contain the powers of x minus a which will be vanished when you substitute x equal to a.

So, we will left with only this term and this is equals to as per our assumption this is equals to this thing. So, this implies c k will be equal to k-th derivative at x equals to a upon factorial k and k varying from one two up to n.

So, we can say that f x that P n x will be what.

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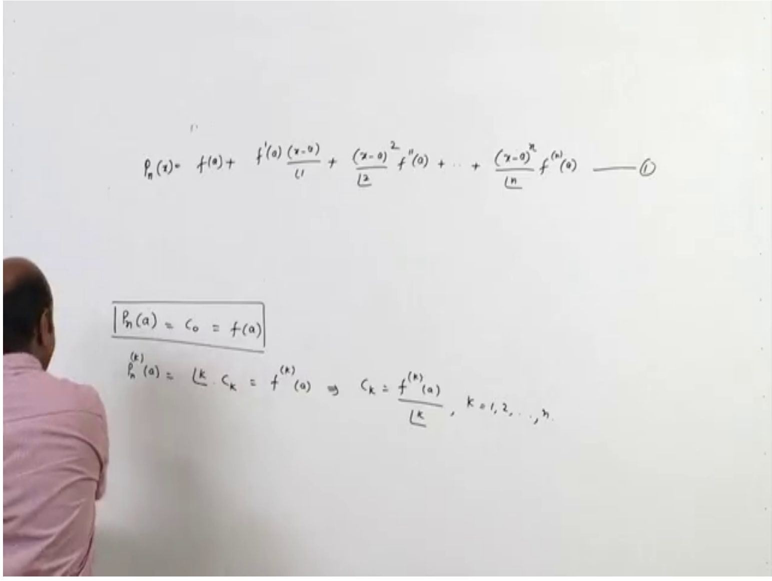

So, we can say that P n x will be f a plus, so c 1 will be f dash a into x minus a upon factorial 1 plus x minus a whole square upon factorial 2 f double dash a and so on plus x minus a whole raise to power n upon factorial n n-th derivative at x equal to a. So, this is first thing. Very substitute c equal to 1, we obtain this term, c equal to 2 this term this upon this, c equal to n sorry, k equal to n this upon this. Now, so this is the polynomial which satisfy those two properties. So, we have seen that R n x will be nothing, but f x minus P n x.

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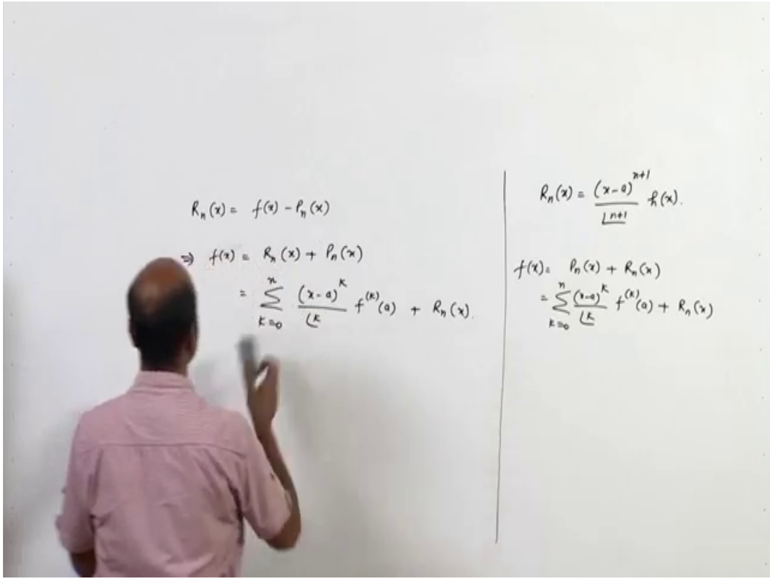

So, this implies f x will be R n x plus P n x and this P n x is nothing, but summation k from 0 to n it is x minus a whole raise to power k upon factorial k k-th derivative at x equals to a plus R n x now only thing to find. Now the expression of R n x the remainder term or the error term ok, then this will prove this theorem.

Now let us suppose the expression of R n x is x minus a whole raise to power n plus 1 upon factorial n plus 1 into h x. So, we have to find out the expression of h x. So, this will complete this theorem this will complete the proof. So, how we will how we will find the expression for h x, let us see. So, we have seen that f x is this term, f x is this is P n x plus R n x where P n x is summation k from 0 to n x minus a raise to power k upon factorial k into k-th derivative at x equals to a plus R n x. Now to find out the expression of h x, let us define capital F t.

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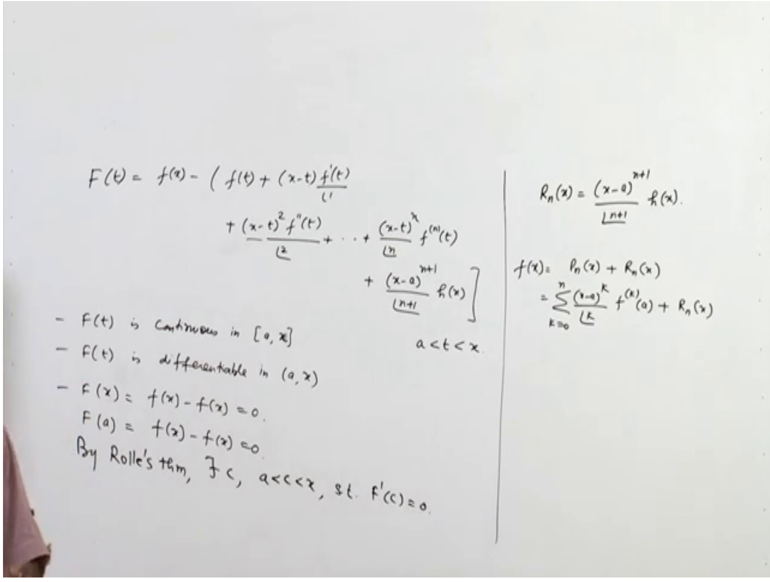

As f x minus ft plus x minus t into f dash t upon factorial 1 plus x minus t whole square f double dash t upon factorial 2 plus and so on x minus t whole raise to power n upon factorial n nth derivative at t and plus R n x. R n x is x minus a whole raise to power n plus 1 upon factorial n plus 1 into h x. So, that is how we defined we defined capital F t where t is between a to x.

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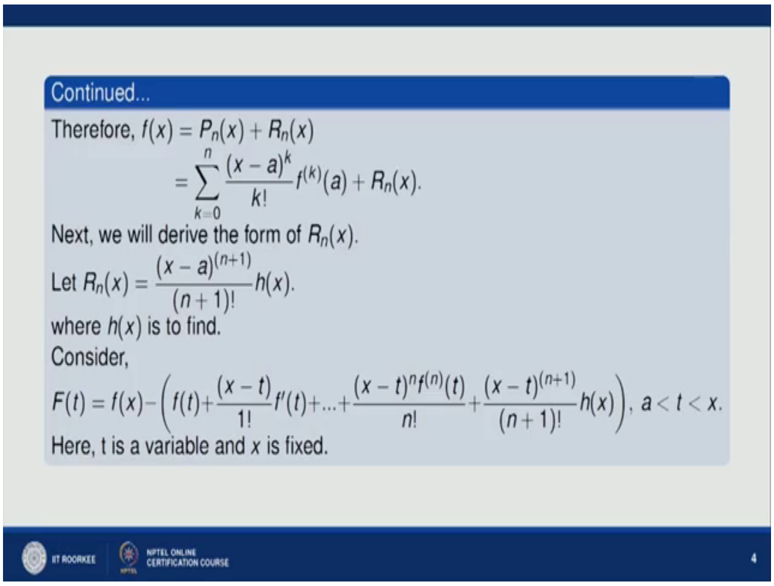

Now, here we are taking x as a constant and t is a variable. Now this function f is; obviously, continuous is; obviously, continuous this ft is continuous in close interval a to x. This F t is differentiable also because it is a polynomial in open interval a to x and also when you take say f of x f at t equal to x. So, you replace t by x when you replace t by x this will be 0 all terms will be 0 only this term left which is f x and f x minus f x will be 0 and when you take f at a. So, you simply replace t by a when you replace t by a. So, this term where as is nothing but f x. So, this will be f x minus f x again which is 0 when you replace t by a here. So, this expression from this is nothing, but f x. So, f x minus f x will be 0. So, all the 3 axioms or the properties of Rolle's Theorem are satisfied in the interval a to x. So, we can say by the Rolle's theorem that, there will exist some c in the open interval a to x such that f dash c will be 0, now when you take f dash of this term first respect to t and then replace t by c.

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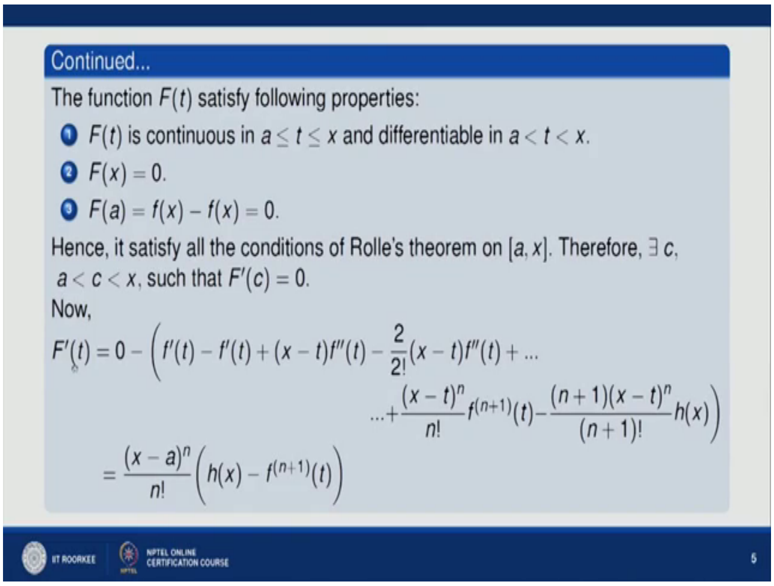

So, what you will obtain now from here when you take f dash t and put it equal to 0. So, we got this expression is equal to 0.

You simply you can simply check you simply find capital derivative of capital F respect to t put it equal to 0, we obtain that this is equal to 0.

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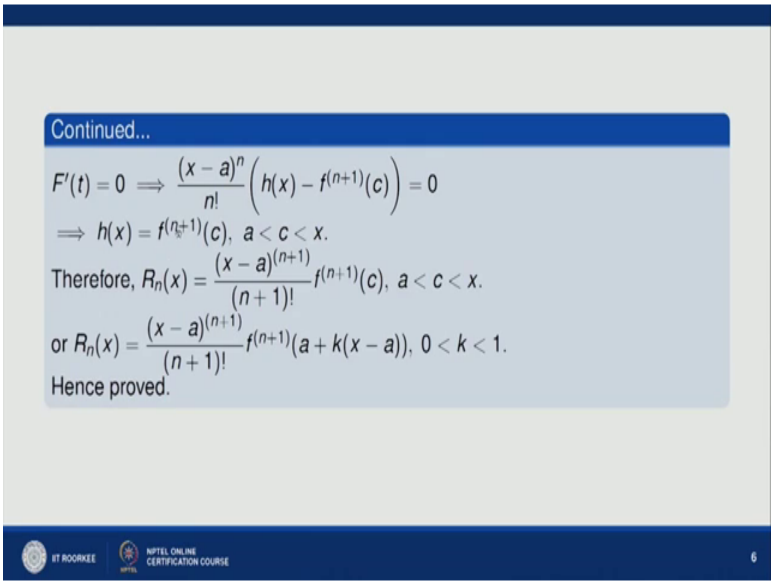

So, from here we obtain h x is equals to n plus 1th derivative at x equal to at x equal to c where c is between a to x and hence R n x will be x minus a whole raise to power n plus 1 upon factor n plus 1 n plus 1th derivative of f at x equal to c.

So, this is the same expression which is which is here in the statement. So, hence, we have proved the theorem, this c can also be replaced by this expression a plus k into x minus a where k is between 0 and 1 because when you substitute k equal to 0. This is a and when you substitute k equal to one this is x. So, c is between a to x the same thing. So, we have proved this theorem; now when you substitute a equal to 0.

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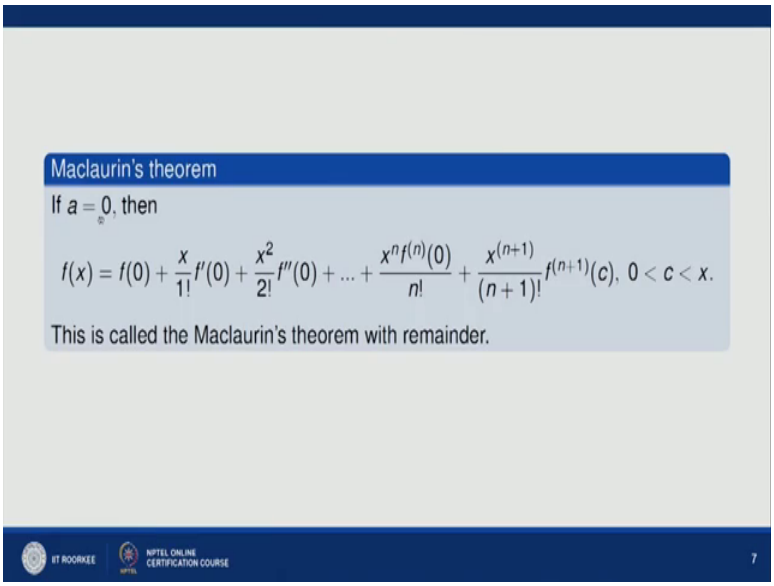

In the Taylor's theorem with remainder we will obtain this expansion this series which we call as Maclaurin theorem with remainder when you substitute a equal to 0. Now how to find a bound of an error you see this is an error term this is an error term.

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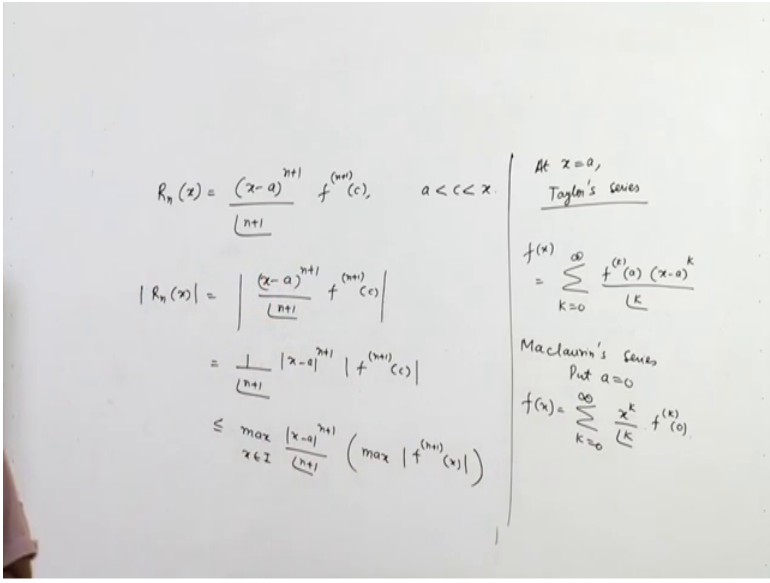

So, what is an error term it is R n x which is x minus a whole raise to power n plus 1 upon factor n plus 1 into n plus 1th derivative at c when c is between a to x. So, the upper bound of this term will give the bound of the error.

So, how do you find that you take mod of R n x this is equals to mod of this expression this is equals to one upon fettle n plus 1 mod of x minus a whole raise to power n plus 1 mod of n plus 1th derivative at x equal to c and this will be less than equals to maximum of mod of x minus a whole raise to power n plus 1 upon fettle n plus 1 into maximum of mod of n plus 1th derivative at x or c in the interval I. So, this will give and bound of an error, now what is an Taylor series?

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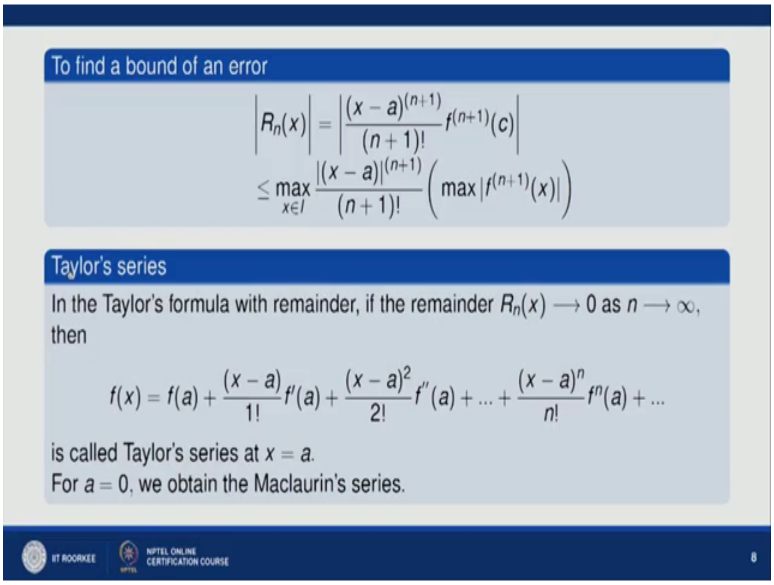

Now, if this remainder term if this remainder term will tend to 0 as an tending to infinity then this Taylor's theorem.

With remainder will go up to infinite terms and that is called Taylor series. So, what Taylor series at x equal to a is x equal to a at x equal to a Taylor series is given by it is f x is equal to summation k from 0 to infinity f k derivative at x equals to a x minus a whole raise to power k upon factorial k. So, this will be the Taylor series of f at x equals to a and of course, when you substitute a equal to 0 in this expansion. So, we obtain the corresponding Maclaurin series. So, for Maclaurin series Maclaurin series put a equal to 0. So, this f x will be summation k from 0 to infinity this is x raised to power k upon factorial k into k-th derivative at k equal to 0.

So, this is basically the Maclaurin series representation of the function f. Now, let us try a problem based on this suppose a function is cube root of x.

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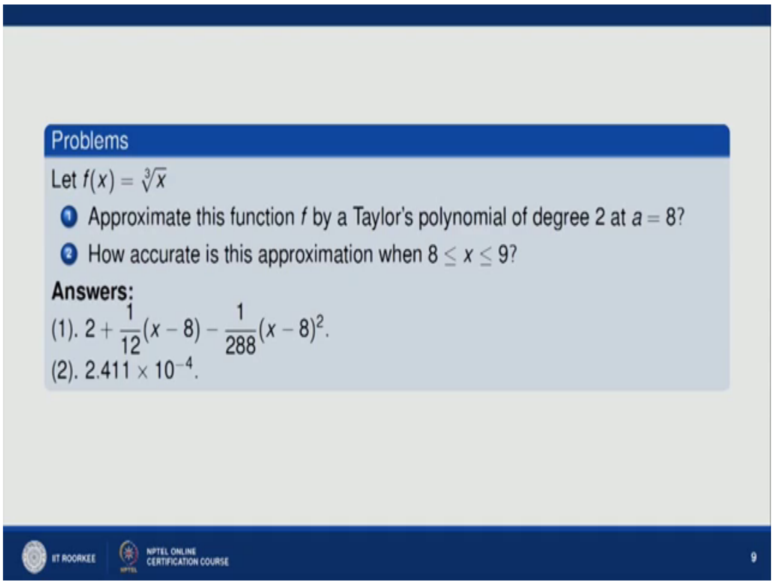

And what we have to find we have to approximate this function f by a Taylor series by Taylor's polynomial of degree 2 at x equals 2 at a equals to 8 and how accurate is this approximation when x lying between a to 9. So, let us try to solve this problem based on Taylor's theorem. Now here we want to approximate cube root of x by a polynomial of degree 2 at a equal to 8.

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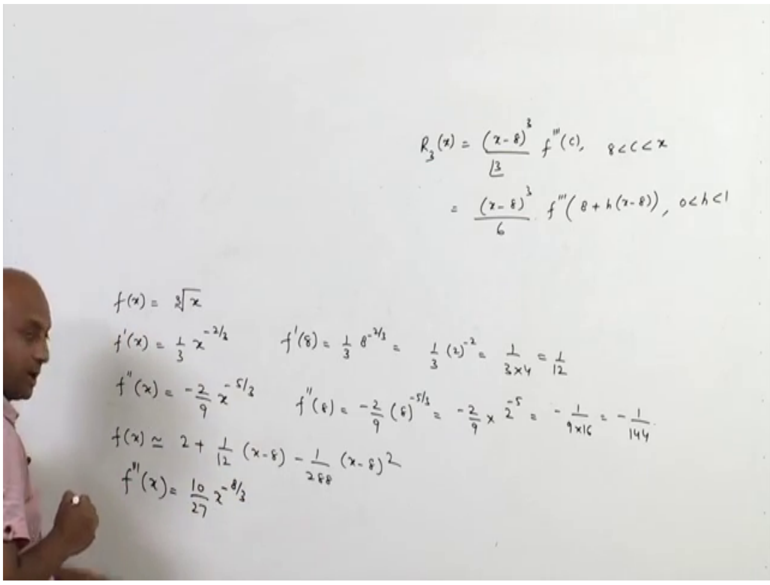

So, we write f x as f a plus x minus a into f dash a upon factorial one plus x minus a whole square f double dash a upon factorial 2 plus r 3 x.

Now, this term is basically approximation of this f by a polynomial of degree 2 ok, approximation of f by a polynomial of degree 2 if we take it up to here. So, this will be the approximation of this f linear polynomial this is the linearization of this f and this term will not be there then we go to the remainder term r to x and this is called corresponding remainder term or the error term. Now what will the approximation of this f at x at a equal to 8. So, first you find what is f x? F x is simply cube root of x. So, what is f dash x it is 1 by 3 x raised to power minus 2 by 3 what is f double dash x.

It is minus 2 by 9 x raised to power minus 5 by 3 what is f dash at a it is 1 by 3 a is 8 a is 8. So, it is 8 raised to power minus 2 by 3 which is 1 by 3 into 2 raised to power minus 2 which is 1 by 3 into 4 which is 1 by 12 and this is f double dash at 8 will be minus 2 by 9 into 8 raised to power minus 5 by 3 which is minus 2 by 9 into it is 2 raised to power minus 5 and this will be this will be minus 1 upon 9 into 16 which is minus 1 upon 288 no, no, it is 144.

So, what will be the approximation of this f by a polynomial of degree 2? So, approximation will be given by f a f at 8 is 2 2 plus f dash a f hash 8 is 1 by 12. 1 by 12 into x minus 2 plus f double dash 8 it is minus 144 upon 2 that is minus 1 upon 288 into x minus 8 whole square. It is 8 because a is 8. So, this is the quadratic approximation or a approximation of this f by a polynomial of degree 2. Now we have to find out; how accurate is this approximation this also we have to find out. So, how can you find this?

So, we can use reminded theorem reminder term remainder term is x minus 8 whole raise to power 3 now ns 3 upon factorial 3, it is third derivative at c where c is between a to x and a here is 8 or we can say it is x minus 8 whole raise to power 3 upon 6 into f third derivative at 8 plus h into x minus 8 where h is between 0 and 1 this also we can say. Now what is a third derivative of this f third derivative will be given by it is 10 by 27 x raised to power minus 8 by 3 minus 8 by 3.

Now, let us find this approximation now here x is between x is between 8 to 9.

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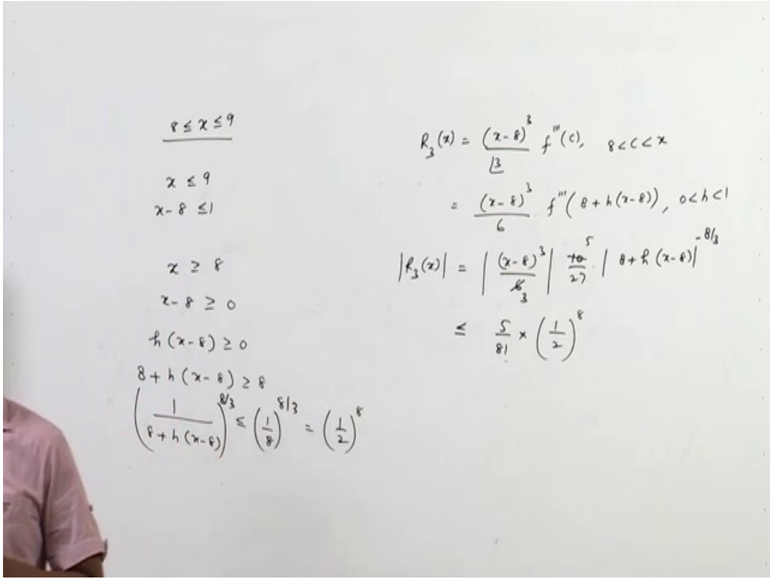

We have to find out the approximation when x is between 8 to 9. So, R n x r 3 x, its modulus will be equal to mod of this and third derivative is given by this expression.

So, we can write 10 by 27 times mod of 8 plus h into x minus 8 whole raise to power minus 8 by 3 where h is between 0 to 1. Now x is less than equals to 9. So, x minus 8 will be less than equals to 1. So, this term will be less than equals to 1.

So, it is less than equals to. So, it is 5 by 81 into now we have to find out the bound of this. So, x is greater equals to 8. So, x minus 8 will be greater equal to 0 h time x minus a 2 is also greater equal to 0 because at h is between 0 and 1 and 8 plus h into x minus 8 will be greater equal to 8. So, 1 upon 8 plus h into x minus 8 will be less than equal to 1 by 8 and whole raise to power 8 by 3 will be less than equals to whole raise to power 8 by 3 that a simply, it is one by 2 whole raised to power 3. So, 1 by 2 whole raise to power 8.

So, this would be less than equals to one by 2 whole raise to power 8. So, this is a bound of this approximation and this is approximately 2 point 4 1 into 10 raise to power minus 4 that you can check. So, in this way if you want to approximate a function by Taylor's theorem we can do that and how accurate is our approximation is that is the bound of the error that also we can find out using the reminder term or the error term. Now how can we can we do the Taylor's theorem for 2 variable functions. So, let us discuss this here now at a function f x y we define.

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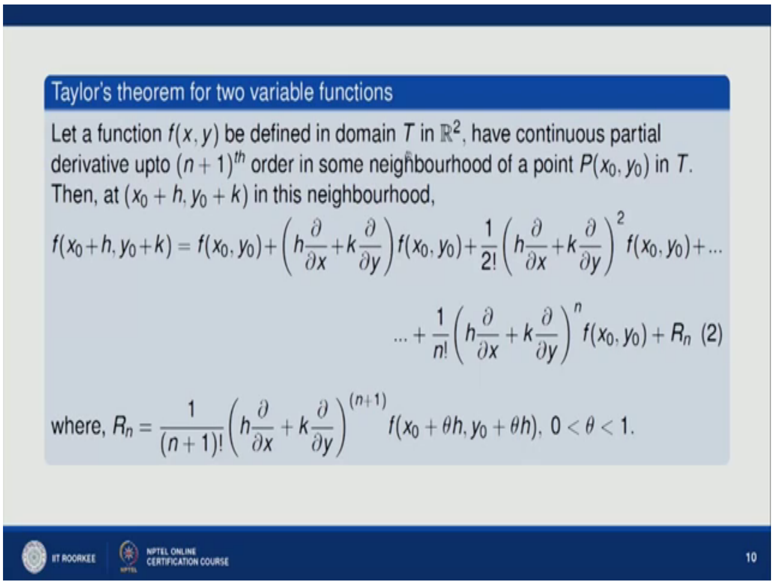

A domain T in R 2 have continuous partial derivatives up to n plus 1th order in some neighbourhood of a point p x naught y naught in t in the domain t then at x naught plus h and y not plus a k in this neighbourhood the expansion is given by this expression 2 where r n is the remainder term or the error term which is given by this expression where theta h between 0 to 1. So, this is simply a extension of one variable Taylor's theorem with remainder to 2 variable Taylor's theorem with remainder you can simply see from here, now you see.

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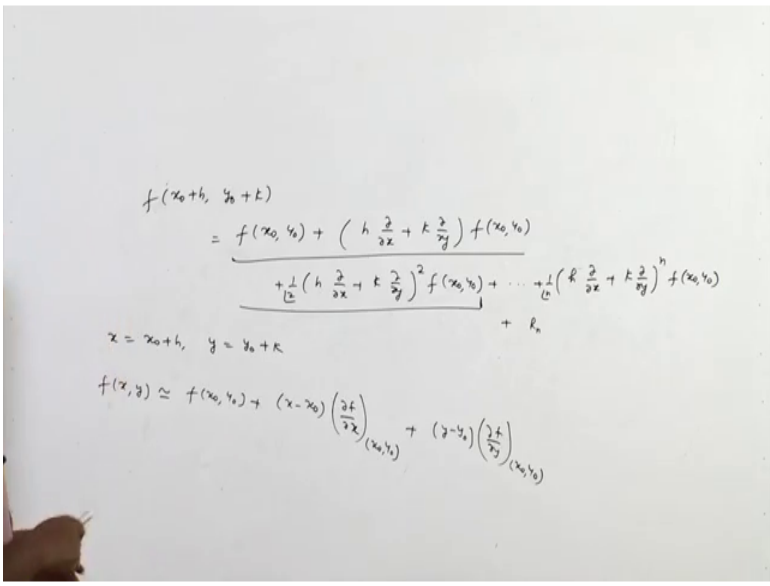

What is f of x naught plus h y naught plus k it is f of x naught y naught plus h del by del x plus k del by del y of f at x naught y naught plus h del by del x plus k del by del y whole square f at x naught y naught plus and so on plus. So, it is one by factorial 2 is also here and it is 1 by factorial n h del by del x plus k del by del y whole raise to power n f at x naught y naught plus remainder term ok, this is the Taylor's theorem with remainder for 2 variable function. Now, suppose we want to approximate this f by a linear polynomial or you want to linearize this f, we want to find out the linear approximation of this f. So, how can I do that? So, we will take terms up to here only.

We will take terms up to here only this will give linear approximation of f for that you simply, let x is equals to x naught plus h and y is equals to y naught plus k. So, what will be f of x y this will be approximately equals to f of x naught y naught h will be x minus x naught this f will come here. So, it is del by a del f by del x at x naught y naught plus this is y minus y naught del f by del y at x naught y naught. So, this will give linear approximation of this f at x naught y naught this is f x y is really equal to f x naught y naught plus x minus x naught del by del x of, this is del by del x of f at x naught y naught plus y minus y naught del f by del y of x naught at x naught y naught.

Now, suppose you want to find out quadratic approximation of this f at x naught y naught. So, we will take terms up to here not up to here we will take terms up to here, this will give quality approximation of this f and the error term will be given by either remainder term. Now, how we will find the quality approximation there of this f this is a linear approximation. So, for quality of approximation, first, we will solve this term and then we will replace h by x minus x naught and k by y minus y naught which will gave quality approximation of this f at x naught y naught. So, first we will solve this term.

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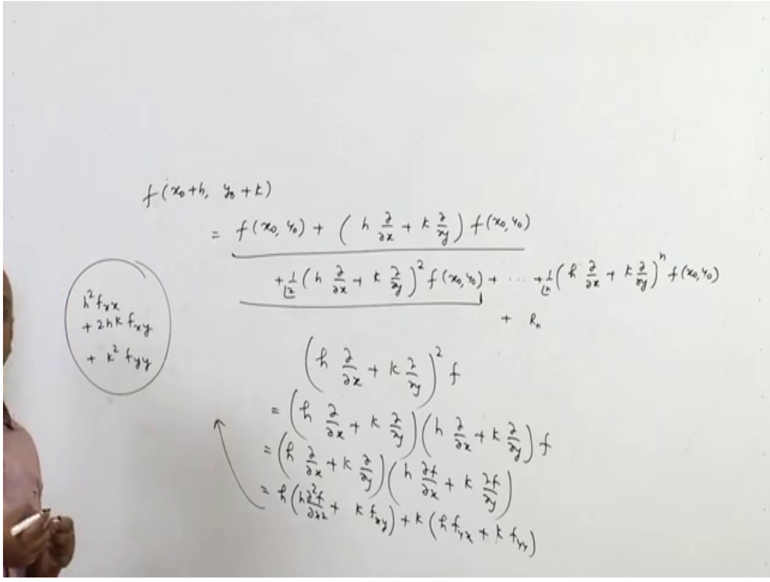

How we solve this? So, this term is this at x naught y naught. So, this is equal to h del by del x plus k del by del y a square means multiply 2 times now this is equals to h del by del x plus k del by del y now you take this f inside this is h del f by del x plus k del f by del y this is equals to now del by del x of this term is del h del square f by del x square plus h k it is f x y plus k times; it is h f y x plus k square f y y, it is k h is we are taking h common. So, h will not be here. Now function is having continuous partial order derivatives. So, this is this will be equal to this.

So, we can take we can take this expression as h square f x x plus 2 h k f x y plus k square f y y and f x x, f x y, f y y is to be computed at x naught y naught. So, what will be the quadratic approximation?

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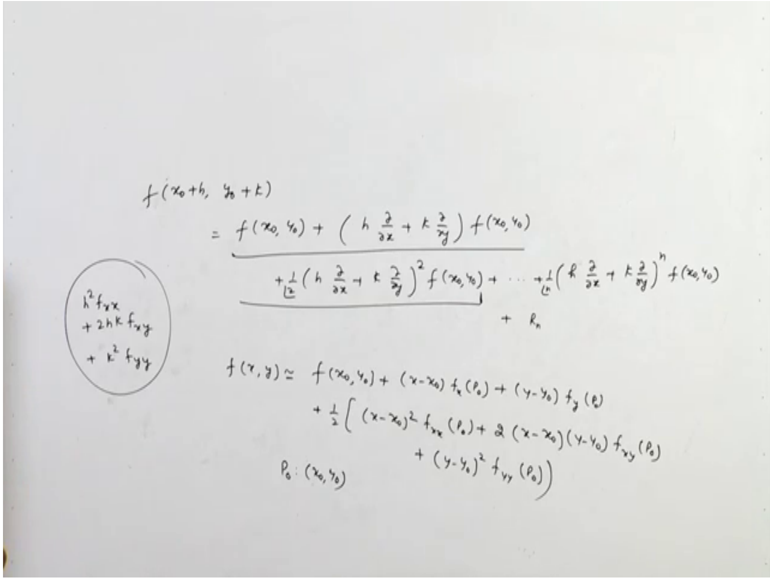

So, the quadratic approximation of this f will be now I simply replace h by x minus x naught and y by and k by y minus y naught. So, this will be f x naught y naught plus x minus x naught f x at p naught plus y minus y naught f y at p naught plus 1 by 2 x minus x naught whole square, this term f x x at p naught plus 2 x minus x naught y minus y naught f x y at p naught plus y minus y naught whole square f y y at p naught where p naught is x naught y naught. So, this is the quadratic approximation of this f at x naught y naught.

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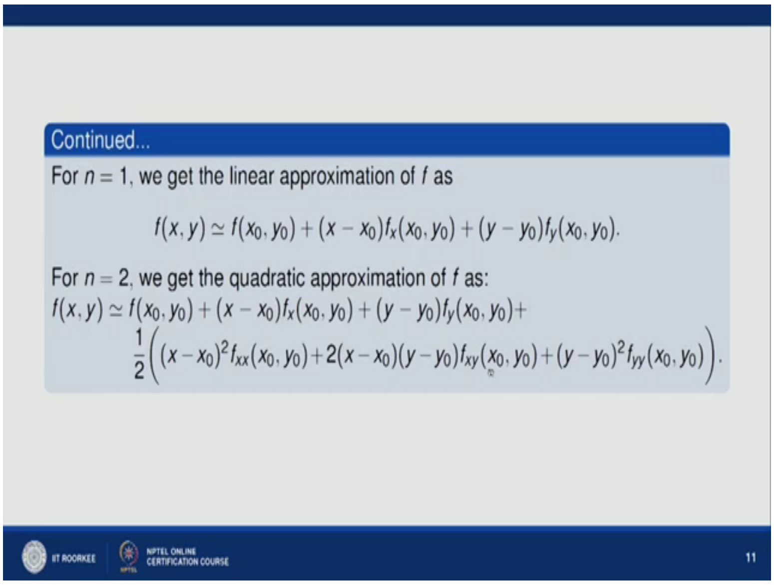

So, this is what is this slide is similarly, if you want cubic approximation of this f. So, we will take terms up to up to power 3, we first expand that term and then replace h by x minus x naught and k by y minus y naught.

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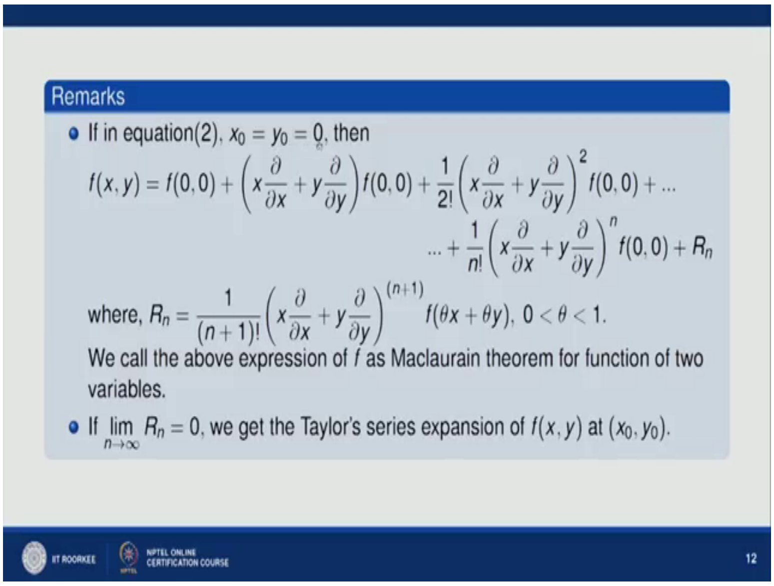

So, if we substitute x not equal to y not equal to 0 in equation 2 that is in the Taylor's theorem we will get the corresponding Maclaurin theorem for 2 variable function now if limit n tends to infinity R n will be 0 then we get that Taylor series expansion of f x y at x naught y naught which is an infinite series now suppose you want to solve the first problem linearization of this f at x at 1 comma 1 for that we easily solved is what is f?

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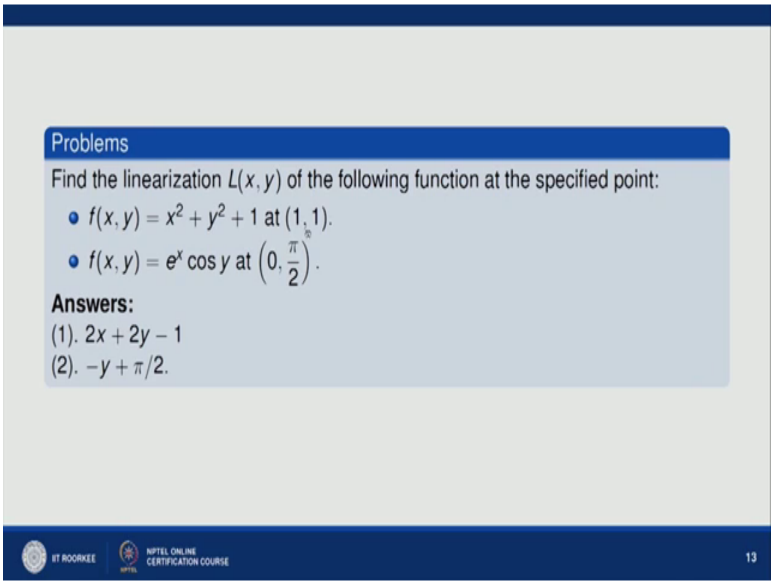

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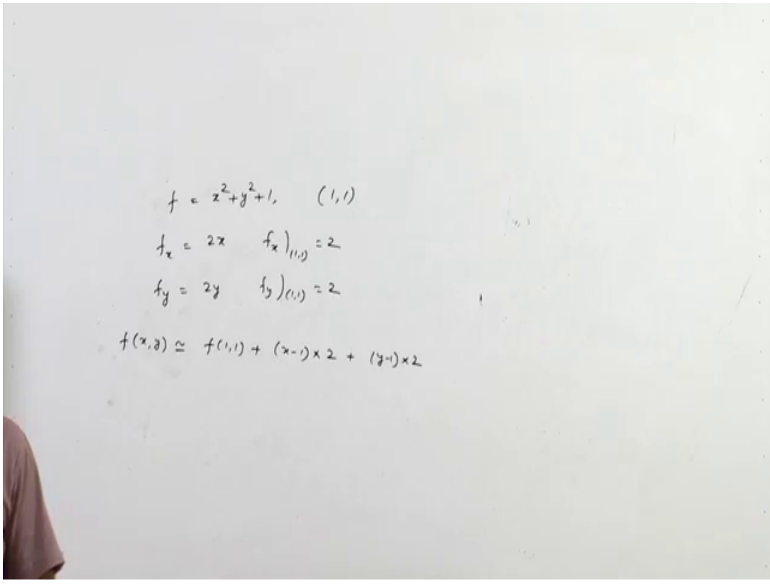

Here, what is the point; point is one comma 1, what is f x is 2 x and f x at 1 comma 1 is 2 what is f y is again 2 y and f y at 1 comma 1 is 2. So, the linear approximation of this f will be f 1 1 plus x minus one f x at 1 1 plus y minus 1 f y at 1 1. So, we can simply solve this and find out the linear approximation of this f at 1 comma 1, similarly we can solve the second problem.

Thank you.