Multivariable Calculus

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Lecture - 13
Tangent Planes and Normal Lines

Hello friends. So, welcome to lecture series on multivariable calculus. So, we have already discussed so many properties of several functions in this lecture we deal with tangent planes and normal lines how can you find out tangent plane to a surface and a normal line at a point ok.

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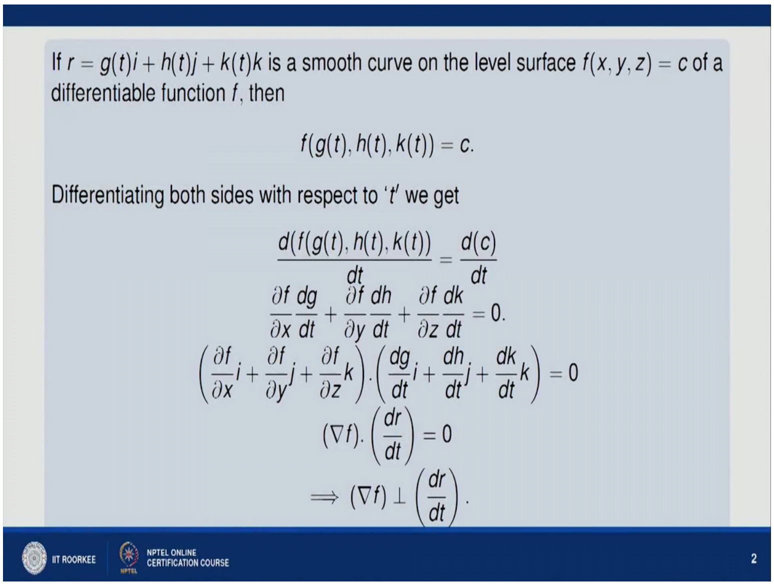

So, first let us discuss this proof let r be given by a g t i cap plus h t j cap plus k t k cap is a smooth curve on a level surface f x y z is equal to c on a differentiable function f ok.

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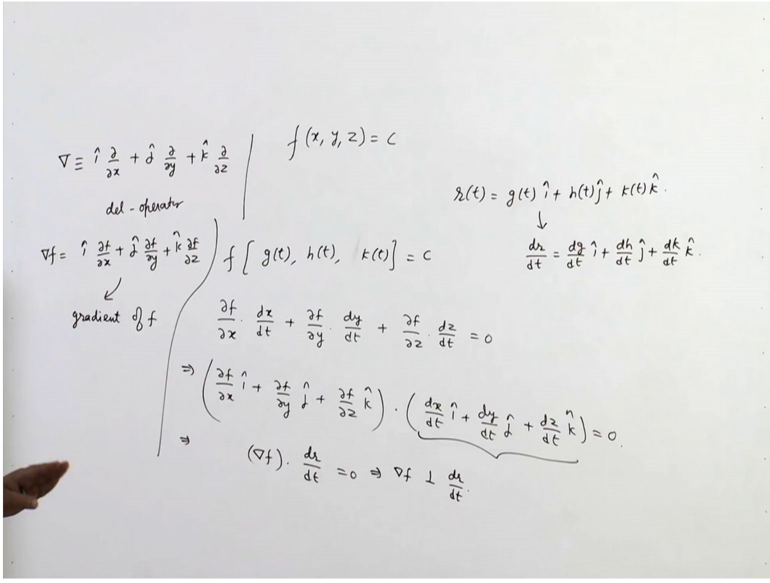

So, suppose you have some surface f x y z equal to c suppose r t which is given by g t i cap plus h t j cap plus k t k cap ok, say this smooth curve passes through this surface. So, what will be the value of function on this surface? So, where f on this surface will be given by f you simply replace x by a g t y by h t and z by k t. So, what you will obtain it is g t h t and k t k t is equals to c.

Now, you differentiate both sides respect to t when you differentiate both sides respect to t what will obtain. Now, we will apply a chain rule on this side this is del f upon del x because f is a function of x y z and x y z are intern are the function of t variable t. So, what you will obtain del f by del x into d x by d t plus del f by del y into d y by d t plus del f by del z into d z by d t which is equals to 0. So, I differentiate both sides respect to t ok. Now this can be written as del f by del x i cap plus del f by del y j cap plus del f by del z k cap and dot with d x by d t i cap plus d y by d t j cap plus d z by d t k cap is equal to 0.

So, basically what is d r by d t d r by d t is same as d x by d t is same as d g t by d t because x is simply g t and y is simply h t z is k t. So, d x by d t is simply d g by d t ok. Now what this is? This we call it we call this as gradient of f, we call this as gradient of f what is what is del operator del is i cap del by del x plus j cap del by del y plus k cap del by del z, this operator is called as del operator and del of f will be i cap del f by del x plus j cap del f by del y plus k cap del f by del z ok.

So, this is simply this thing and we call it gradient of f ok. So, this is this we call as gradient of f dot with now what this is now when you find d r by d t of this equation of this vector. So, d r by d t will be d g by d t i cap plus d h by d t j cap plus d k by d t k cap now. So, this is same as this vector because d x by d t is nothing, but d g by d t because x is nothing, but g.

So, we can say that this vector is nothing, but d r by d t and this equal to 0 ok. Now this dot part is 0 this means the angle between them is dot product a dot b is simply mod of a mod of b cos of angle between them and if cos of angle between them that is cos theta is 0. This means theta is pi by two; that means, this vector is perpendicular to this vector. So, this implies gradient of f is perpendicular to d r upon d t.

Now, what d r upon d t represent d r upon d t represent velocity vector velocity vector of this smooth curve or we can say tangent at a point t d r upon d t; I mean a slope, sorry, which we can say a slope ok. Now this is perpendicular to del f; that means, del f is normal to a surface because this is basically perpendicular to a tangent and if this is perpendicular tangent this means del of f give normal to a surface if you have this type of surface this is f basically of 3 variable x y z then gradient of f at a point x y z simply give normal to the surface ok. So, we can say that at every point along the curve.

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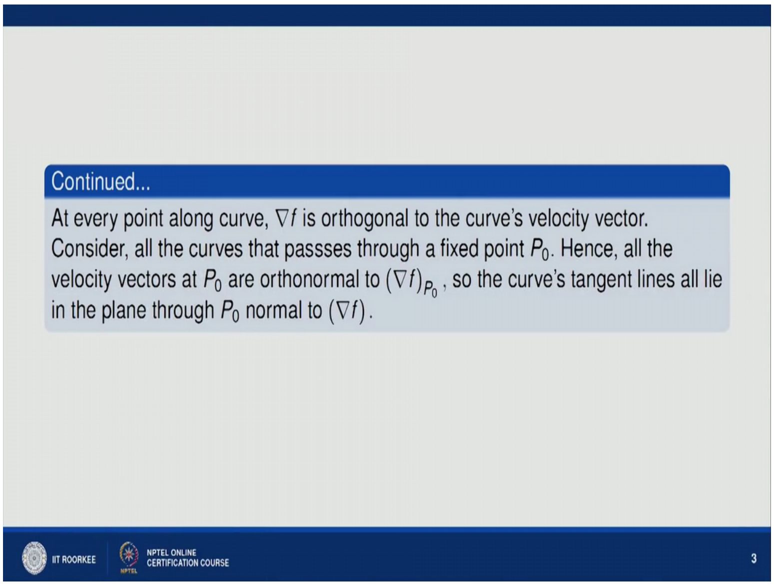

Gradient of f is orthogonal to the curves velocity vector if we consider all the curves it passes through a fixed point p naught hence all the velocity vector at p naught are orthogonal to gradient of f at p naught. So, the curves tangent lines all line the plane through p naught is normal to gradient of f ok.

So, what gradient of f gives at a point on the surface gradient of f simply gives a vector normal to a surface now suppose you want to find out a tangent plane at a point x naught y naught z naught. So, how will you find now you have a surface you have a some surface f x y z is equal to c and at some point x naught y naught z naught you want to find out a tangent plane now the tangent plane we know that gradient of f at a point x naught y naught z naught is normal to a surface. So, it will be a normal to a tangent plane also. So, direction ratios of the normal to at I mean direction ratios of normal to a tangent plane are del f del f by del x at p naught del f by del y at p naught and del z by del f by del z at p naught ok.

Now, suppose you want to find out a normal line at a point x naught y naught z naught now this is a surface and you want to find out a normal line. Now gradient of f is normal to a surface at p naught. So, gradient of f is also normal and normal line is also perpendicular to the surface at that point. So, gradient of f will be parallel to I mean I mean normal line will be parallel to gradient of f at p naught now how can. So, these are geometrical representation if we have a surface this ok.

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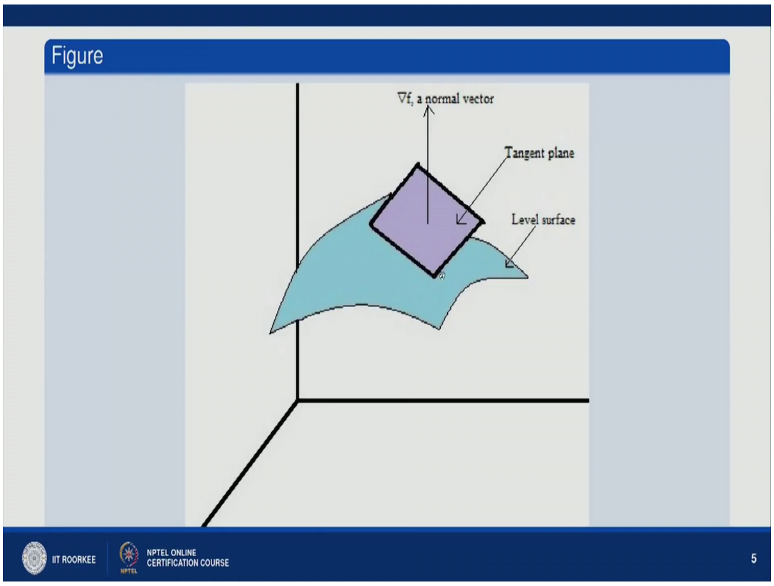

This is a tangent plane.

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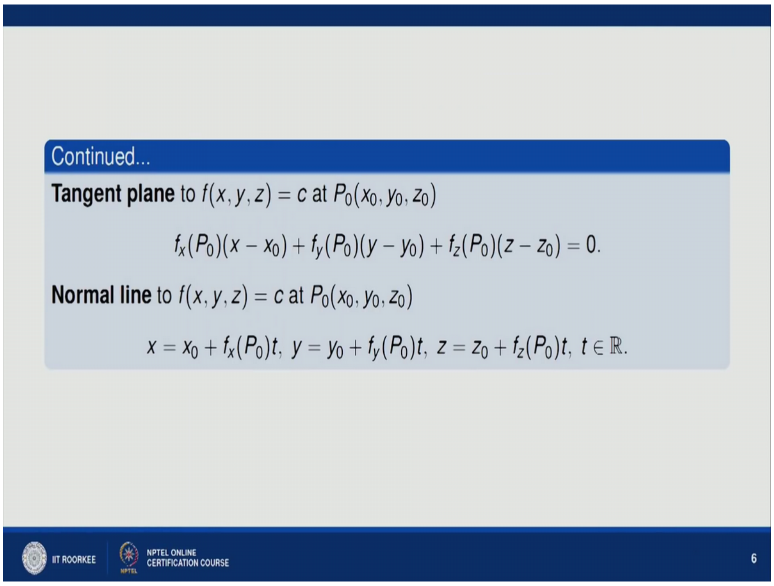

So, gradient of f will be a vector normal to a surface. Now how can you find the equation of tangent plane and a normal line?

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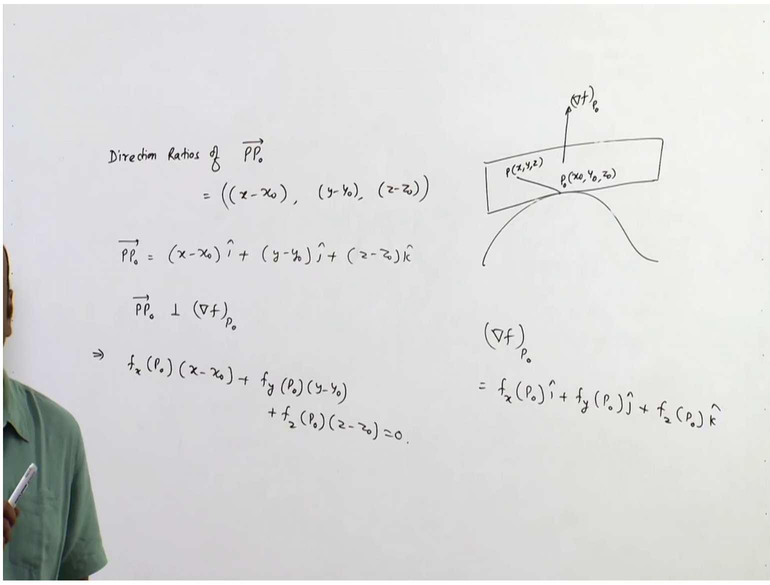

Now, suppose this is a surface at some point p which is x naught y naught z naught you are interested to find out the equation of tangent plane the equation of tangent plane at this at this point p now gradient of f will be normal to this point gradient of f at p naught at p gradient of f at. So, we are calling this point as p naught. So, it is p naught ok. So, gradient of f at p naught will be a vector normal to a surface at p naught ok.

Now, let us take an arbitrary point on this plane say this point is x y z p point let us take an arbitrary point x y z on the plane p on the plane on a tangent plane. So, what will be the direction what will be the direction ratios of this vector what will be the direction ratios of this vector? So, direction ratios direction ratios of p p naught will be x minus x naught y minus y naught and z minus z naught this will direction ratios of this vector this is x minus x naught y minus y naught z minus z naught.

What is gradient of f at p naught ? This will be f x at p naught i cap plus f y at p naught j cap plus f z at p naught k cap ok. Now what this p p naught will be x minus x naught i cap plus y minus y naught j cap plus z minus z naught k cap. Now since gradient of f at p naught is perpendicular to this line, you see you have a tangent plane you have a tangent plane like this ok. On this surface you have a tangent plane and you have taken arbitrary point on tangent plane and this is x naught y naught z naught is a fixed point on the plane.

So, this vector this p p naught will lie on the tangent plane and gradient of f is normal to the plane. So, gradient of f will also be normal to p p to the vector p p naught. So, we can easily say that p p naught vector will be perpendicular to gradient of f at p naught. So, this implies f x at p naught into x minus x naught plus f y at p naught into y minus y naught plus f z at p naught into z minus z naught will be 0 and this would be the equation of tangent plane at p naught.

So, if you if you are interested to find out the equation of tangent plane at a point p naught or the surface f x y z equal to c. So, we can simply apply this equation to find out the equation of the tangent plane now let us think about normal line. Now if we want to find out the equation of normal line on the surface on the surface.

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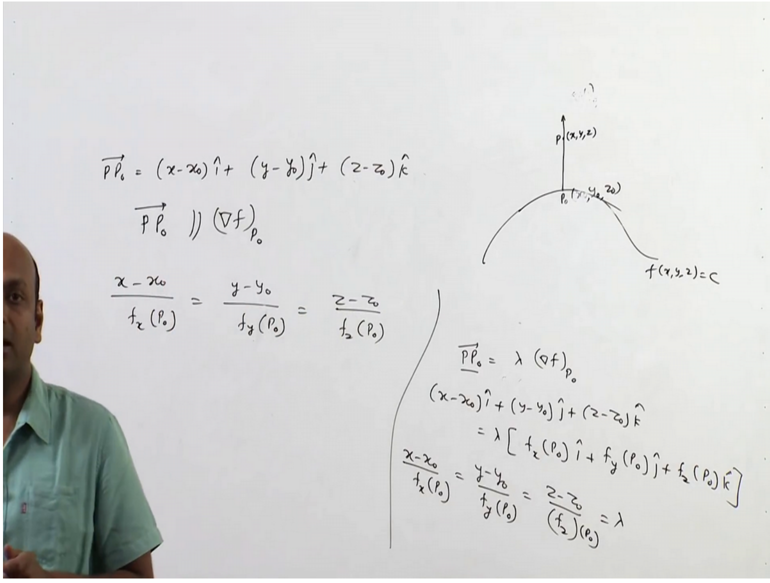

F x y z is equal to c at a point p naught which is x naught y naught z naught, we are want we want to find out the equation of tan normal line. So, so you have a this surface ok. Now this is p naught which are fixed point on the surface and you want to find out the equation of normal line. So, take a arbitrary point on the normal line take arbitrary point on the normal line x y z say p p may be any point on the normal line.

So, again what will be the vector p p naught the vector p p naught will be x minus x naught i cap plus y minus y naught j cap plus z minus z naught k cap now this vector is where this vector is on the normal line normal to a surface and gradient of f is also normal to a surface at p naught. So, both vector are normal to a surface; that means, both are parallel ok. So, we can simply say that p p naught vector will be parallel to gradient of f at p naught. Now since they are parallel this means this means they have a same ratios the ratios will be same; that means, that means x minus x naught upon f x at p naught will be same as y minus y naught upon f y at p naught and z minus z naught upon f z at p naught because this vector is parallel to this vector.

Now, if 2 vectors are parallel this means one will be lambda times other vector if this vector is parallel to this vector this means this vector any one of the vector will be sum lambda times other vector now this vector is x minus x naught i cap plus y minus y naught j cap plus z minus z naught k cap will be equals to lambda times gradient of f will be f x at p naught i cap plus f y at p naught j cap plus f z at p naught k cap.

So, from here if we equate the coefficients; so, x minus x naught upon f x at p naught will be equals to y minus y naught upon f y at p naught will be equals to z minus z naught upon f z at p naught and that will be equals to lambda. So, the same equation is here ok. So, in this way we can easily find out equation of normal line. So, that will be normal line you can take t or lambda it hardly matters because a parameter.

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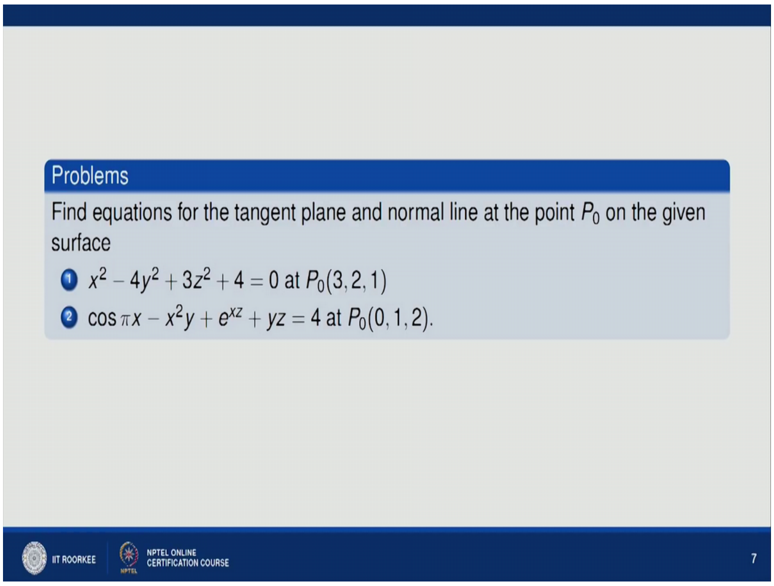

So, these are some problem let a try let us try to solve these problems.

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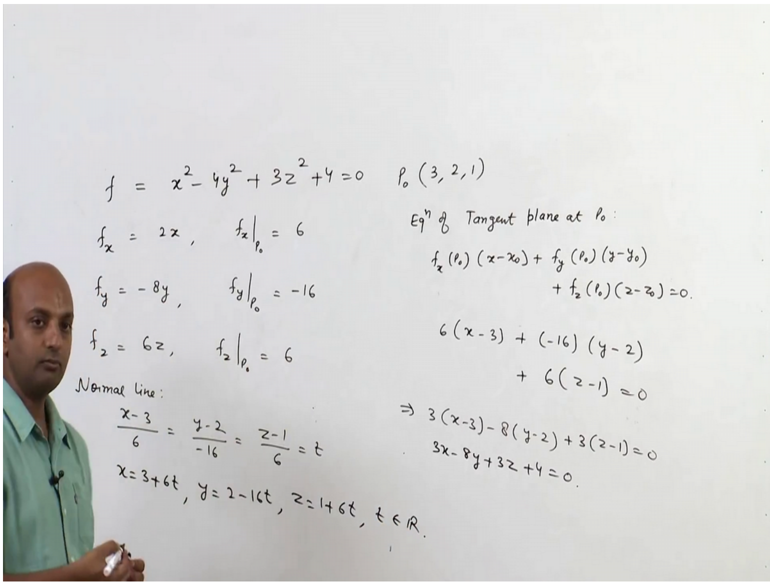

The first problem is x square minus 4 y square plus 3 z square is equal to plus 4 is equal to 0 and the point p naught is 3, 2, 1 ok. So, this is basically f. Now you have to find out the equation of tangent plane a normal line at a point p naught ok. So, first we will find del f del f by del x, it is 2 x and del f by del x at p naught will be 6 f y which is minus 8 y. So, f y at p naught will be will be simply minus 16 f z f z is 6 z and f z at p naught will be 6. So, what will be the equation of tangent line at p naught it will be equation of tangent line at p naught will be it is f x at p naught x minus x naught plus f y at p naught y minus y naught plus f z at p naught z minus z naught will be equals to 0.

To remember this equation you have to simply remember that the gradient of f is always normal to the plane i mean normal to a surface at a point p naught. So, if it is normal to a surface at a p naught. So, if you take a vector x minus x naught y minus y naught z minus z naught on the tangent plane. So, both are perpendicular; that means, dot product will be 0. So, if you take the dot product you will simply get back to this equation what is f x f x is 2 x which is 6 minus 8 y which is 16 and 6 z which is 6 yeah tangent plane; so, equation of tangent plane ok.

Now, what is f x at p naught is 6 x minus x naught is 3 it is x naught y naught z naught f y is minus 16 y minus y naught plus f z is 6 and z minus 1 equal to 0, you divide the entire equation by 2 which is 3 x minus 3 minus 8 y minus 2 plus 3 z minus 1 equal to 0 it is 3 x minus 8 y plus 3 z it is minus nine minus 16 and minus 3.

Student: Plus 16.

Yeah, it is plus 16 ok. So, it is plus 16. So, it is minus 12 plus 16 is plus 4 equal to 0. So, this will be the equation of a tangent plane now how can you find the normal line. So, normal line will be x minus x naught upon f x y minus y naught upon f y at p naught and z minus z naught upon f z will be equals to t. So, x will be given as 3 plus 6 t y is equals to 2 minus 16 t and z equal to 1 plus 6 t where t is any real value ok. So, this is a normal a line now for the second problem.

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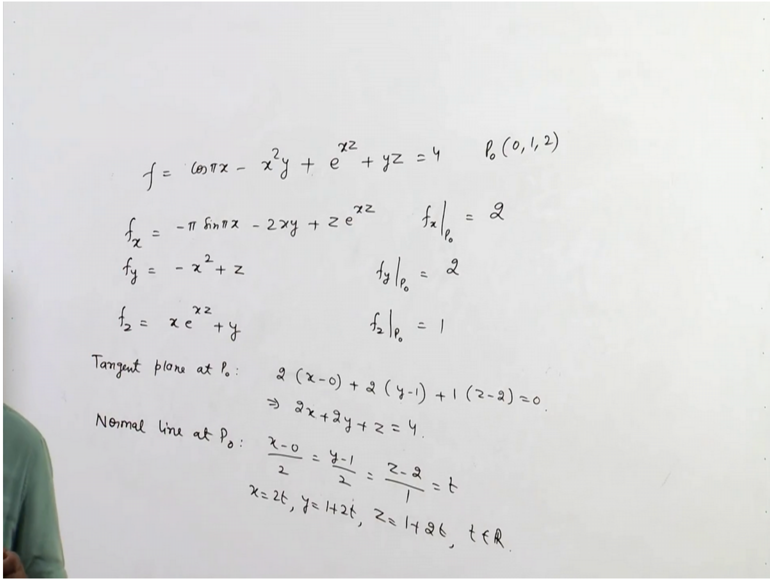

So, what is f here for a second problem it is cos pi x minus it is x square y plus e raise to power x z plus y z is equal to 4 and the point is 0 one 2. So, first we will find f x what will be f x f x is minus pi sin pi x minus 2 x y plus z times e raise to power x z and f x at point p naught will be when you substitute x as 0 y as 1 and z as 2. So, this is 0 this is 0 and this will be one and z is 2. So, it is 2.

Now, what is f y f y will be a minus x square plus z and f y at p naught will be x is 0 and z is 2. So, it is again 2. Now f z f z is x e raise to power x z plus y and x is 0 and y is one. So, f z at p naught is one. So, equation of tangent plane will be at p naught will be, it will be 0 into x minus x one it will be f x f x at p naught that is 2 2 x minus 0 plus 2 y minus one plus 1 z minus 2 equal to 0.

So, this equation will be simply 2 x plus 2 y plus z and it is minus 2 minus 2 that is equal to 4 and equation of normal line at p naught normal line at p naught will be it will be x minus 0 upon 2 y minus 1 upon 2 and z minus 2 upon 1, let us suppose equal to t. So, x will be 2 t y equals to 1 plus 2 t and z equals to 1 plus 2 t where t is any real number. So, this will be equation of tangent plane and normal line.

Now, if we have a equation or surface like z equals to f x y, ok.

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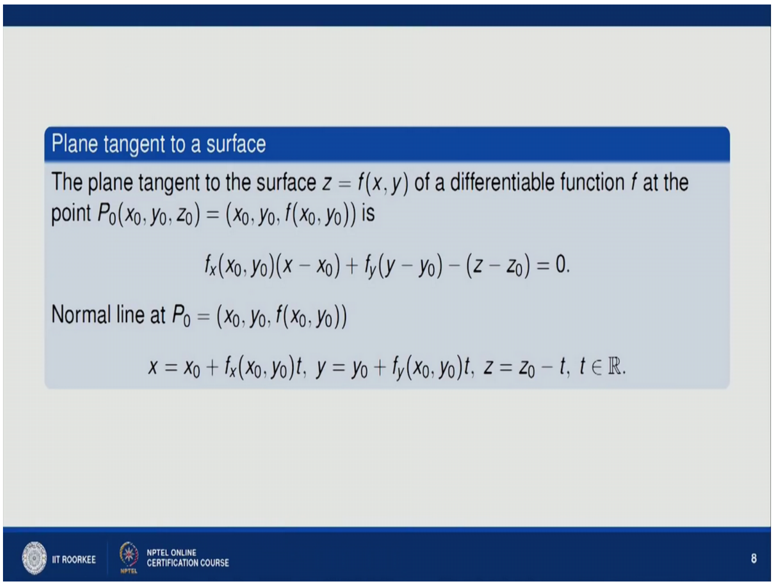

If we have the equation surface like z equals to f x y.

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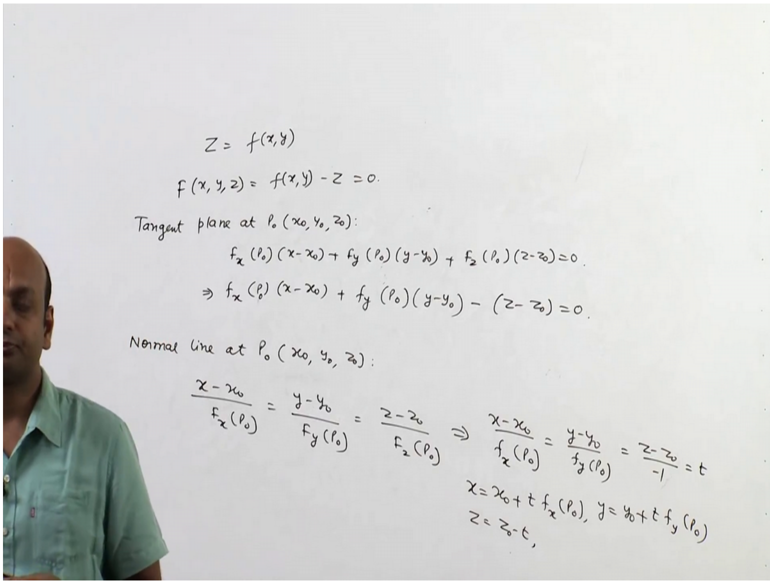

So, so take all the things on the one side. So, what will be capital F x y z it will be f x y minus z equal to 0 and suppose at a point p naught which is x naught y naught and z naught, you want to find out equation of tangent plane and normal line. So, what will be the equation of tangent plane a normal line for this curve. So, basically this is a special case of the first part.

So, this will be f x at p naught x minus x naught this will be f y at p naught y minus y naught plus f z at p naught into z minus z naught equal to 0. So, what is f x at p naught from this equation it will be f x small f at if small f x at p naught x minus x naught plus small y f y at p naught into y minus y naught and what will be f z at p naught it is minus 1 always. So, it is minus z minus z naught equal to 0.

And similarly the equation of normal line at p naught which is x naught y naught z naught will be x minus x naught upon f x at p naught y minus y naught upon f y at p naught and z minus z naught upon f z at p naught. So, this implies this is x minus x naught and f x at p naught is a small f a small f x at p naught, this is y minus y naught and similarly it is f y at p naught and it is z minus z naught upon minus one and say it is t f z is minus one. So, x will be x naught plus t times f x at p naught y will be y naught plus t times f y at p naught and what will be z z will be z naught minus t where t is any real number. So, this will be equation of tangent plane and a normal line at a point p naught for this surface.

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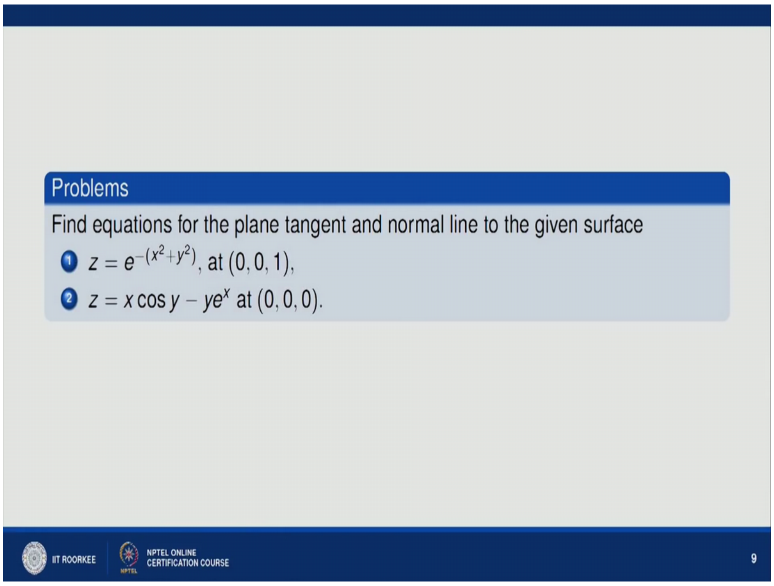

Suppose you want to solve this problem find the equations of tangent plane a normal line to the given surface, say we take the first problem similarly we can solve the second problem also.

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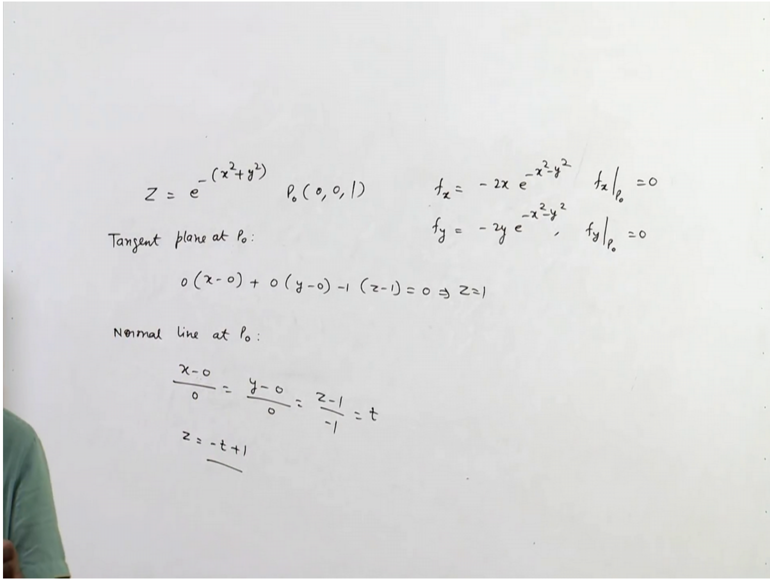

So, it is z is equals to e raise to power minus x square plus y square and the point is 0 0 0 0 0 1. So, how can we solve this? So, tangent plane will be say tangent plane at p naught will be. So, you first find f x. So,; so, f x simply this function ok. So, that will be minus 2 x e raise to power minus x square minus y square and at 0 0 1 it is 0 and f y will be again minus 2 y e raise to power minus x square minus y square and f y at p naught will again be 0.

So, the equation of tangent plane will be 0 into x minus x naught plus 0 into y minus y naught minus 1 into z minus z 1 z naught. So, this implies z equals to 1. So, these are equation of tangent plane at this point for this curve and normal line normal line at p naught will be x minus 0 upon 0 y minus 0 upon 0 and z minus 1 upon minus 1 equal to t. So, you can take this will be z is equals to minus t plus 1 for any for any t, similarly we can solve the next problem.

Now, how to find tangent line to the curve intersection of surfaces ok? Now let us discuss this thing.

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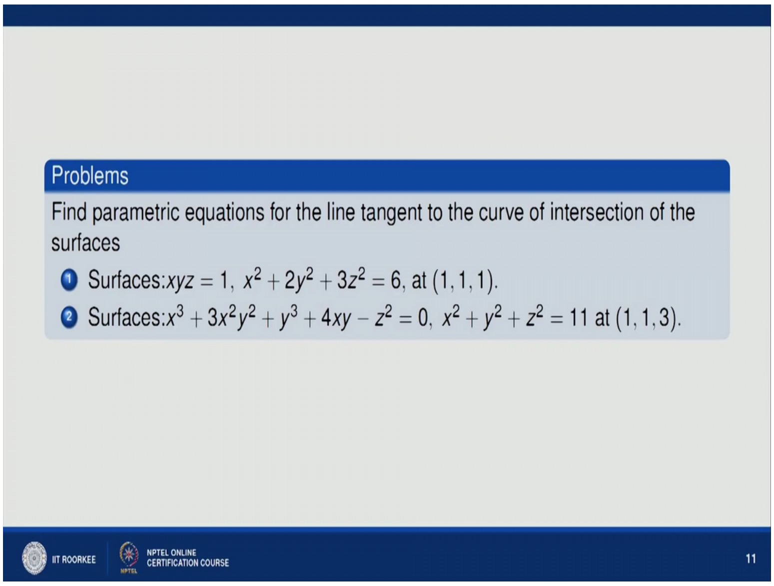

Now suppose you have 2 surfaces, ok, suppose you have 2 surfaces.

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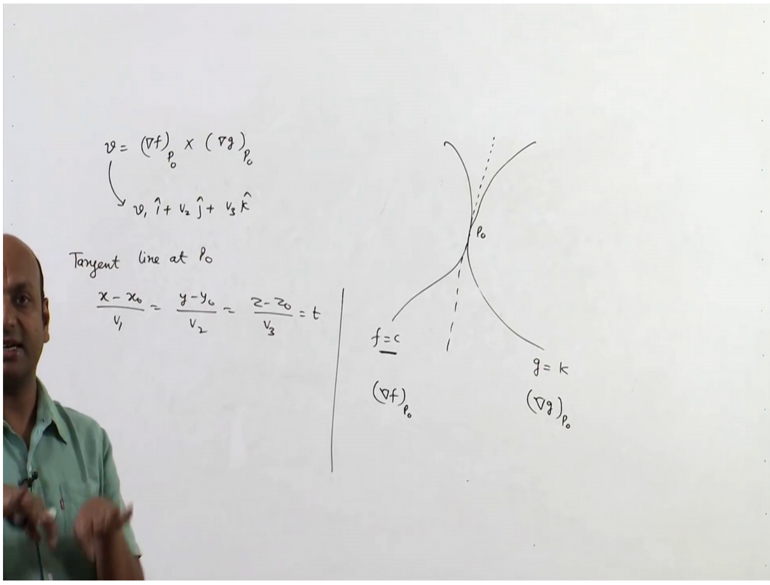

This is one surface and this is second surface this is f equal to c and this is g equal to some k and this is the point of intersection at this point of intersection, you are interested to find out the equation of tangent line, it will be some tangent line because 2 surfaces are intersecting each other. So, it will it will give a tangent it will give a tangent line now how to find the equation of tangent line at this point.

Now, gradient of gradient of f suppose this point is p naught at p naught will be will give will give a vector normal to the surface at p naught ok. So, it will be definitely normal to a tangent line also similarly gradient of t at p naught will give a vector normal to the surface g equal to k at p naught. So, it will be perpendicular to or it will be normal to tangent line also. So, this is also normal to tangent line at p naught and this is also normal to tangent line at p naught now the cross product of these 2 vector will give a vector normal to I mean when you are having 2 vector a and b cross product of this give a vector normal to both a and b ok.

So, cross product of these 2 will give a vector normal to both the vectors normal to this vector and this vector and hence nor and hence when it is normal to this and normal to this. So, parallel to this because these 2 because these 2 are normal to this tangent line is it clear because this is normal vector normal to f at p naught to. So, normal to tangent plane also similarly this is also normal to a tangent at p naught and the cross product give a vector normal to both the vector this and this. So, the vector which is obtained by the cross product these 2 vector will be parallel to a tangent line ok.

So, in this way we can get the equation of tangent line say v is the vector which is cross product of these 2 vector and say this say this vector is v 1 i cap plus v 2 j cap plus v 3 k cap. So, what will be the equation of tangent line at p naught will be x minus x naught upon v one y minus y naught upon v 2 and z minus z naught upon v 3 because both are parallel you see x minus x naught y minus y naught z minus z naught is a vector lie on the tangent line and vector v is also parallel to tangent line. So, both will parallel this means ratios will be same. So, in this way we can find out the equation of tangent line when 2 curves are intersecting.

Say we have first problem so.

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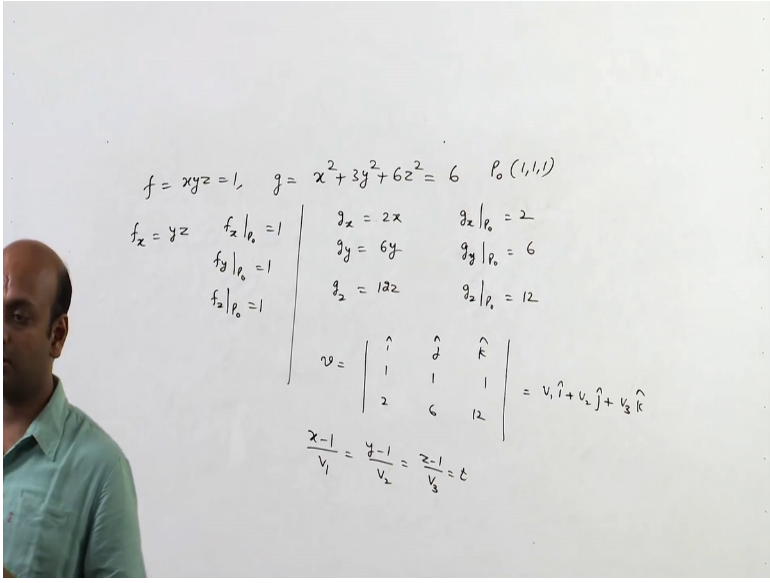

f here is x y z equal to o1ne and g here is x square plus 3 y square plus 6 z square equals to equals to 6 and the point is 1 1 1. So, what will be f x it is y z f x at p naught is 1. Similarly f y at p naught is 1 and similarly f z at p naught is 1 what is g x g x will be 2 x g y will be 6 y and g z will be 12 z. So, g x at p naught will be 2 g x g x at g x g y sorry at p naught will be 6 and g z at p naught will be 12 ok.

So, how to find v, now v will be equals to cross product of these 2 vectors that is i cap j cap k cap this is 1 1 1, this is 2 6 12 the cross product of gradient of f at p naught and gradient of g at p naught this will be v. So, this vector will be the suppose this vector is v 1 i cap plus v 2 j cap plus v 3 k cap that we can easily find ok, then the equation of tangent line will be x minus one upon v one y minus 1 upon v 2 and z minus 1 upon v 3 which is t say. So, this will be equation of tangent line at p 1. So, similarly we can solve the second part also. So, these can also be solved using the properties of tangent plane and normal line these problems so.

Thank you very much.