Multivariable Calculus
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Lecture - 15
Extreme Values-II

Hello friends! So, welcome to lecture series on Multivariable Calculus. So, in the last lecture we have seen that, how can we find maxima minima for 2 variable functions. We have seen, that if we have a function of this form Z is equals to f x, y.

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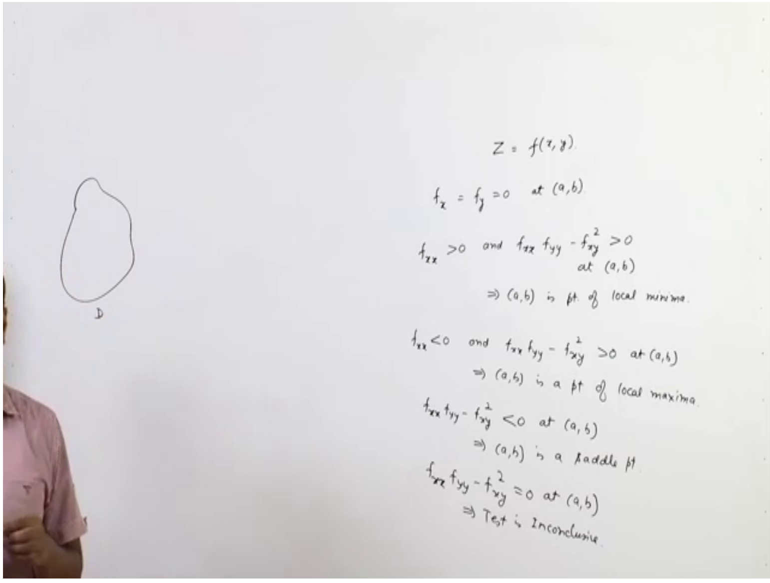

Then, suppose f x is equals to f y equal to 0 at a comma b then, to check whether this point a, b is a point of local maxima, local minima, saddle point, we have to go for the second order, partial derivative test.

What is that test? In that test, we simply find f xx. if f xx is positive and f xx f yy minus f xy square is also positive at a, b this implies a, b is a point of local minima. Ok, now if, f xx is negative at a, b and f xx into f yy minus f xy square is positive at a, b, then this implies a, b is a point of local maxima and if, f xx into f yy minus f xy square is less than 0 at a, b implies a b is a saddle point. And if this quantity is equal to 0, so test is inconclusive ok. If, f xx into f yy minus f xy square is equals to 0 at a, b and this implies test is inconclusive. Inconclusive means, we have to go for the higher order partial derivative test, to check whether a, b is the point of local maxima, local minima or saddle point.

So, this we have already seen in the last lecture, that if this condition hold, then we can say the point a b which is a critical point at a, b f x equal to f y equal to 0, ok, then this point the point of local maxima, local minima or saddle point. This can we can see by second or derivative test. Now, in this lecture, we will deal that how can you find an absolute maximum or absolute minima if we have a closed bounded region, ok.

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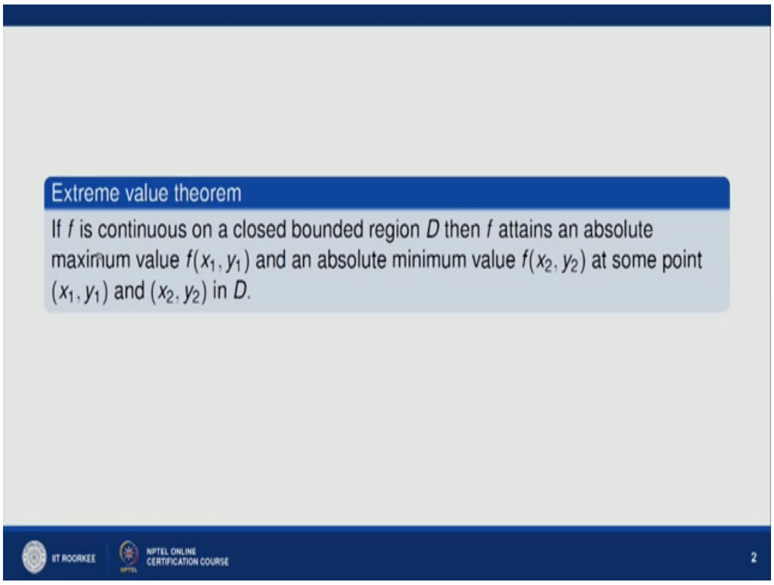

Now, first we have Extreme value theorem. What it states? It states that, if f is a continuous function on a closed and bounded region D, then f always attains an absolute maximum value f at x 1, y 1 and in absolute minimum value f x 2, y 2 at some point x 1, y 1 and x 2 y 2 in t; that means, if you have a close and bounded domain D, of a function f, then there will always exist some point x 1, y 1 and some point x 2, y 2 in this region D, where function will have absolute minimum and absolute maximum value. There always exists such point in a close and bounded region, ok.

Now, how to find absolute maxima and absolute minima if we have a close and bounded region D?

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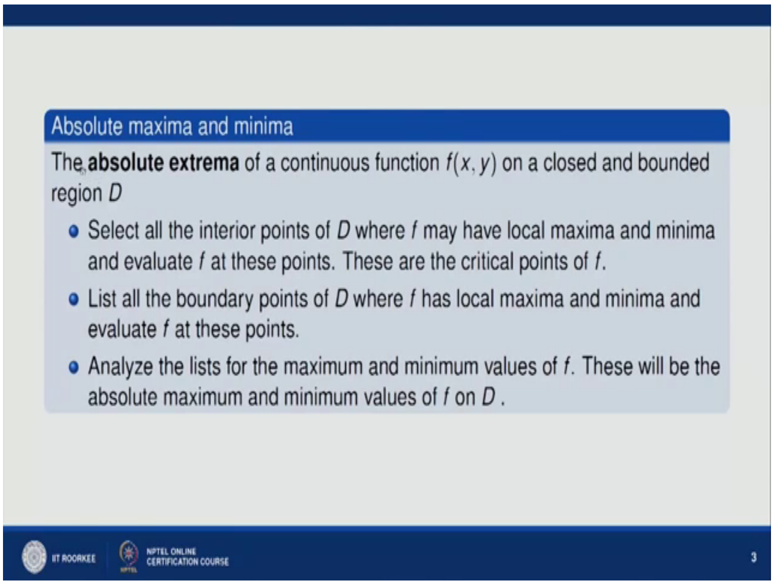

So, for that, now suppose, the absolute extreme of a continuous function f x,y on a closed and bounded region D, how to find that, we first select all the interior points of D, where f may have local maxima and minima and evaluate f at these points, ok. These are simply the critical points of f. The first step is this thing. The second step is, list all the boundary points of D, where the where f has local maxima and local minima and evaluate f at these points. Then we will find out the points where f attains local maxima, local minima on the boundary, ok.

Analyze the list of list for the maximum and minimum values of f; these will be the absolute maxima and absolute minimum values of f on D. So, let us understand these steps by an example. Say we have a first example, first problem, where we have to find out the absolute maximum and absolute minimum of the function, 2x square minus y square plus 6y, region is R is equals to all those x, y where x square plus y square is less than equal to 16. That is, all the points on the circle of center 0, 0 and radius 4.

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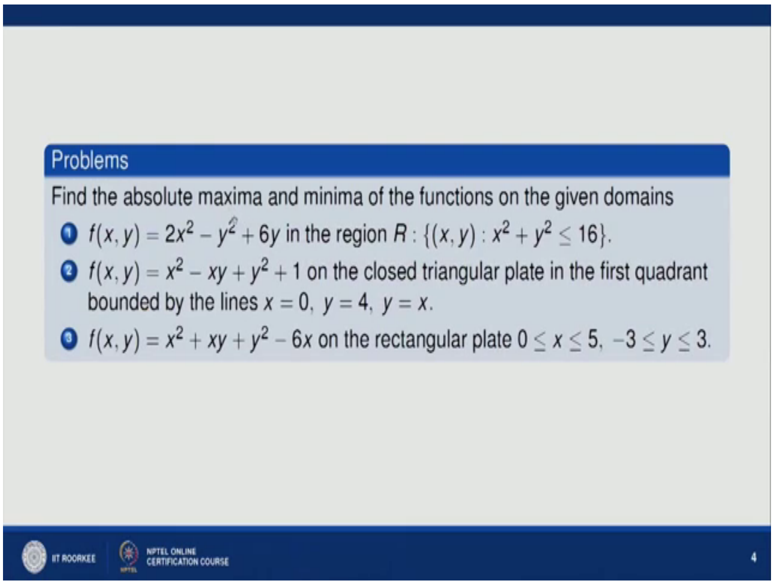

So, how can you find out maxima, minima for this.

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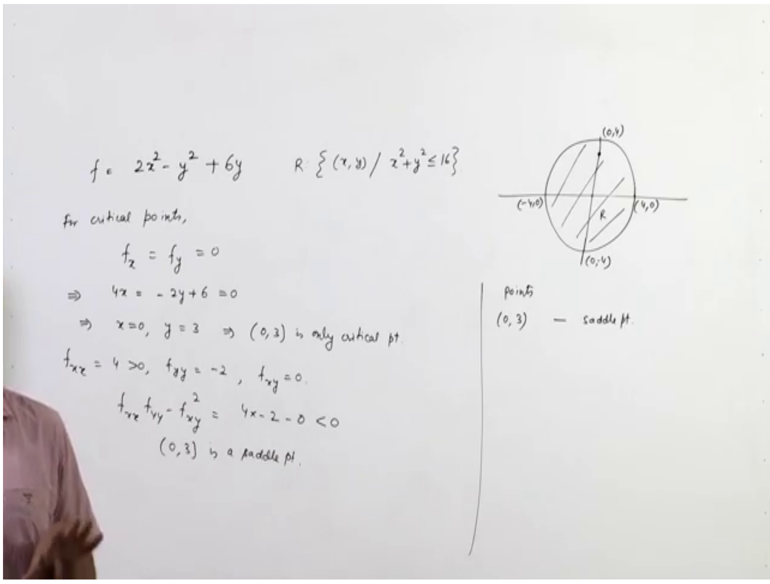

Now, here, what is f? f is, 2x square minus y square plus 6y and region is all those x, y, such that x square plus y square is less than equal to 16. So, basically region is a circle, is a circle, of radius 0 0 and center 4. So, this is R. This region is R, ok, this point is 4, 0 this point is minus 4, 0, this point is 0, 4 and this point is 0, this point is 0, minus 4, this point is 0, 4.

So, first you find out all the critical point of this function, ok. So, for critical points, what we are having? For critical points, f x is equals to f y should be 0. So, this implies, 4x is equals to minus 2y plus 6 should be 0 and this implies, x is 0 and y is 3; so that means, this implies, 0, 3 is the only critical point, ok. Now we have to check that this point is this point lies in the interior of the region R? So, this point is somewhat here; 0, 3 is somewhat here. So yes, this point lies on the interior of the region R, ok.

Now, we have to check, where this for the point of local maxima, minima or saddle point, ok. So, we find second order, partial derivatives. Say what is f xx? f xx is 4, which is positive, f yy is minus 2 ok, f xy is 0 and f xx f yy minus f xy square, if you calculate this value, then it is 4 into minus 2 minus 0 that is negative. So, this point 0 comma 3 is a saddle point, ok. So, we have points, point is 0, 3 this is the interior point and what is the nature? This is a saddle point, ok. The first point, the first critical point which we have find, is the saddle point, which is 0, 3 lies inside the region R.

Now, we move along the boundary of the region. The boundary of the region is a circle; x square plus y square is equal to 16. The boundary is; x square plus y square equal to 16.

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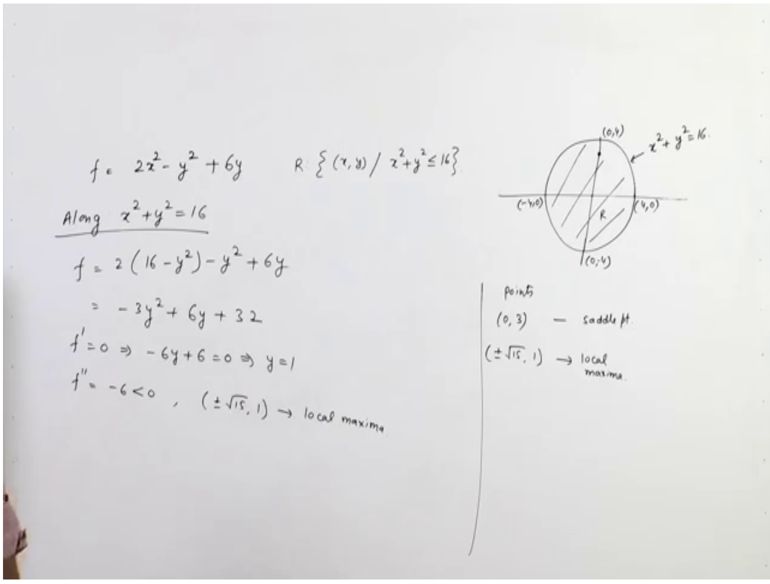

Now, we move along the boundary, ok. Along the boundary; along x square plus y square equal to 16; Along this f will be, you simply replace x square by 16 minus y square, this will be, minus 3y square plus 6y plus 32. Now this is a single variable function, you can simply differentiate; first derivative put it equal to 0 and then you can find out second derivative, to find out whether the point is a point of local maxima or local minima.

So, what is the first derivative? First derivative put it equal to 0 this implies minus 6 y plus 6 equal to 0. So, this implies, y is equal to 1 and the second derivative is, minus 6 which is negative. So, this point is a point of; is a point of local maxima, ok. Now, when y is 1; when y is 1 then x will be; then x will be plus minus under root 15 and 1. So, this point is the point of local maxima, ok. So, we have find two more point; plus minus into root 15, 1 and these point are point of local maxima, ok.

So, first we have to find an interior point; by finding the critical point, check the nature of that point, then we move along the boundary of the region; the boundary may be a circle, may be a rectangle, may be a triangle or may be some other shape ok, you, you have to move along the boundary of the region and then you have to find f on the vertices; all the vertices. Now here, you have this point 0, minus 4.

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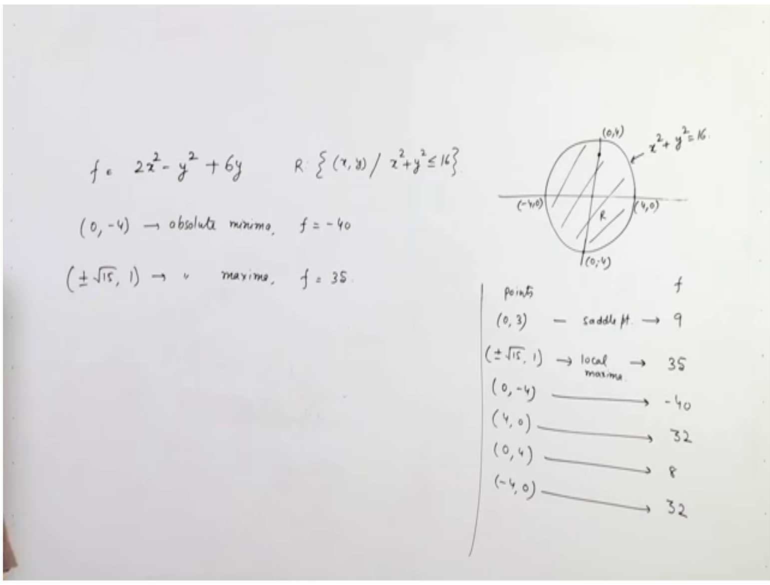

As this, where this circle intersect y axis, then 4, 0 then 0, 4 and then minus 4, 0; so, these are all the possible points, where the function may attain maxima or minima, ok.

Now, at this point, at this point, now the value of function; value of the function at this point is; you simply substitute it here, it is 2 into 15 2 into 15 minus 1 plus 6. That will be 36 minus 1 that is 35. So, f is 35 at this point, ok. Now at 0, minus 4, f is; when you substitute x as 0 and y as minus 4, so this will be simply minus 16 minus 24, ok, when you substitute y as minus 4. So, it will be minus 40 at this point; at this point, value of f will be 16 into 2 that is 32; at this point f will be minus 16 plus 24 that is 8; at this point f will be; the simply substitute x by this. So, that will remain; that will remain the same point that is 32.

So, you have listed out all the points; all the points and find out the corresponding value of the f. Now from all these values, the minimum value of f is; let us find the value of f at this point also; however, it is a saddle point. So, it will be simply minus 9 plus 18 that is 9 only, ok. Now from all this point the minima value of the f is minus 40, which is at this point. So, this point is a point of absolute minima, ok.

So, 0, minus 4; 0 minus 4 is a point of absolute minima and absolute minimum value is minus 40, and plus minus under root 15 and 1 is a point of absolute maxima which is 35. So, plus minus under root 15, 1 is a point of absolute maxima and the absolute maximum value is 35. So, that is how we can find out absolute maxima and absolute minima of a function when some region is given to us; close and bounded region is given to us.

So, you have to first find out the critical points. The interior point where function may attain maxima minima or saddle point, then move along the boundary; find out all the vertices; list out all the point, at the corresponding functional values and check from all those values where the function has absolute maxima and absolute minima. So, let us try second problem based on this.

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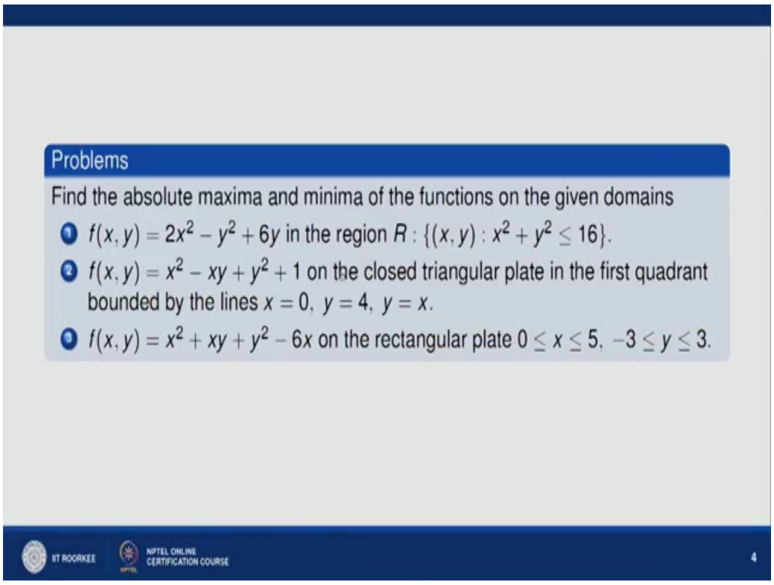

The second problem is. So, here f is x square minus xy plus, it is y square plus 1.

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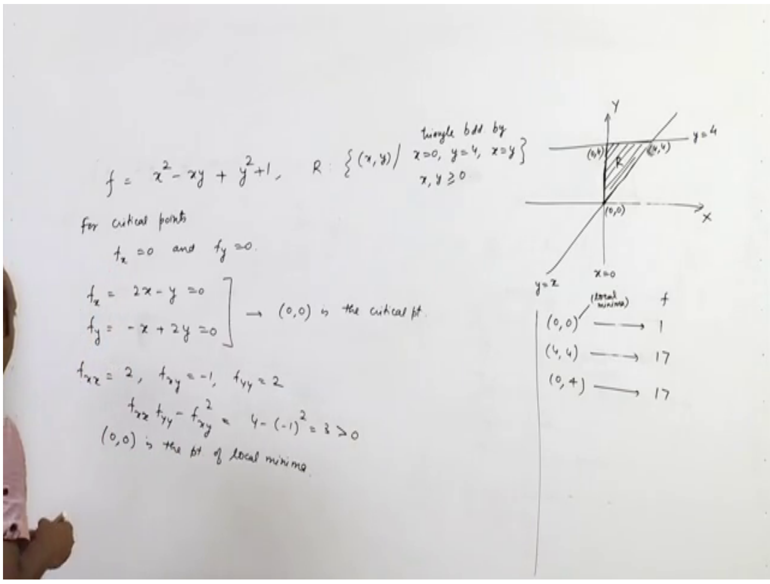

And region R is, all those x, y, such that x equal to 0 y is equals to 4 and x equals to y ; that means, a triangle bounded by, triangle bounded by these 3 lines, ok. All those x, y where triangle bounded by this and lying in first quadrant means x, y, z must be non-negative, oh z is not there. So, x, y must be non-negative. That is, the closed triangular plate in the first quadrant bounded by these 3 lines.

So, what this triangle is basically ? You see, it is x axis and it is y axis. This is x equal to 0 line ; y equal to 4 is suppose this line, this is y equal to 4, y equal to x is this line ; this is y equal to x, this is x equal to 0. So, this is the region which is in first quadrant and bounded by these 3 lines. This is the first line, the second line and the third line. So, this region is suppose R ok. So, clearly you can find out the intersection point. This is z this is this is 4, 4. This point is 0, 4 and this point is 0, 0.

So, first you can see is these 3 are vertices are 0, 0 ; 4, 4 and 0, 4, ok. These are the vertices and the value of f will be ; here the value of f is 1 ; here the value of f is it is 16 minus 16, cancel out 16 plus 1 is 17 and here the value of f will be, when you substitute x as 0, y as 4. So it is simply, again 17, ok. Now you find out critical points or the points inside the region R, where the function having local minima or local maxima, ok.

So, for critical points again f x equal to 0 and f y equal to 0. Now what is f x? f x is 2x minus y equal to 0 and f y means minus y, oh sorry; minus x plus 2y is equals to 0. So, solving these 2 equations, we directly can obtain 0, 0 is the critical point, ok. Now we have to find out the nature of this point ok. So, what is f xx? F xx is 2, f xy is minus 1 and f yy is again 2. So, this is positive and f xx into f yy minus f xy square is simply 4 minus, minus 1 square that is 3 which is again positive. So, this is positive and this is positive; this means 0, 0 the point of local minima, ok. So, that point is here, on the boundary itself and this point is a point of local minima ok. So, this one is a point of local minima ; however, this point is is on the boundary itself, ok.

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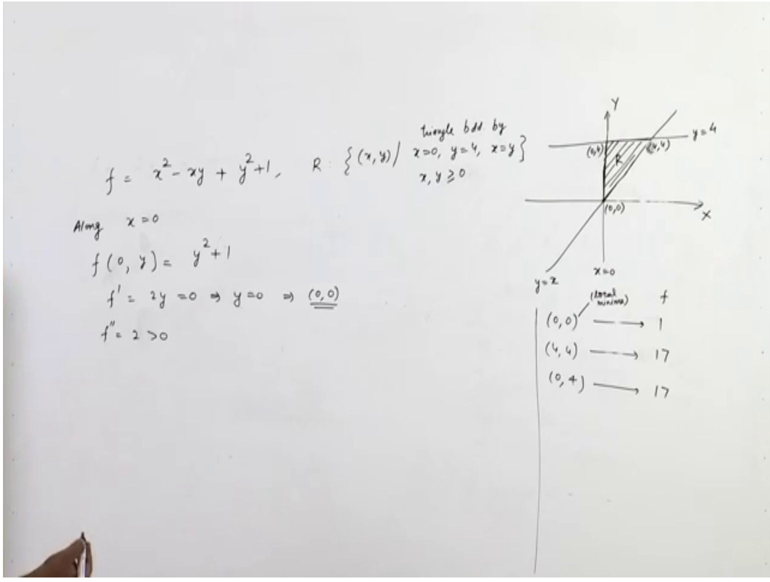

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Now, we will move along the boundary. First we move along. On the boundary they have 3 lines; x equal to 0, y equal to 4 and y equal to x. First we move along. Say x equal to 0. When you move along x equal to 0, so f at 0 comma y will be y square plus 1 and now it is a single variable function to find out maxima minima of this simply differentiate; put it equal to 0. So, derivative of this will be 2y; which is equal to 0, implies y is equal to 0. So, x equals 0 and y equal to 0 implies 0, 0 is a point, and 0, 0 we have already checked; however, 0, 0 we have already checked. You can again see from here the second derivative is 2, which is positive means a point to the point of local minima. So, that we have already deal with.

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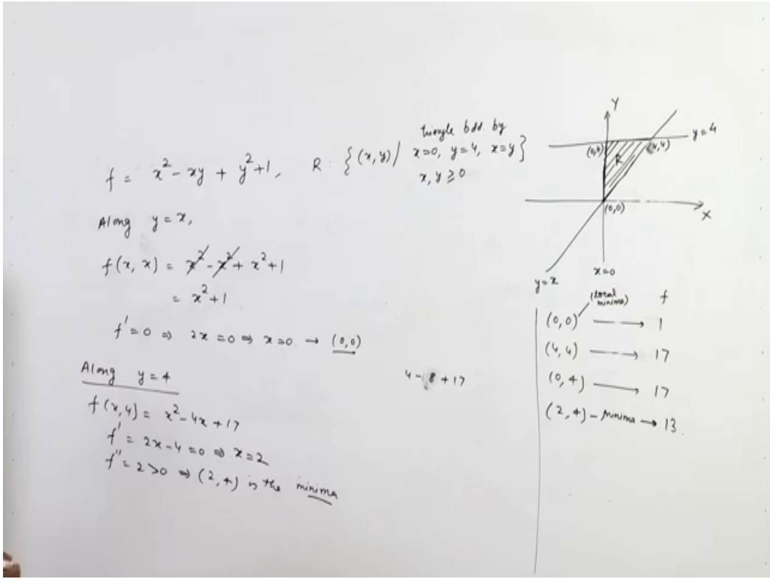

Now, we move along; now we move along, say x equals to y or y equals to x. So, at y equals to x; it would be x square minus x square plus x square plus 1. So, it is x square plus 1; again single variable function derivative, put it equal to 0. So, 2x equal to 0 implies x equal to 0. So, again 0, 0 because x equal to 0 means y equal to 0 so again we have a point 0, 0. So, that the nature of this point we have already deal with, ok. Then you move along y equals to 4; now along y equals to 4, what we are having; it is x square minus 4x plus 16 plus 1 equal to 17. We simply substitute y equal to 4 because now we are moving along one y equal to 4.

So, now put now take the derivative. It is 2x minus 4; put it equal to 0 implies, x equal to 2 and the second derivative is 2, which is positive means 2 and 4 2, 4 is a point of is a point of minima ok. So, you have to check; you have to find the value of the function at this point also; if this is point of local minima and at this point value the function will be, it is it is 4 minus minus 8 ok; 4 minus 8 plus 16 plus 1, 17. So, it is minus 4, 13 it is 13.

So, now we have listed out all the possible points, where the function may attain absolute minima or maxima on this; on this region R. So, from all these points, the value where the function attain absolute minima 1 that is at 0, 0 and the maxima will be a maximum value of the function will be 17 at these points. So, we can say, so we can say that this function, this function has absolute minimum value of f; will be 1 at 0, 0 and absolute maximum value of f is equals to 17 at 4, 4 and 0, 4. So, that is how we can find out absolute maxima or absolute minima of a function f along a closed and bounded region R.

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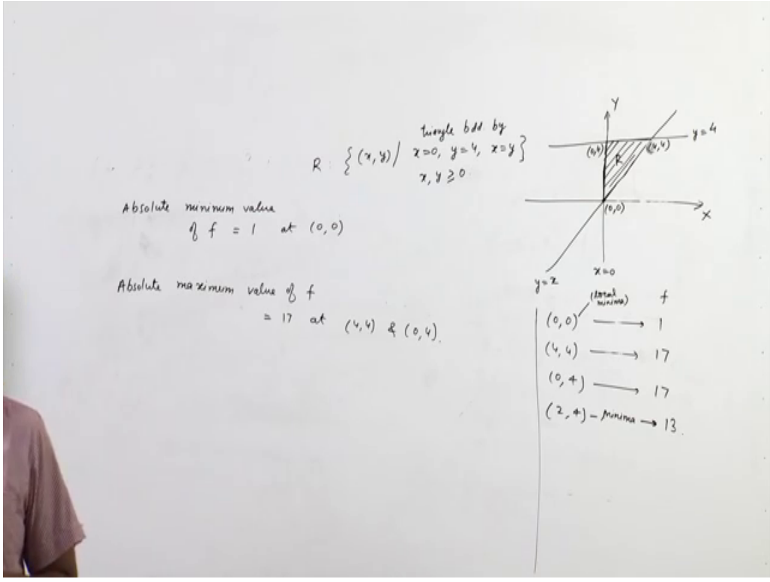

Now, in the third example, in the third problem, we have a function which we have to find the absolute maxima, absolute minima on a rectangular plate.

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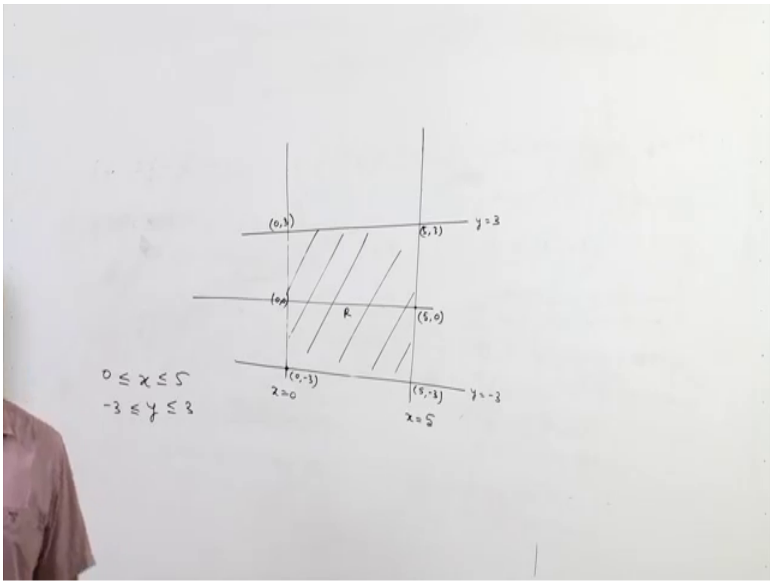

Now here, what is a rectangular plate ? So, here rectangular plate is x is less than equals to 5 greater than 0 and y is less than equals to 3 and greater than minus 3. So, x is lying between 0 and 5. This is x equals 0 and this somewhat x equal to 5 ; y is less than equal to minus 3 and greater equal greater equal to minus 3 and less than equal to 3. So, suppose y equal to minus 3 is somewhat here. So, y equal to minus 3 and y equal to 3 is somewhat here. So, this is the region R, and we have to find out absolute maxima, absolute minima of the function this with on this rectangular plate.

So, first we are having these vertices, these vertexes 0, minus 3 ; this is 5, minus 3 ; this point is this point is this is x equals to 5 and y is 0 ; this point is 5, 3; this point is 0, 3 ; this point is 0, 0. So, we have 1, 2, 3, 4, 5, 6 points. We find the functional value at these 6 points, ok. Then for critical point, we take f x equal to f y equal to 0; find out the critical point and the nature can be determined by finding second order partial derivatives at that point, ok.

Then, we move along the boundary; we move along this line y equal to minus 3; we move along x equal to 5; we move along y equal to 3 and we move along x equal to 0, all the 4 boundaries of the rectangular plate and find out the possible points where the function may attain local maxima or local minima. There we list out all those points and find out the absolute maxima and absolute minima seeing the functional value of f at the respective points. The the point where they have attained, the least value will be the will be the absolute minima value of the f, at that point and the point where the where the function at where the function has a maximum value, will be the absolute maxima of the function in that region R, ok. So, in this way we can solve this problem also.

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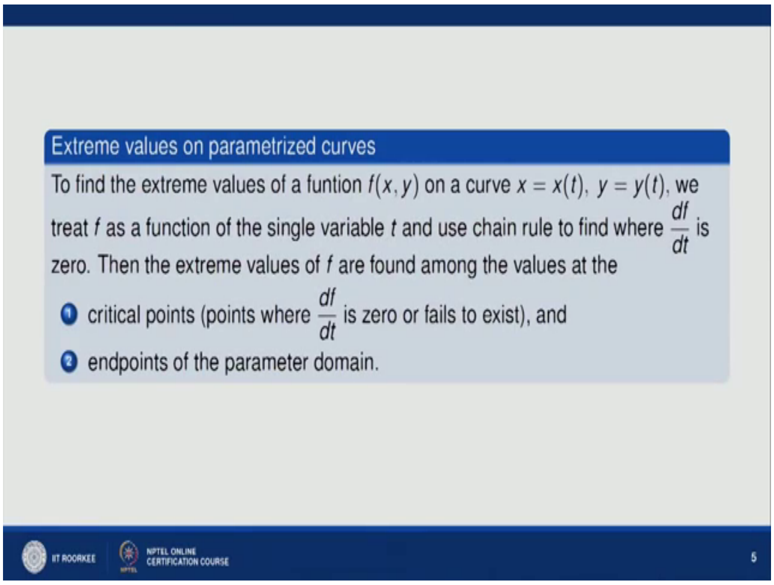

Now, let us find out extreme values on parameterized curves. To find out the extreme value of function f x, y on a curve x equal to x t; where x is a function of t and t is a parameter and y is also a function of t, we treat f as a function of single variable t and use chain rule to find where d f upon d t is 0. That is the critical points. Then, the extreme values of f are found among the values at the critical points and endpoints of the parameter domain, if the domain is bounded. So, we have to find out the endpoints also. So, in this way, we can find out extreme values on parameterized curve. Let us discuss this by an example.

The first problem is: f is equal to x square plus y square.

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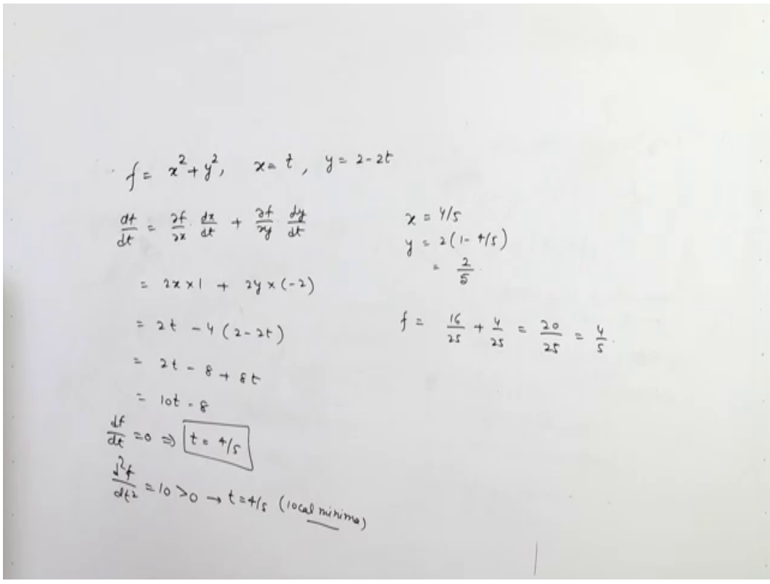

And x is given as t; function of t and y is simply 2 minus 2 t. And you have to find out the maximum, minimum value of this function on this curve. Now how can you find this? This is you can find d f upon d t, which is del f upon del x into d x upon d t plus del f upon del y into d y by d t. This is by chain rule, ok. So, it is 2x d x by d t is 1 plus 2y this is minus 2, x is t and y is 2 minus 2t. So, this is 2t minus 8 plus 8t. So, that is 10t minus 8 and d f by d t equal to 0 implies t is equals to 4 upon 5, ok.

Now, the second derivative second derivative respect to t. Second derivative respect to t is simply 10 ; which is positive. So, this point is a point of local minima and local minima value is, to find out the local minima value you simply substitute t equal to t equal to 4 by 5 here. So, you can get x and y; substitute this x and y over here. So, you can find out a local minimum value of this function f. So, what will be x? x will be simply 4 by 5 ; y will be 2 into 1 minus 4 by 5 ; that is simply 2 by 5, ok. So, function value will be 16 by 25 plus 4 by 25 that is 20 by 25 or 4 by 5. So, this is the local minima value of this function at this point.

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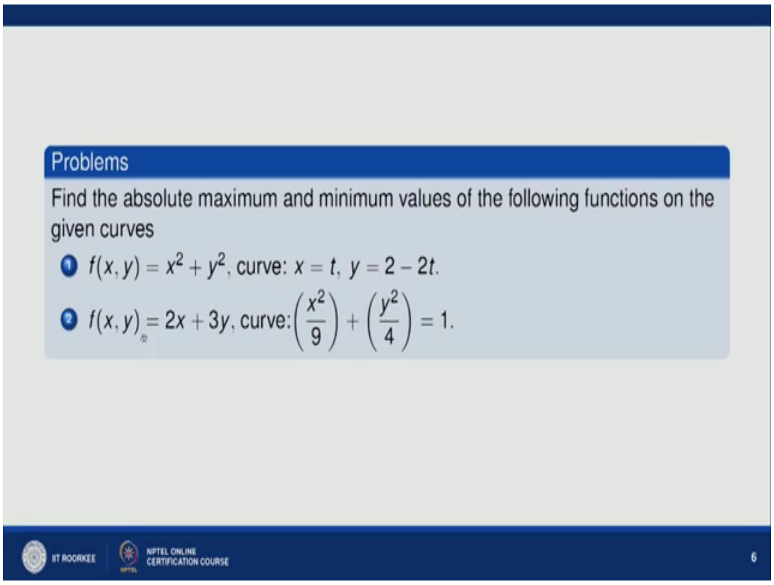

Similarly, if you deal with this problem, function is 2x plus 3y and curve is given by this expression. So, here you can easily find out the parametric representation of this curve. What is the function here?.

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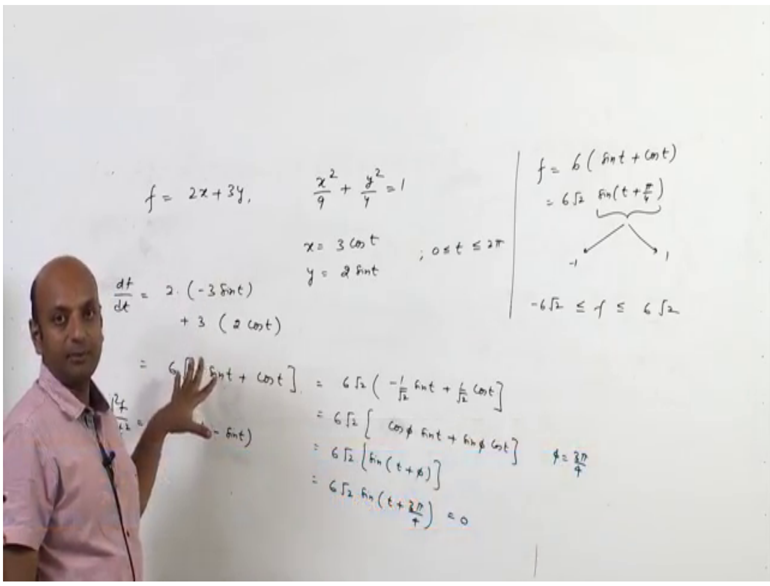

Function is, 2x plus 3y and the parametric curve is x square by 9 plus y square by 4 is equals to 1. So, you can take x as 3 cos t and y as 2 sine t, and t is varying from 0 to 2 pi. So, what are the entire ellipse should be covered? Ok. So, this is a parametric representation of this curve. Now you can simply find del f d f by d t, which is simply del f by del x into d x by d t plus del f by del y into d y by d t and it is simply 6 times minus sine t plus cos t and you simply differentiate it respect to, that you have that you have simply find out. So, this this is the nothing but you can easily write this as, 6 by 6 under root 2 into minus 1 by under root 2 sin t plus 1 by under root 2 cos t, and this will be, 6 under root 2 this is this is say sin phi or cos phi sin t plus sin phi cos t.

And this is 6 under root 2 sin t plus phi. So, what should be phi? Sin phi is 1 by under root 2 and cos phi is minus 1 by under root 2, that is phi lies in second coordinate. So, that will be simply, phi minus pi by 4 that is c phi by 4. So, phi will be 3 phi by 4. So, you can simply take it is 6 under root 2 sin t plus 3 phi by 4 now the maximum value, ok. So now, you can simply see that you guys simply find out where it is 0, ok. You can find it directly also. When you when you put this equal to 0; that means, tan t equal to 1, and you and list out all the points where this is 0; you can find second derivative, ok. What is second derivative of this function? Second derivative is simply 6 minus sin t minus cos minus sin t minus cos t minus sin t.

So, what I want to say, you can simplify you have find d f by d t; you simply put it equal to 0, this equal to 0 or this is equal to 0. Find out all the t all the values of t where this is equal to 0, and then put it in the second derivative, to find out the to check whether that put the point of local maxima, local minima and then you can find out the functional value at that point, otherwise this question can be solved directly also. How? You see, what is f? f is you simply substitute this over here and y over here.

So, this would be 6 times sin t plus cos t and this is 6 under root 2 sin t plus phi by 4 and the minimum value of this is 1 oh sorry maximum value is 1 and minimum value is minus 1. So, f will lying between 6 under root 2 to minus 6 under root 2 from minus 6 under root 2 to 6 under root 2. So, minimum value of this function is this and a maximum is 6 under root 2, ok. So, this is simply an illustration that how you can solve such type of problems using derivative so.

Thank you very much.