Multivariable Calculus
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Lecture - 14
Extreme Values-I

Hello friends. Welcome to lecture series on multivariable calculus. So, today we will discuss on extremum values; that is, how to find maximum or minimum values of a function of several variables. So, let us first recall with a function of single variable.

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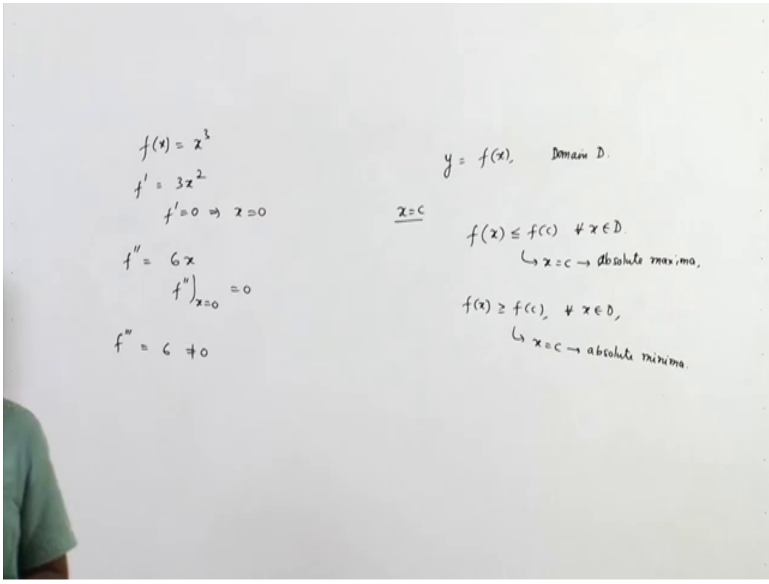

So, we know that if we have a function say y equals to f x, a single variable function y equal to f x. And domain is D supposed of this function. Then a point x equal to c is set to be a point of absolute maxima of this function, if f of x will be a less than equals to f c for every x in D ok.

If this hold, then we say that x equal to c is a point of absolute maxima. Similarly, if f x is greater than equals f c for every x and D then x equal to c recall as absolute minima. Now how can we check whether points are point of local maxima or local minima? So, we have a second derivative test. So, x equal to a is a point of local maxima, if f dash a equal to 0 and f double dash a is less than 0. We already know this result ok if a is the point of local maxima, then the first derivative at a will be 0, and second derivative at a will be less than 0.

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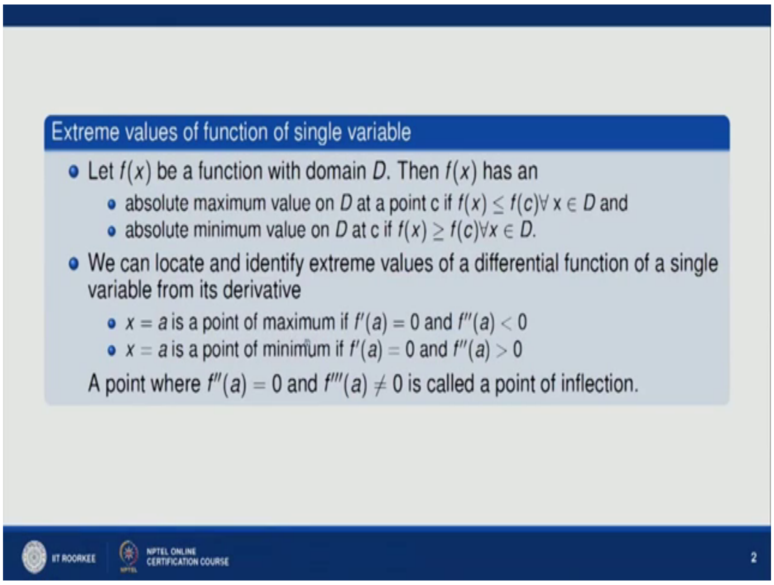

We are assuming that the function is differentiable. x equal to a is a point of local minima if f dash a equal to 0, and f double dash a is greater than 0. A point where f f double dash a equal to 0, and f double f triple dash a is not equal to 0 is called point of inflection ok. Say we have this example f x equal to x cube. So, when we take first derivative it is 3 x square, and f dash equal to 0 implies x equal to 0. So, x equal to 0 is a critical point basically.

Now, to check whether this point is the point of maxima minima or point inflection, we find second derivative. Now second derivative is 6 x, and second derivative at x equal to 0 is again 0. Find third derivative third derivative is 6, which is not equal to 0 at x equal to 0. So, this means this point x equal to 0 is the point of inflection.

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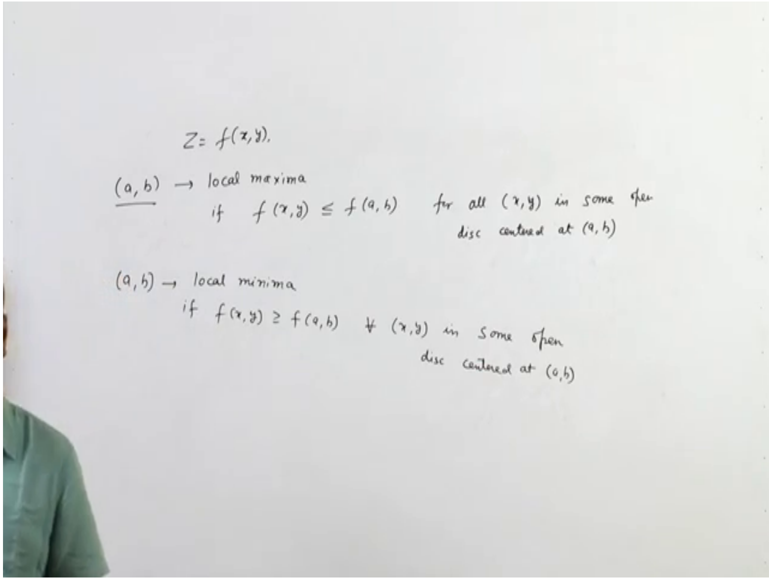

Now, let us come to maxima minima of 2 variable functions. Say you have a function of 2 variable ok. So, a point a comma b will be a point of local maxima, will be a point of local maxima if f of x y will be less than equals to f of a b, for some disc for all x y yeah, for all x y in some disc centered at a comma b ok.

You take a point a comma b and there exists a disc. So, centered a comma b, and if for all x y belongs to that disc f x y is less than equal to f a b, then we say that a comma b is a point of local maxima. Similarly, this a b is a point of local minima if f x y is greater than equals to f a b for all x y in some open disc centered at a comma b. It must be open disc. So, if this if this inequality hold for every x y in some open disc centered at a comma b, then we say that a comma b is a point of local minima ok.

So, we first have first derivative test. What is the first derivative test for local extremum values?

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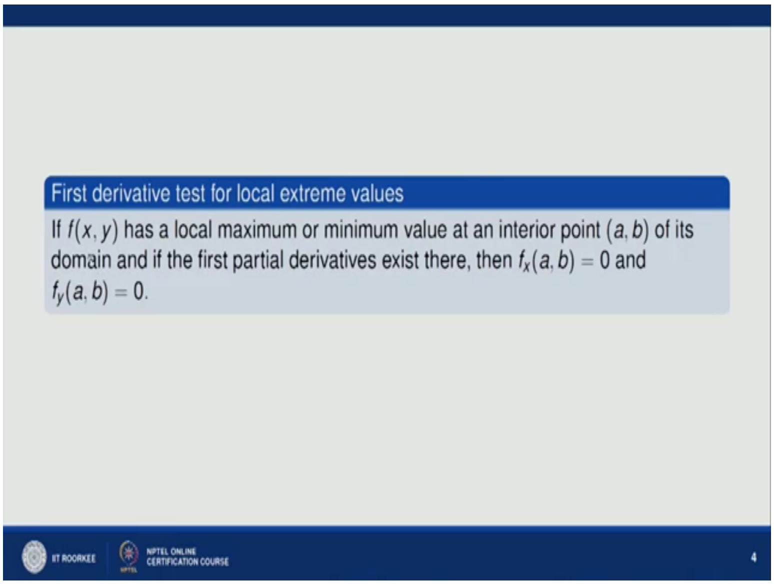

So, if f x y has a local maxima or minimum value, at an interior point a comma b of it is domain, and if the first partial derivative exist there, then f x at a b is equals to f y at a b will be 0 ok. Like in the single variable function, a function is differentiable, and x equal to a is a point of local maxima or local minima then f dash a will be 0 ok. In a similar way, here for the 2 variable function if the function if a point a comma b; which is interior point is a point of local maxima local minima, and it is first order partial derivative exist at this point, then f x at a b is equal to 0 and f y at a b is also equal to 0.

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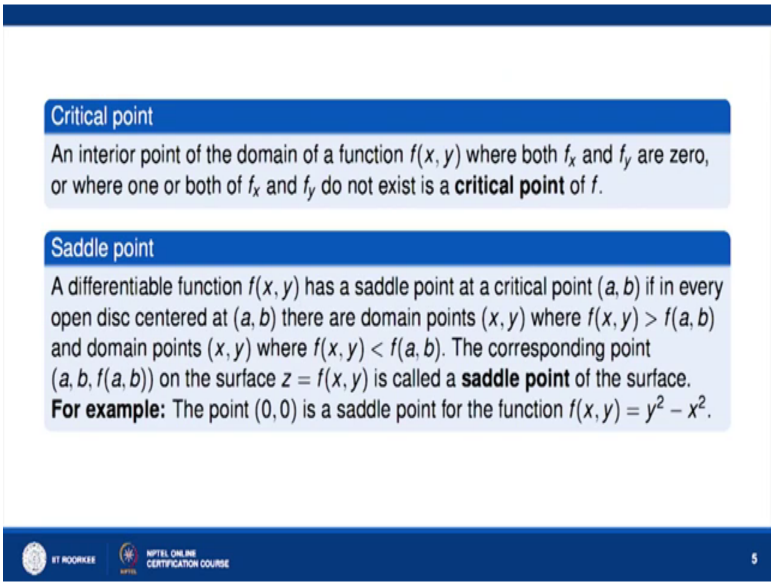

Now, what is critical point, and interior point of the domain of the function f x y, we are both f x and f y are 0, or we are one or both of f x or f y do not exist is called the critical point. So, basically critical points of point, where either f x equal to f y equal to 0, or f x or f y do not exist. Now next is saddle point. A differentiable function f x y has a saddle point at a critical point a comma b, if in every open disc centered at a b there are points x y where f x y is greater than f a b and x y, where f x y is less than a b. This point is called saddle point of the surface.

So, in order to understand saddle point, let us discuss this example. So, f x y is suppose y square minus x square.

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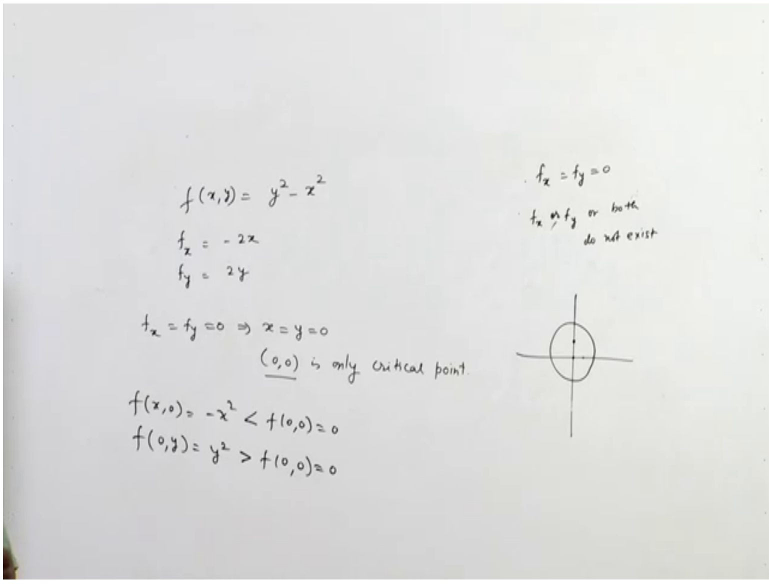

So, first of all saddle point is a critical point. So, first you find the critical point of this function. For critical point we have 3 conditions ok. We have either f x equal to f y equal of 0, or f x or f y or both do not exist. So, the point where these 2 result these 2 conditions hold this or this condition hold, we say that point as critical point. Now how to find the critical point of this function is clearly differentiable? So, you can find the f x as minus 2 x and f y as 2 y, and f x equal to f y equal to 0 implies x equal to y equal to 0. So, the point is 0 0 0 0 is the only critical point.

Now, this point is a critical point ok. Now if you take 0 comma 0, and take any neighborhood of 0 comma 0, I mean any open disc centered at a centered at 0 comma 0 any open disc. If you take any open disc centered at 0 comma 0. So, there are some point which are on the x axis, and there are always some point which are on the y axis ok. So, f at x comma 0 will be for this function will be minus x square which is always less than f 0 0, which is 0 ok. There are there is always some point on the x axis where the value of the function is less than 0. And if you take on the y axis it is 0 comma y which is y square here, and it is always greater than f 0 0 which is 0.

If you take a point on the y axis, there are some points where the value of the function is greater than 0 ok. No matter which open disc you are taking, whatever for every open disc centered this point they are always some point on the x axis, and they are always some point on the y axis such that such that at some point this value is less than 0 and some point this values greater than 0. So, this means this point is a saddle point ok.

Now, if you find f x in f y ok, after a find the critical points how can we check that a point is a point of local maxima I mean local maxima local minima or saddle point. So, what is the second derivative test for 2 variable functions?

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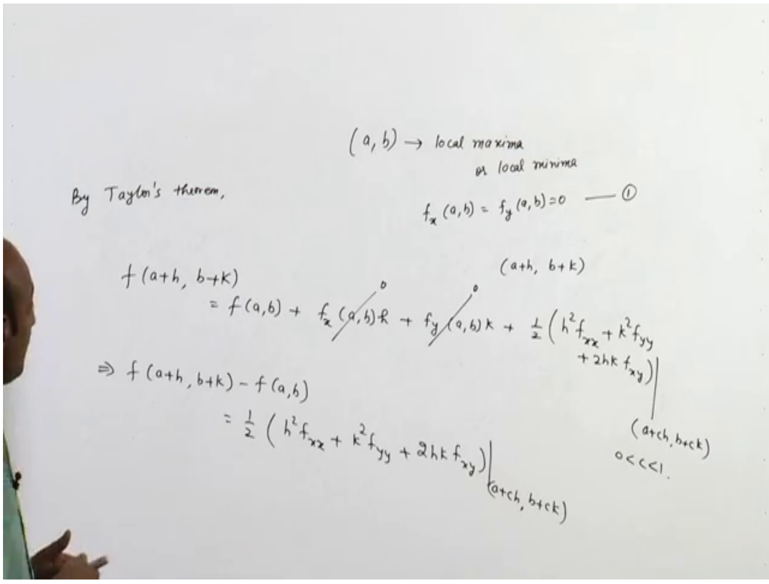

So, let us suppose a comma b is a point of local maxima or local minima. So, if it is a point of local maxima and local minima, and the first and second of partial derivative exist, then f x at a comma b will be equals to f y at a comma b will be equals to 0. This is by the first derivative test of local maxima local minima ok.

Now, take a point in the in the open disc of centered at a comma b, or take a point say a plus such an b plus k, which is near to a comma b. Take a point a plus such and b plus k which is in the neighborhood of I mean which is in some open disc centered at a comma b. You can always find such point in the open disc centered at a comma b ok. Now by the Taylor's theorem for 2 variable functions, we know that f of a plus h b plus k will be equals to f a b plus f x at a b into h plus f y at a b into k plus 1 by 2, h square f x x plus k square f y y plus 2 h k f x y at a plus c h and b plus c k, where c lies between 0 and 1 .

So, this is by the Taylor's theorem of 2 variable functions ok. So, f of a plus h b plus k is equal to this result. Now this is 0, because this point is a point of local maxima and local minima and this is also 0. So, what you obtain from here. So, this implies f of a plus h b plus k minus f a b will be equals to 1 by 2, h square f x x plus k square f y y plus 2 h k f x y at a plus c h and b plus c k ok. Now to see whether this point to the point of local maxima local minima or saddle point. We have to see the sin of this. If a comma b is a point of local maxima suppose ok.

If a, a b is a point of local maxima; that means, that means the value of this the value of function at this point will be less than this. Because, because a comma b is a point of local maxima ok. So, the value of function at this point will be less than this. So, this value will be less than 0. And hence this must be less than 0. If a b is a point of local minima, then in the in the point at the point near to a b the value of the function will be more than value is a function at a b. So, the difference will be greater than 0 ok. So, that means, this must be greater than 0 ok. And a saddle point for some h k, this will be less than 0, and for some other values of h and k this will be less greater than 0, for saddle point.

For saddle point for some h and k because h and k you can vary in that open disc they may be positive and they may be negative for. For those for some h and k this value may be positive and some other values of h and k these values maybe negative for saddle point ok. Hence this values. Now we have to only see the sin of this at this point. So, this will guarantee whether up whether a points of point of local maxima local minima or saddle point.

So, how can we say for this point? I mean this is a function.

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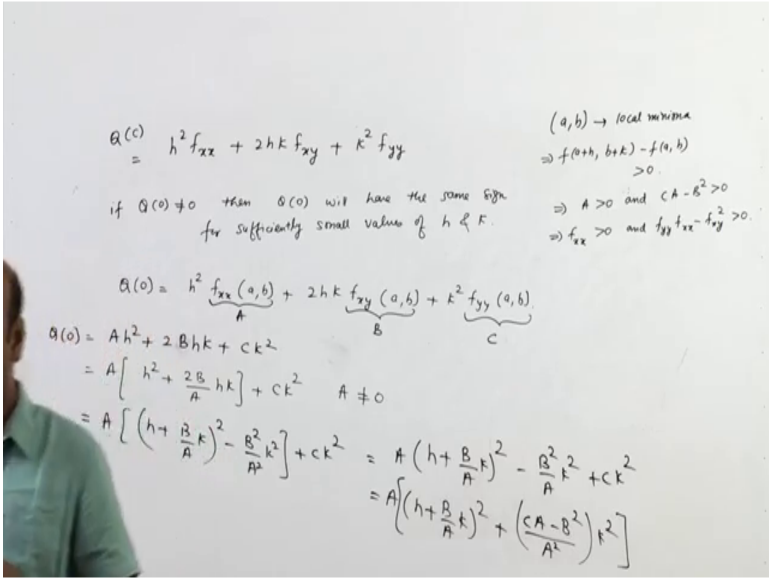

Now, let us let us their this term as say q c, because it involved c. So, let us call it q c which is h square f x x plus 2 h k f x y plus k square f y y at this point. Now if q c is not equal to 0, then q c will have the same sign for sufficiently small values of h and k .

If q c if Q 0 if Q 0 yeah, Q 0 you can say Q 0 take say c a 0. If Q 0 is not equal to 0, then this will have the same sign or Q 0, you can say Q 0 Q 0 will have the same sign, for a sufficiently someone values of h and k. You will always take the small values of h and k for which q c will have the same sign, no matter what, what c you are taking ok. You can take Q 0. So, Q 0 will be h square f x x at a comma b plus 2 h k f x y at a comma b, plus k square f y y at a comma b, because when c is 0 the point will be nothing but a comma b.

Now, we will simply see the sign of this function I mean this value this expression is a sign of this expression is positive. No matter I mean for some small values of h and k ok. If the sign of this expression is positive for small values of h and k, then this means a comma b is a point of local minima if this is negative. This means a comma b is a point of local maxima. So now, let us let us find out the sign of this how can we do this let us see. So, let us suppose this is a h square plus 2 h k 2 b h k plus c k square.

Suppose a is a suppose this is a capital A, suppose this is capital B, and suppose this is capital C. Now make perfect square. You take a common assuming A is not equal to 0. So, this will be h square plus 2 B by A h k plus C k square. Assuming a is not equal to 0.

So, this implies this is A h plus B by A into k whole square minus B square by A square into k square plus C k square. Which is further equal to this is A into this A into h plus b by a into k whole square, minus B square by A into k square plus C k square.

So, this will be equal to A h plus B by A into k whole square plus this is C A minus B square by A into k square. So, if you take a common from the entire expression, if you take a common from the entire expression. So, this will be a square. Now what is what is A? A is f x x at a comma B. What is B? B is this value and c is this value.

Now, no matter whatever h and k you are taking in some open disc centered at a comma b. If you what if you are calling a comma b is a point of is a point of local minima say, if this is a point of local minima then this implies f of a plus h b plus k minus f of a b will be will be greater than 0. Because this is the point of local minima; that means, in the in the in the point near to a comma b the value of the function will be more than this value. So, difference will be positive.

Difference will be positive means this must be positive, because this is equal to this value ok. So, this must be positive. Now this must be positive, for this must be positive no matter what h and k you are taking. So, A, A must be positive. This implies a is positive. And this must be positive. And this implies f x x at that point must be positive, and what is c? C is f y y minus f x x minus f x y square must be positive. So, this is the first condition for local minima ok.

Now, if you if you check for local maxima for local maxima if you are calling a b as a point of local maxima.

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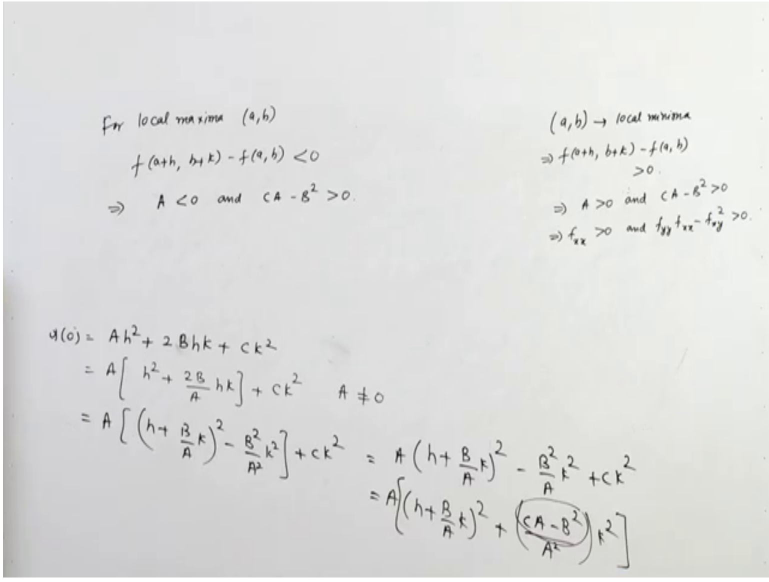

Then f of a plus h b plus k minus f of a b must be less than 0. And this implies this quantity must be less than 0. For this quantity must be less than 0 what we are having ah? We are having a must be negative and this must be positive. Because positive plus positive is always positive, and multiplied by negative values always negative. If you take this value as negative value, then for some h and k this maybe positive and for some h and k this maybe negative.

So, we cannot guaranty that whatever h and k we are taking, this will always be negative. So, in order to guaranty this, we take a less than 0 and c a minus b square must be greater than 0. That is the first condition. Now if this value, if this value is negative. Whatever a maybe A may be, positive or negative it hardly matters. If this value is negative, then for some h and k, this minus this, may be negative and for some h and k this and this, this minus this may be positive. So, the in that case it will be a saddle point. Because for some value some for some h and k this is positive, and some for some h and k this is negative. So, a comma b will be a saddle point in that case.

And if it is 0, if it is 0 so, test will be in conclusive. We have to go for the higher order test to find out the nature of a and b ok. So, that is how we can find out whether a point x a b is a point of local maxima local minima or saddle point ok. So, let us discuss first problem based on this.

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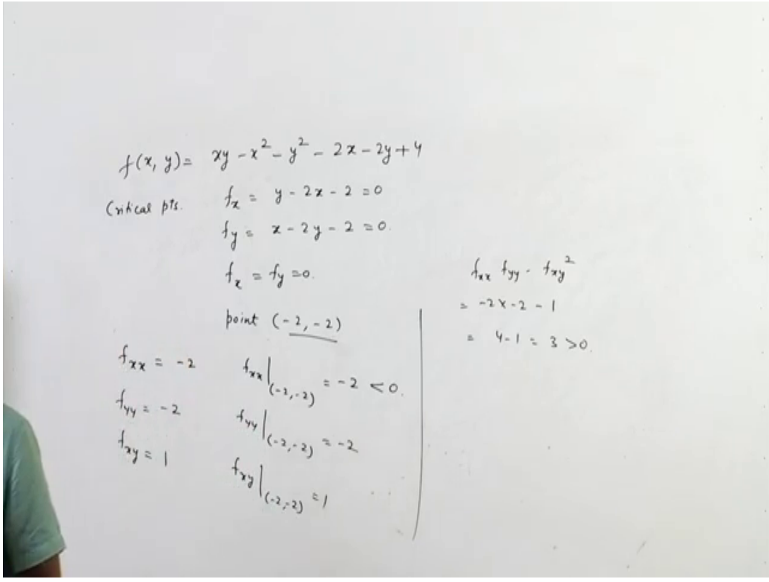

So, first problem is, it is x y minus x square minus y square, minus 2 x minus 4 y plus 4, minus 2 y plus 4. Now you have to find local maxima local minima and saddle point if exist for the following functions.

So, first find critical points. So, how to find critical points? This function is very nice function this is differentiable. So, you can find f x, f x is y minus 2 x minus 2 equal to 0. And find f y f y is x minus 2 y minus 2 equal to 0. You have to find that point where f x equal to f y equal to 0 both are 0. So, that will be a simply point of intersection of these 2 lines, these 2 equations ok. So, what is the point of intersection of these 2 equations? This you can solve so, point will be minus 2 minus 2 you very easily, check you can check also. So, point would be a minus 2 minus 2 yeah it is clear.

Now, we have to check where this point to the point of local maxima local minima or saddle point ok. So, we first find f x x, what is the f x x? F x x is simply you differentiate partially this again respect to it is a minus 2. So, f x x at point minus 2 minus 2 is minus 2 it is a negative. Now what is f y y f y y is again minus 2. So, f y y at this point is minus 2. And what is f x y? F x y is 1. So, f x y at this point is also 1.

Now, what is what is f x x into f y y minus f x y square, this is equals to this is minus 2 into minus 2 minus 1. That will be 4 minus 1 it is 3 it is positive. So, f x x is negative, and this value is positive; that means, this point the point of from this result, this point will be a point of local maxima ok.

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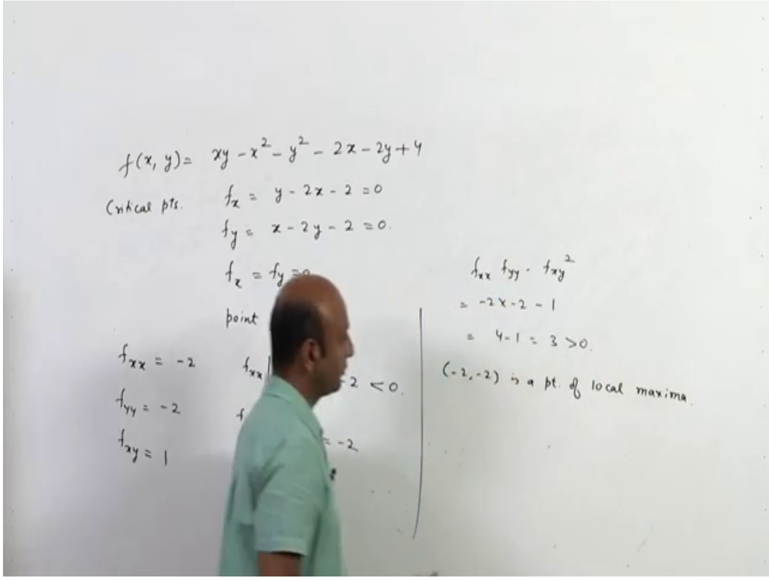

So, point minus 2 minus 2 is a point of local maxima ok. Now let us try to solve the second problem.

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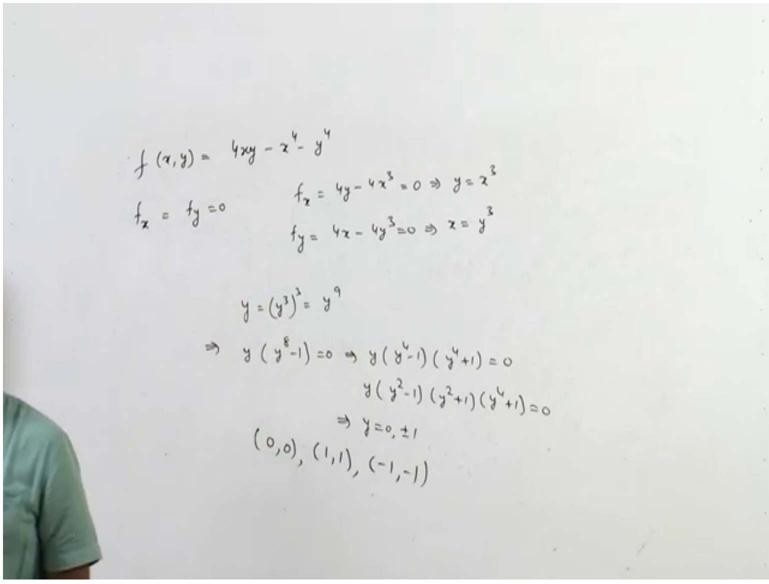

The second problem is f x y is equal to this is 4 x y minus x raise to power 4 minus y raise to power 4.

So, what is f x for critical point f x equal to? F y equal to 0 so, this implies first you find f x. What is the f x ? It is 4 y minus 4 x cube which is equal to 0 implies y equal to x cube. Now f y f y is equals to 4 x minus 4 y cube equal to 0 implies x equal to y cube. Now we have to solve the values of x and y from these 2 equations ok. So, you can substitute x equal to y in this equation. So, what you will obtain y is equals to y cube whole cube, that is y raise to power 9, and this implies y into y raise to power 8 minus 1 will be 0. This implies y y raise to power 4 minus 1 y raise to power 4 plus 1 equal to 0.

Y into y square minus 1 y square plus 1 y raise to the power 4 plus 1 equal to 0, and this implies y equal to 0 or plus minus 1 0 and plus minus 1 no real y here no real y here. So, when y is 0 x is 0. So, points are 0 comma 0, when one it is one, and when minus 1 it is minus 1. So, these are the 3 critical points. Now we have to check which point the point of local maxima minima or saddle point ok.

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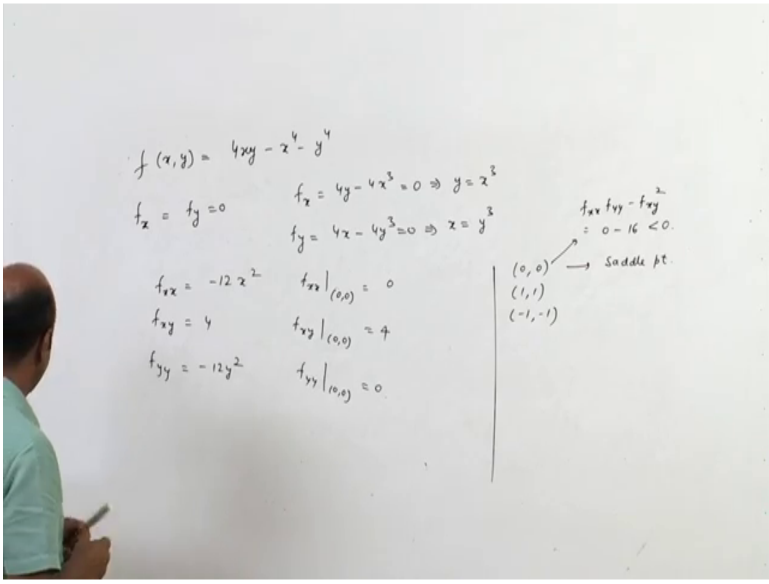

So, we are having 3 points here 0 0 1 1 and minus 1 minus 1. So, find f x x, what is f x x? It is minus 12 x square. And f x x at first we check for 0 0 ok. First, we check for 0 0, at 0 0 it is 0 what is f x y? F x y is 4.

So, f x y at 0 0 is also 4, and what is f y y? F y y is minus 12 y square which is f y y at 0 comma 0 is 0. So, you find f x x into f y y minus f x y square, if you find f x x for this point, if you find f x x into f y y, minus f x y square. So, this is 0 minus 16, which is less than 0 this implies, this point is a; this point is a saddle point.

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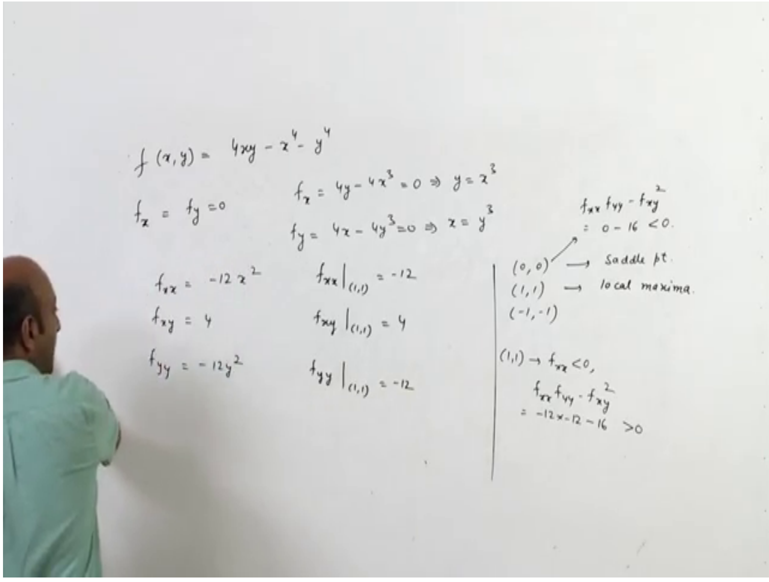

Now, check for 1 comma 1. At 1 comma 1 this is minus 12, this remains same, and this value is again minus 12. So, at 1 comma 1, f x x is negative. And what about this value? This value is minus 12 into minus 12 minus 16.

Of course, it is positive. This is negative this is positive that mean this point the point of local maxima. Now again check for minus 1 minus 1. So, what is f x x at minus 1 minus 1? It is remain 12 for minus 1 minus 1 it is again 4.

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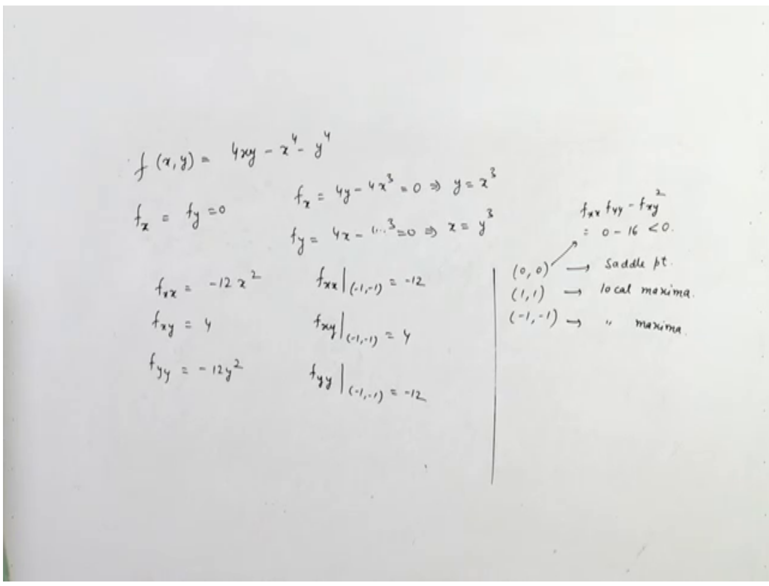

For minus 1 minus 1 it is minus 12. So, the analysis remains the same, it is negative and f x x into f y y minus f x y square is positive. So, this point is again a point of local maxima. So, in this way we can find out the nature of the points, nature of the critical points. So, similarly we can also solve question number 3 and question number 4 to find out which point the point of local maxima local minima or saddle point.

So, thank you very much.