Multivariable Calculus
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Lecture – 16
Lagrange Multipliers

Hello friends. So, welcome to a lecture series on multivariable calculus. So, we were dealing with maxima minima of 2 or more than 2 variable functions, now I will deal with Lagrange multipliers. So, what Lagrange multiplier method is let us see. In many practical problems we need to find the maximum or minimum value of a function.

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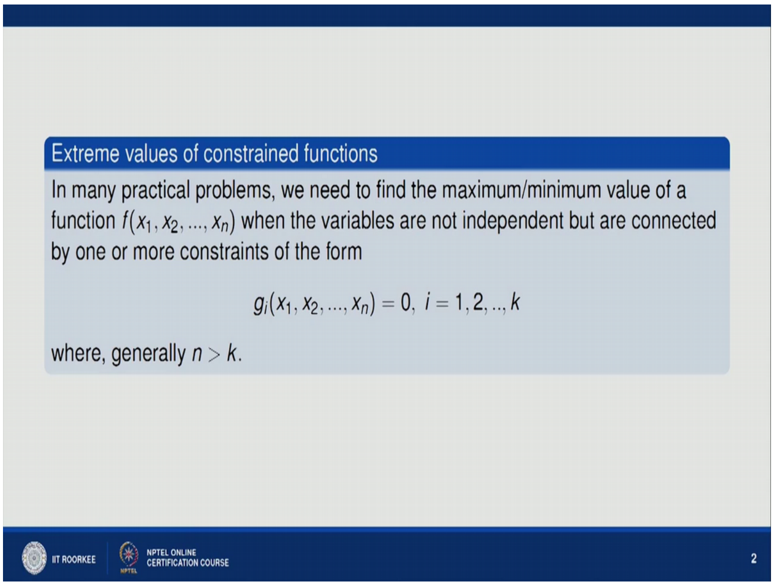

F X 1, X 2 up to X n when the variables are not independent, but they are connected by one or more constraints of the form gi X 1, X 2 up to Xn equal to 0 where i from 1 to k and generally n is greater than k.

Now, suppose you have a constraint optimization problem, we call such problems as constrained optimization problem that is; you have to find out maximum and minimum value of some function f subject to some conditions some conditions are given to you. Conditions are gi of X 1, X 2 up to Xn equal to 0 ok. So, hence these variables X 1, X 2 up to Xn are not independent they are connected by some relations. So, if they are connected by some relations. So, how can you find out maximum or minimum value of function subject to those conditions? How can we solve those problems?

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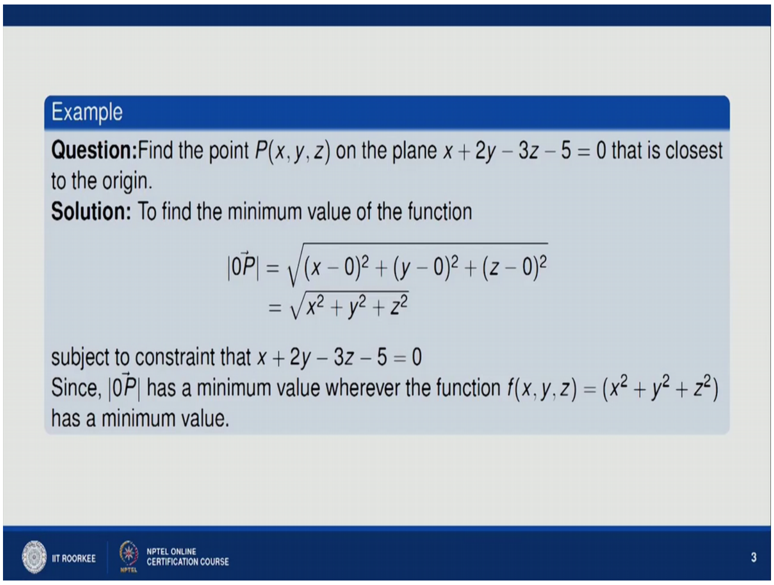

Suppose you have to find a point P xyz on the plane x plus 2 y minus 3 z equal to 5 that is closest to the origin.

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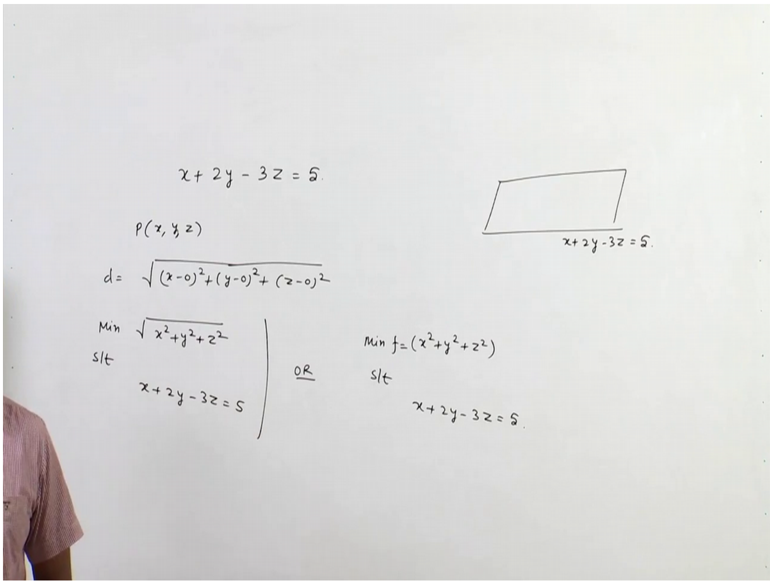

So, what is the plane? Plane is x plus 2, y minus 3, z equals to 5. Now in this plane there are so many points ok. You see this plane this is the equation of plane is x plus 2, y minus 3, z equals to 5. From all those points we have to find out that point which is closest or nearest from the origin ok.

So, this is this is some origin. So, let us suppose that point is x, y, z. So, distance from the origin will be under root x minus 0 whole square, plus y minus 0 whole square plus z minus 0 whole square. So, you have to you have to find out the minimum value of this d subject to this condition ok. So, what is our problem? Problem is minimizing this objective under root x square plus y square plus z square subject to x plus 2, y minus 3, z equals to 5 or this problem can be rewritten as minimum of. Now finding minimum of under root or finding minimum of the inside expression I wanted the same thing the problems are equivalent.

So, we can say that finding minimum of x square plus y square plus z square subject to x plus 2, y minus 3, z is equal to 5. Now the first way out is the method of substitution. So, what is a method of substitution? Now this is an equation having 3 unknowns, you simply eliminate one other variable in this expression in this expression with the help of this expression.

So, let us compute say x.

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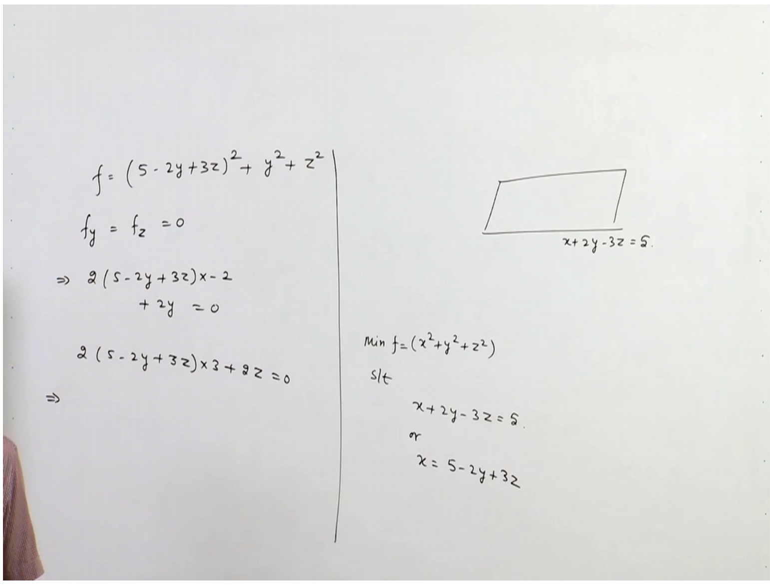

So, what will be x? It is 5 minus 2, y plus 3 z; so, what will be f? F will be 5 minus 2 y plus 3 z whole square plus y square plus z square. So, now, it is something like unconstrained problem, I mean now it is a 2 variable problem without any condition and you have to simply find a minimum value of this. So, the our old technique will work out that is you find out the critical point of this function ok. Another using the second order partial derivative test you can check where that point is a point of local minima or local maxima. So, that will give the that will give the minimum value of this function.

So, what is fy and fz put it equal to 0. So, this implies it is simply 2 times 5 minus 2 y plus 3 z into minus 2 plus 2 y is equal to 0 and the second equation is 2 times 5 minus 2 y plus 3 z into 3 plus 2 z equal to 0. So, the equations are simply. So, equations are it is you can easily find out the 2 equation, one equation is this second equation is this and when you solve these 2 equations. So, you can find out the values of y and z of course, the second or derivative test will give the minimum value I mean ah point as a local minima substituting those y and z over here you can simply find out x.

So, that xyz will be a point; which is closest from the origin lying on the plane. So, what is that point?

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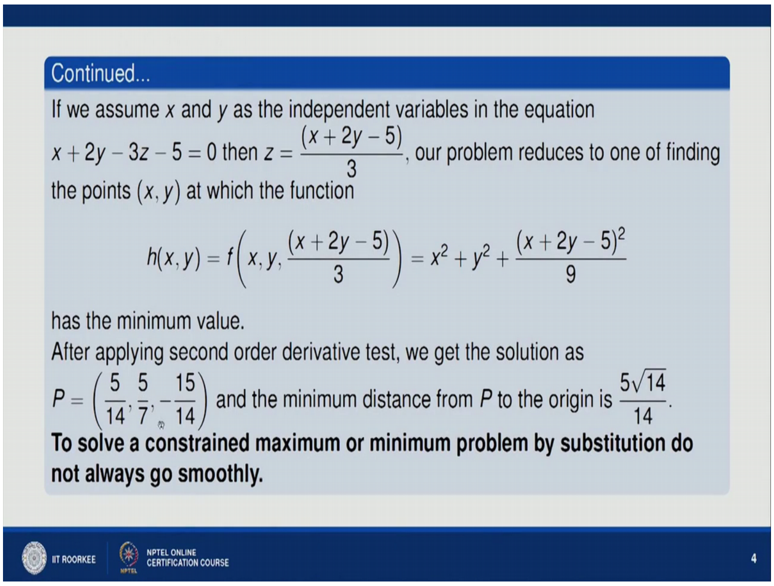

That point is simply 5 by 14, 5 by 7 minus 15 by 14 that you can easily find out solving these 2 equations simultaneously and then find the values of y and z you simply substitute it over here find the value of x ok.

Now, to solve a constraint maximum or minimum problem by substitution do not always work smoothly you see. Here we have a linear equation. So, we have easily substituted we have easily find out one variable respect to other 2 and simply substituted over here, but this may not always work. We have some non-linear terms also which is difficult to remove from this expression in this expression you this relation ok. So, we have some method to find to solve all those cases. So, that method is Lagrange multiplier method. Now what is that method?

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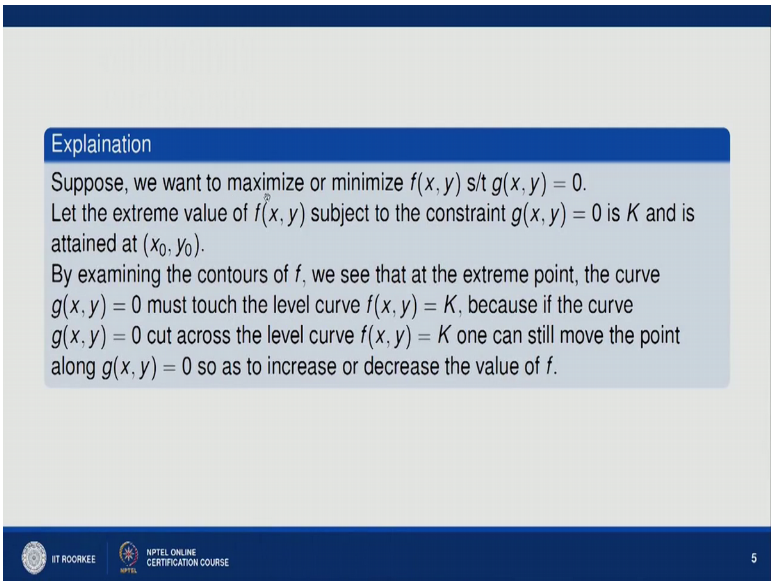

Now, suppose we want to maximize or minimize the function fxy, we are considering here 2 variable function the same process will go for 3 or more variable functions also subject to g x equal to 0 g x, y equal to 0. Now let the extreme point of f x, y subject to the constraint g x, y equal to 0 is k and is attained at x naught y naught. By examining the contours of f; we can see that at the extreme point the curve g x, y equal to 0 must touch the level curve f x, y equal to k because if the curve g x, y equal to 0 cut cross the level curve then one can a still move the point along g x, y equal to 0. So, as to increase the decrease the value of f.

So, what does it mean? Let us see from the graph you see.

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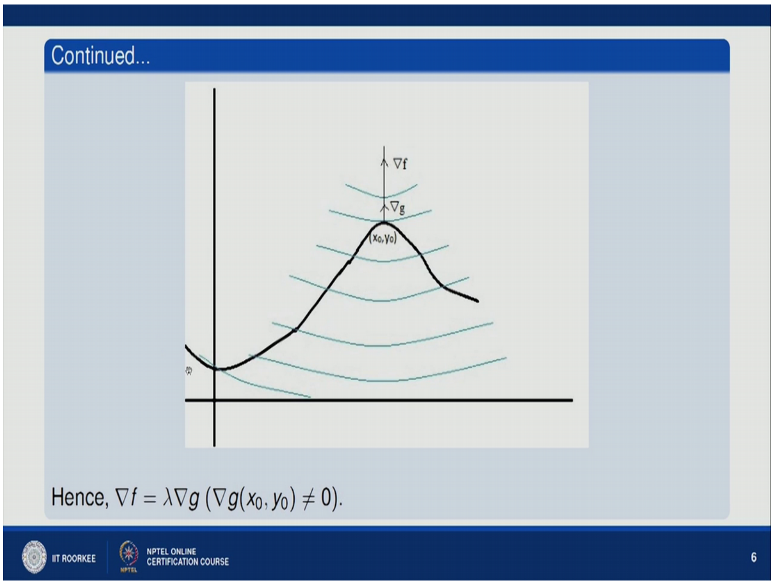

Say this is this is the equation of g x, y equal to 0. This graph which is given here in the bold line say this graph is a graph of g x, y equal to 0 ok. Now you plot different contours of f say f x, y equals to m 1, f x, y equals to m 2 f x, y equal to m 3 and so on ok. We can simply observe that at the point x naught y naught if you draw a contour, which touches at this point will give the stream values. Because if we if we if we go inside the region we are it cuts we are it intersect g x, y equal to 0 then the value of the function may increase may increase or decrease. I mean if you move further the value of value of the function may increase the or decrease further.

Suppose this for the point of local maxima, where the function attained maximum value then if you go if you go in the region below the curve value of the function will decrease and continuously decrease. So, this can be understand by following say the first example let us discuss this example. So, what I want to say basically, you see in the first example we have to find out the maximum minimum value of the function and subject to condition is x square plus y square equal to 1 ok.

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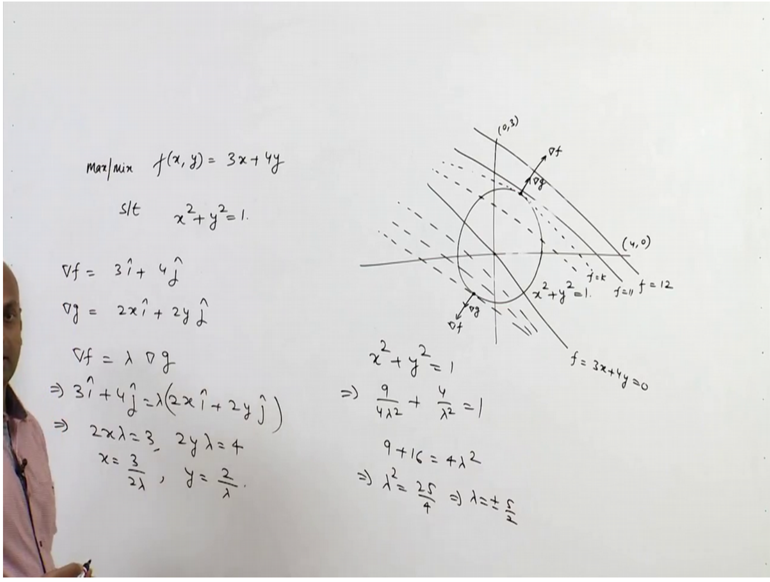

So, now this is our g; now let us draw different contours of f. So, if you have. So, this is a straight line basically. So, add here, here f equal to 3, x plus 4 y is 0; because it is passing through origin ok. Now when which is 1; so suppose at this point it may be 12 when this is 4 and this is 3 at some point here it may be 11, at some point here when it touches the circle it may have some other value say k.

Now when you when you come inside this circle at this point and this point, the function have some value function 3, x plus 4 y have some value. But as you move inside the function the value of the function decreases continuously and it is 0 here. So, function will attain this function will attain maximum value when it touches this circle over here.

And again when you move inside this circle, this will give the minimum value over here where it touches the when it touches the circle over at this point ok. Because further decrement in the value of the function subject to this condition is not possible, because if you take a function over here take a function over here we want we want all those points which lies on the circle subject to this condition we are we are talking about. We are talking about maximum minimum of this f over this condition; I mean all those points lying on this circle.

Now, if we are saying that this is not a point of maxima this is the point of maxima. So, all those values when we move inside the region, the value of f decreases here it is 0 and when you away from this point, value of f increases ok, but we are interested to find out those values those x and y, which are on the circle y square equal to 1. So, now, at this point if you are talking about this point at this point, this will be simply gradient of f; gradient of f is always normal to a surface normal to a curve.

And at this point at this point the this is also the gradient of g, the same direction with the rate of g, g is g here is x square plus y square minus 1 equal to 0. Now at this point gradient of g and gradient of f are same, I mean direction are same this means they are parallel. And parallel means parallel means gradient of f will be some lambda times gradient of g similarly here, gradient of f and gradient g are parallel.

So, this means at the point of at the point of extrema, if you have to maximize or minimize the function subject to some condition g equal to 0; then at that point gradient of f is always equal to lambda times gradient of g, because gradient of f and gradient of g are parallel vectors. So, they are parallel this means one vector can be written as lambda times other vector ok. So, this is the main concept of Lagrange multiplier method basically.

Now, here what is gradient of f? It is 3 i cap plus 4 j cap what is gradient of g? It is 2 xi cap plus 2 yj cap del g by del x i cap plus del g by del y k cap and lambda is called Lagrange multiplier this lambda is called Lagrange multiplier. So, what is gradient of f is equal to lambda tangent of g. So, this implies 3 i cap plus 4 j cap will be equals to 2 xi cap plus 2 yj cap this implies lambda times this implies 2 x lambda will be equals to 3 and 2 y lambda will be equals to 4. So, here x will be equals to 3 upon 2 lambda and here y will be equals to 2 upon lambda.

Now, this point must be on circle also. So, this must satisfy x square plus y square equal to 1 also. So, when you take x square plus y square equal to 1. So, this implies 9 by 4 lambda square plus 4 by lambda square will be equals to 1. This means 9 by 16, 9 plus 16 will be equals to 4 lambda square and this implies lambda square equals to 25 by 4 implies lambda is equals to plus minus 5 by 2 ok.

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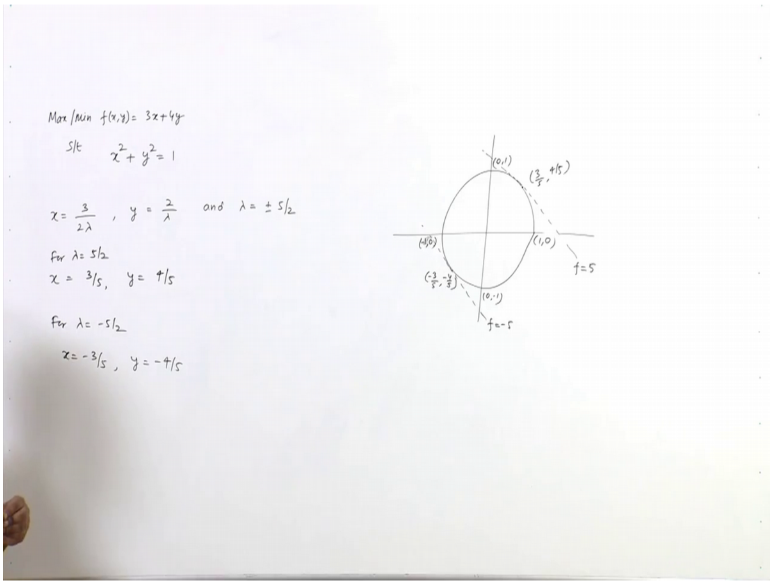

Now, a lambda is 5 by 2. So, we were solving this problem that is maximizing and minimizing f x, y equal to 3 x plus 4 y subject to x square plus y square equal to 1. And we have just obtain that x equals to 3 by 2 lambda and y equal to 2 by lambda where lambda is plus minus 5 by 2. So, what your values of x and y? X will be when you substitute. So, for lambda equals to 5 by 2 x will be equal to it is 3 by 5 and y will be equal to 4 by 5 and for lambda equal to minus 5 by 2 x will be minus 3 by 5 and y will be minus 4 by 5 ok.

So, where these points are? These points are on the circle, they say a unit circle of center 0 radius 1 it is 1 comma 0, minus 1 comma 0 it is 0 comma 1 and 0 comma minus 1, and this point is somewhere here which is 3 by 5 and 4 by 5 and this is a point where this function attains where this function attains maximum value what is that value? When you substitute x as 3 by 5 and y as 4 by 5 it will be nothing, but 9 plus 16, 25 by 5 it is 5.

So, that is the maximum value of f and what is the minimum value of f? Minimum value is obtained at x equals to minus 3 by 5 and y equals to minus 4 by 5 and where this point is? This point is somewhat here which is minus 3 by 5 and minus 4 by 5. So, the this is this is the f which is a minimum value of this function. So, the maximum value of function is 5, which is at this point and the minimum value of function is minus 5 which is at this point.

So, let us try to solve a problem 2; the temperature at a point xy on a metal plate is given by T x, y is equal to 4 x square minus 4 x, y plus y square.

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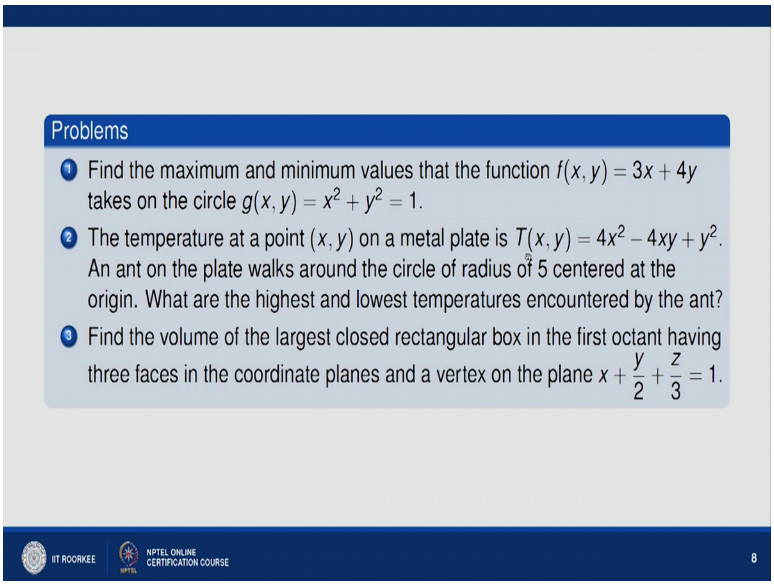

An ant on the plate walks around the circle of radius 5 centered at the origin. So, we have to find the highest and the lowest temperatures encountered by the ant. So, you have a metallic plate, the temperature of the metallic plate is governed by that expression and ant is moving on that plate following a circular path, circular path is centered it origin and radius is 5.

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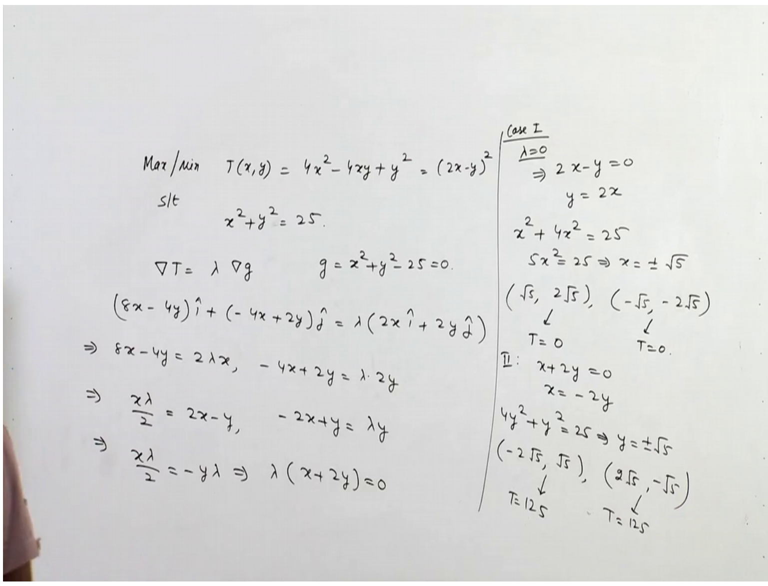

So, basically we have to find out the maximum and minimum value of T x, y which is given as 4 x square minus 4 xy plus y square subject to x square plus y square is equal to 25 because ant is ant is following this root, ant is following this circle is moving on this circle and when ant is feeling the maximum temperature and when it is feeling at the lowest temperature, the points that we have to find out. So basically we have to maximize or minimize this function subject to this condition.

So, how can we do that? We can take gradient of T is equal to lambda times gradient of g; where g is x square plus y square minus 25 is equal to 0 this is by Lagrange multiplier method what is gradient of T? It is 8 x minus 4 y i cap plus minus 4 x plus 2 y j cap is equals to lambda times 2 xi cap plus 2 y j cap. So, this implies 8 x minus 4 y it equals to 2 lambda x and minus 4 y plus 2 y is equals to lambda into 2 y.

So, from these 2 equation this implies x lambda by 2 will be equals to 2 x minus y from here and from here it is minus 2 x plus y which is equals to lambda y. So, from these 2 equation what we have obtained this implies x lambda by 2 is equals to minus y lambda and this implies lambda times x plus 2 y is equal to 0.

So, now we have two possibilities; either lambda equal to 0 or x plus 2 y equal to 0. So, first we will take lambda equal to 0. So, case one when lambda equal to 0. If lambda equal to 0 this means 2 x minus y equal to 0, the same equation obtained from this expression. So, 2 x minus y equal to 0 this implies. So, y is equals to 2 x now this expression this x and y must satisfy this equation also because ant is moving on the circle. So, we substitute y equal to 2 x here. So, it will be x square plus 4 x square is equal to 25. So, 5 x square is equal to 25, and this implies x is equal to under plus minus under root 5.

So, what is the point? Point will be when x is under root 5 y will be 2 under root 5 and when x is minus under root 5, y will be minus 2 under root 5 and what will be T at this point? T at this point will be when you substitute. So, so what is T? T is basically when you carefully see on T; it is 2 x minus y whole square and 2 x minus y is 0. So, at both this point T will be 0. So, this means this point and this point gives the minimum value of the temperature or temperature is lowest at these points ok.

Now, now the case 2 when x plus 2 y equal to 0. If x plus 2 y equal to 0 means x is minus 2 y. Now when you substitute x equal to x equal to minus 2 y here, it is 4 y square plus y square equal to 25. So, this implies y is equal to plus minus under root 5. So, therefore, then the points will be when y is plus under root 5. So, x will be minus 2 under root 5 and when is when it is minus under root 5, then it is 2 under root 5 and values of T at this point will be when you substitute x as minus 2 under root 5 and y as under root 5. So, this value will be this value will be 2 into 2 4 under root 5 plus 5 that is 125 and here also it is 125. So, this is the point these are the points, where T is maximum.

So, when ant is moving on the circle, at when the ant comes at this point or this point the temperature is minimum that is T equal to 0. And when the ant comes at this point or this point then temperature is maximum which is 125 ok. So, that is how we can solve this problem. So, now, next problem find the volume of the largest closed rectangular box in the first octant having 3 faces in the coordinate planes and the vertex on the plane this ok.

So, have you have to find out the volume of the largest rectangular box can satisfy this condition.

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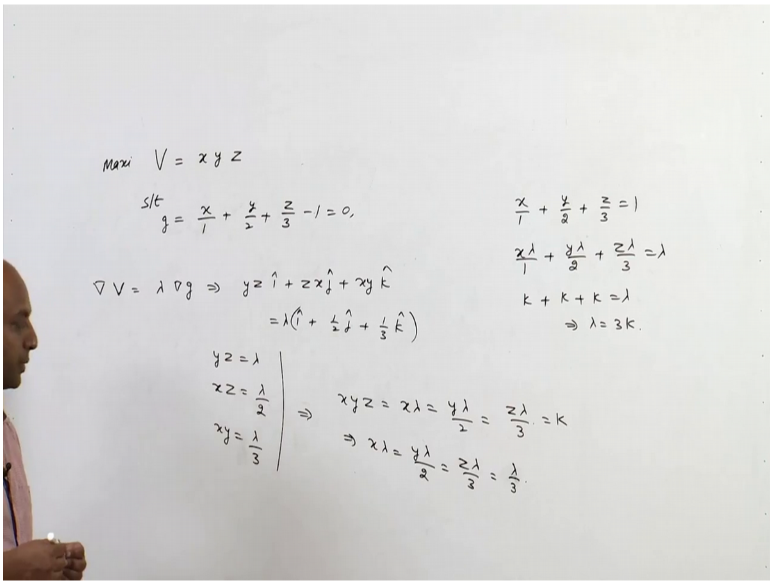

So, volume of a rectangular box will be given by x into y into z length into breadth into height and a subject to what is the condition. So, this we have to maximize yes. So, this we have to maximize and the condition is basically x by 1 plus y by 2, plus z by 3 is equals to 1 a subject to this condition we have to maximize this value.

So, again we will take. So, let us suppose this is g. So, gradient of V will be equals to lambda times gradient of g and this implies yz i cap plus zx j cap plus xy k cap will be equal to i cap plus 1 by 2 j cap plus 1 by 3 k cap lambda times. So, yz equal to lambda xz equals to lambda by 2 and xy is equals to lambda by 3. Now fear from here what we obtain? You multiplied this by x this by y this by z ok. So, what you will obtain? Xyz will be equals to x lambda will be equals to y lambda by 2 will be equals to z lambda by 3.

Now, this point this condition must satisfy this also. So, what we are having from here? It is x by 1 plus y by 2 plus z by 3 is equals to 1 you multiply the entire expression by lambda. So, it is x by lambda into 1 by 1, y lambda by 2 plus z lambda by 3 is equal to lambda say this is say this equals to k. So, this is a k plus k plus k is equal to lambda. So, this implies lambda equals to 3 k ok.

So, from here what we obtain? X lambda is equals to y lambda by 2 is equals to z lambda by 3 will be equals to 3 will be equals to lambda by 3. So, what will be x now?

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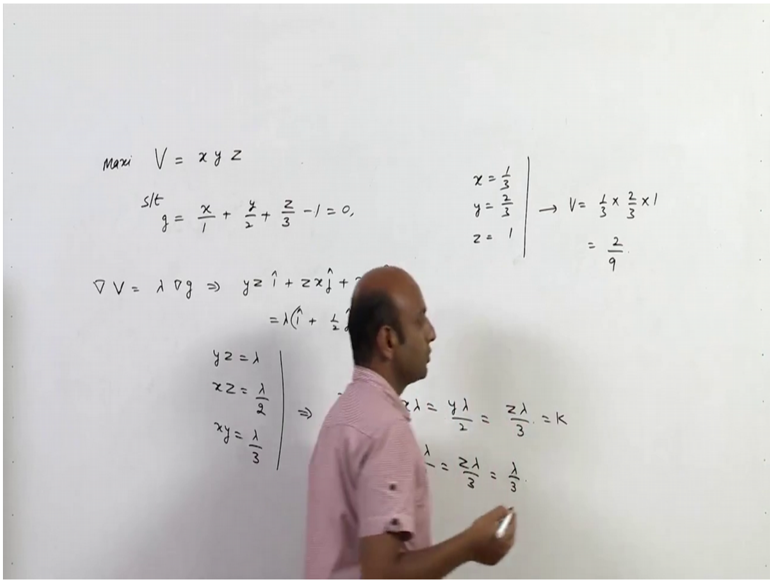

So, x will be x will be equals to 1 by 3, y will be equals to 2 by 3 and z will be equals to 1 ok. So, you can simply substitute you can simply check here. So, x 1 by 3 plus 1 by 3 plus 1 by 3 is one 1 minus 1 is 0. So, satisfy this equation also ok.

So, this is the point where it attains maximum value and the maximum value of v will be nothing, but 1 by 3 into 2 by 3 into 1 that is 2 by 9. So, that is how we can find out we can solve this problem. Now suppose you have 2 constraints, you have 2 maximize maximum of you have to find out the maximum minimum value of function fx yz subject 2 conditions. So, how can you find how can we use Lagrange multiplier in this case?

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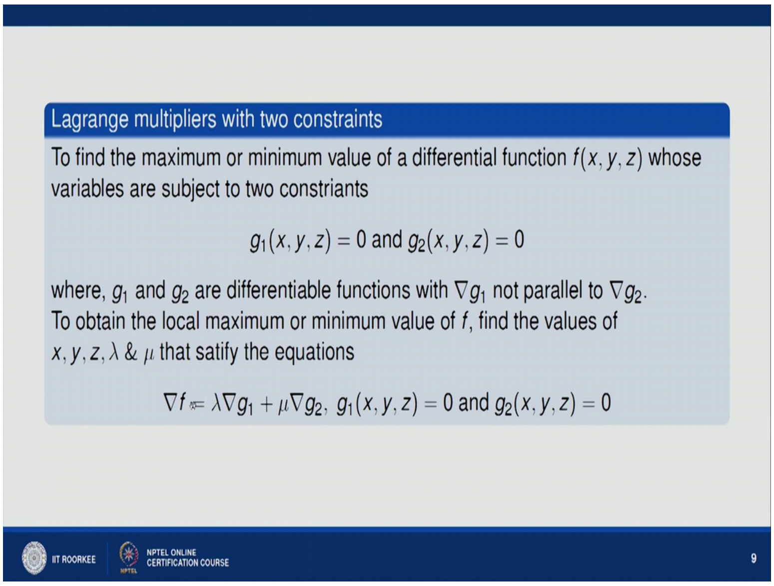

We can simply use we can simply take here gradient of f as linear combinations of gradient of gradient of g 1 plus gradient of g and gradient of g 2; that is lambda time gradient g 1 plus mu times gradient g 2 so, and this condition also must hold that is we take a gradient of f as a linear combination of gradient of the constraints gradient of g 1, gradient g 2, gradient g 3 and so on.

Now, let us solve the first problem in this case. So, here we have to find out the minimum value of this function.

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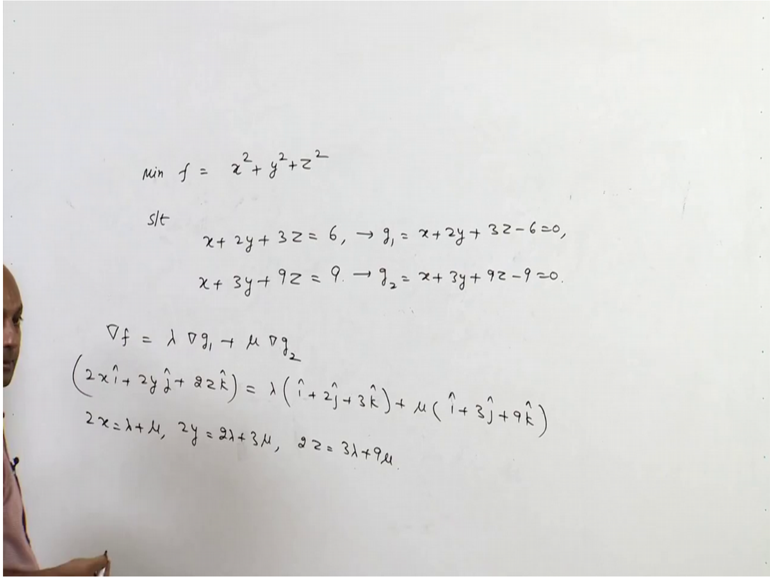

Function is x square plus y square plus z square and subject to condition are conditions are x plus 2 y plus 3 z equals to 6 and second condition is x plus 3 y plus 9 z is equal to a 9. So, how can we solve this? So, gradient of f we can take as lambda time gradient of g 1 plus mu time gradient of g. So, what is g 1 from here? G 1 will be x plus 2 y plus 3 z minus 6 equal to 0 and g 2 from here will be x plus 3 y plus 9 z minus 9 equal to 0.

So, that will be gradient of g will be 2 x i cap plus 2 y j cap plus 2 z k cap is equals to lambda times i cap plus 2 j cap plus 3 k cap plus mu times i cap plus 3 j cap plus 9 k cap. So, you formulate from the equations from here it is 2 x equals to lambda plus mu, 2 y is equals to lambda 2 lambda plus 3 mu, and 2 z equal to 3 lambda plus 9 mu and next is these 2 condition also must hold.

So, you simply substitute x here y here and z here again here also you some simply substitute the values of x y and z. So, you will get back 2 equations in lambda and mu you solve those 2 questions find out the values of lambda and mu. When you get the values lambda and mu is simply substitute here you find the values of xy and z that will give the point where this f attained the minimum value. So, this is how we can solve this type of problems a similarly we can solve the second problem also so.

Thank you.