Multivariable Calculus

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Lecture – 12

Euler's theorem for homogeneous functions

Hello, friends. So, welcome to lecture series on Multivariable Calculus. So, in this lecture we will deal with Euler's theorem for homogeneous functions.

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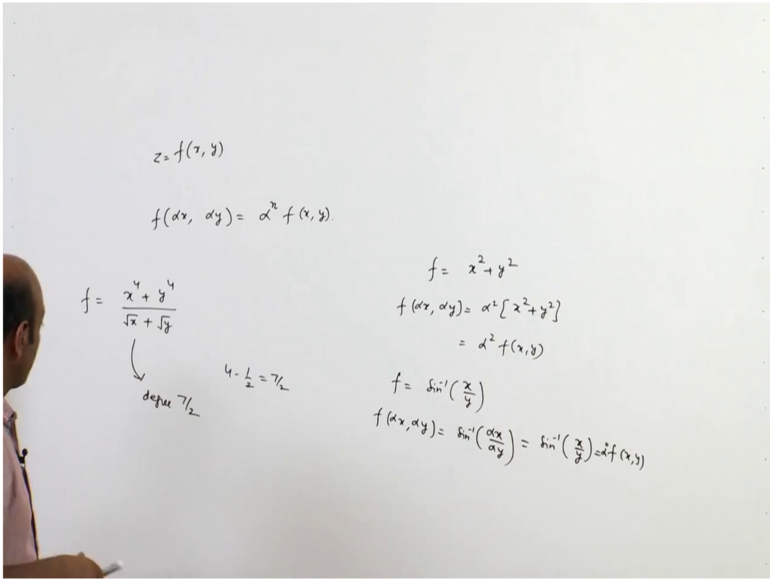

So, what homogeneous functions are first? Now, suppose you have a function of two variable x and y. Now, this function is said to be homogeneous function of degree n if f of alpha x alpha y is equals to alpha raise to power n f of x, y. So, then this if this property hold for a function of two variable f, x, y, then we say that this is a homogeneous function of degree n. Suppose, f is x square plus y square it is clearly homogeneous because if you take alpha x alpha y, so, it is alpha square times x square plus y square you simply replace x by alpha x and y by alpha y. So, alpha square will come out and it is alpha square times f of x y. So, we can say that it is a homogeneous function of degree 2.

Now, say f is sine inverse x upon y, it is also homogeneous because if you replace x by alpha x, y by alpha y, so, it is sine inverse of alpha x upon alpha y which is sine inverse x upon y and it is same as f, x, y. So, it is also homogeneous function of degree 0, because this can be written as alpha raise to power 0. So, it is also homogeneous function of degree 0.

Say, you have this function x raise to power 4 plus y raise to power 4 upon under root x plus under root y. Now, this is also homogeneous function because if we replace x by alpha x and y by alpha y, from the numerator you will be getting alpha raise to power 4 and from the denominator you are getting alpha raise to power half. So, it is 4 minus half that is 7 by 2. So, this is homogeneous function of degree 7 by 2.

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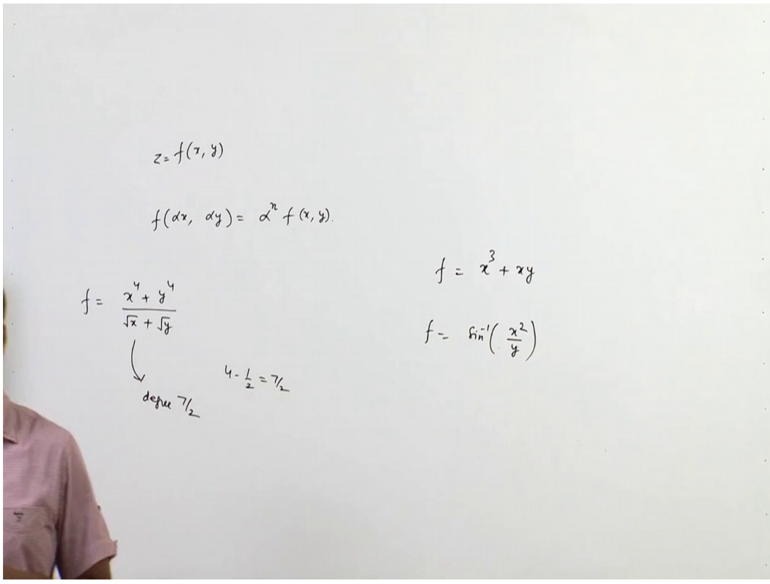

Now, there are some function which are not homogeneous like you are having say x cube plus xy, it is not a homogeneous function because when you replace x by alpha x and y by alpha y from here we are getting alpha cube and from here we are getting alpha square. So, the entire alpha is not coming out, so, that means, this function is not a homogeneous function.

Similarly, suppose you have this function sine inverse of x square upon y it is also not a homogeneous function. So, this is homogeneous function.

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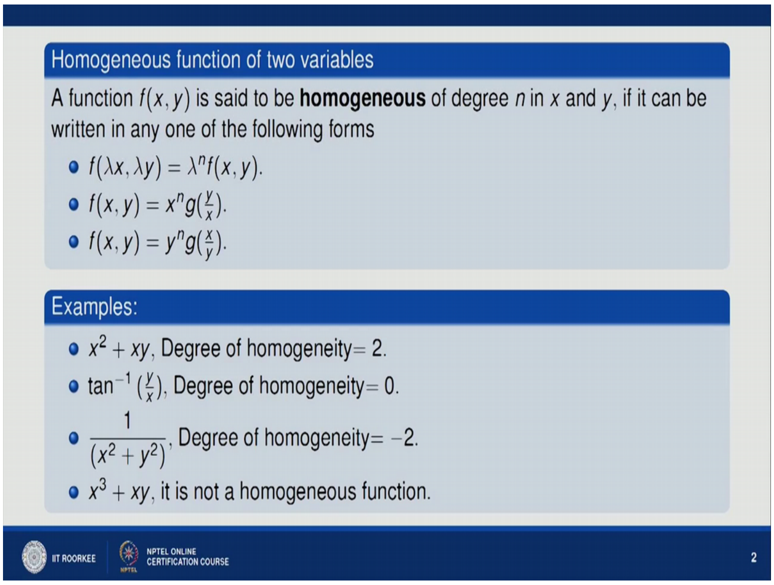

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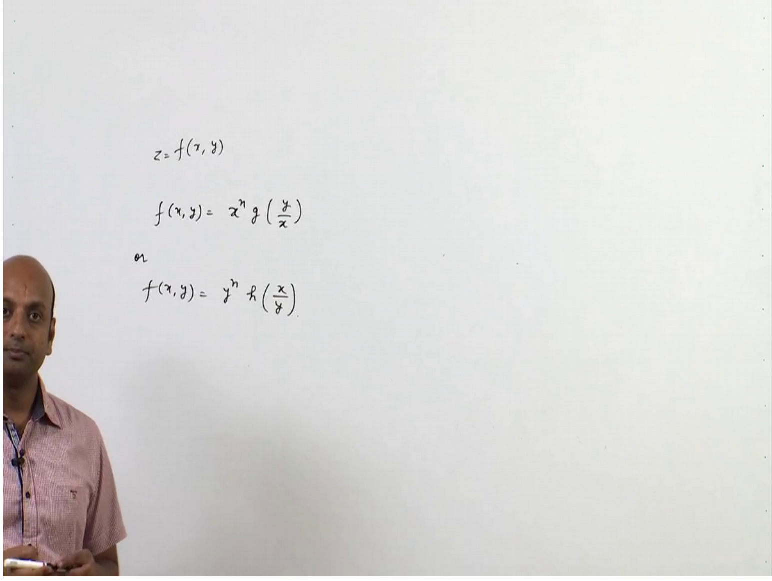

So, a homogeneous function is also expressed as you see if we are saying that f is a homogeneous function of degree n then f can be expressed as x raised to power n some g of y by x or f can be expressed as some y raise to power n, h of x upon y, if the homogeneous function of degree n.

Now, here are some examples; the first example is homogeneous function of degree 2, the second example is homogeneous of degree 0. We can easily verify the third example is homogeneous function of degree minus 2. The third example is not homogeneous.

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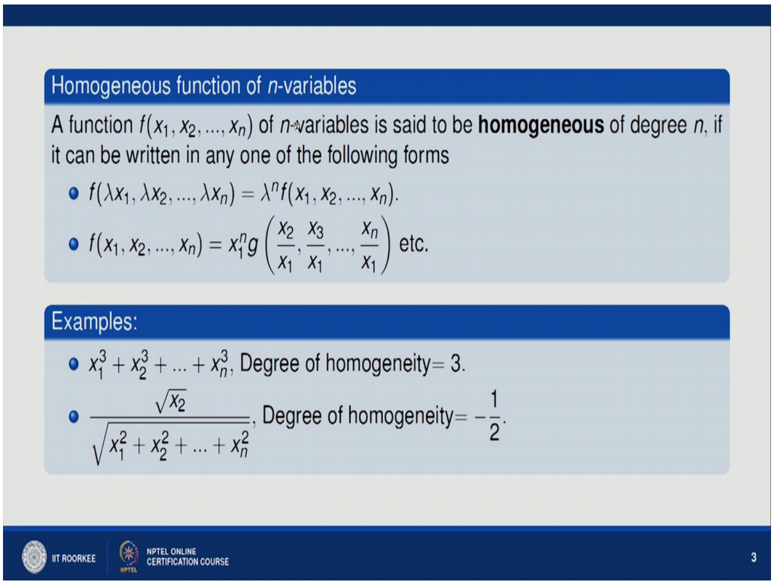

Now, here we have a homogeneous function of n variables the same definition is applicable for homogeneous function for n variables also, you simply replace X 1 by lambda X 1, X 2 by lambda X 2 and so on. If you are getting lambda raised to power n times the same function, that means, the function is homogeneous of degree n.

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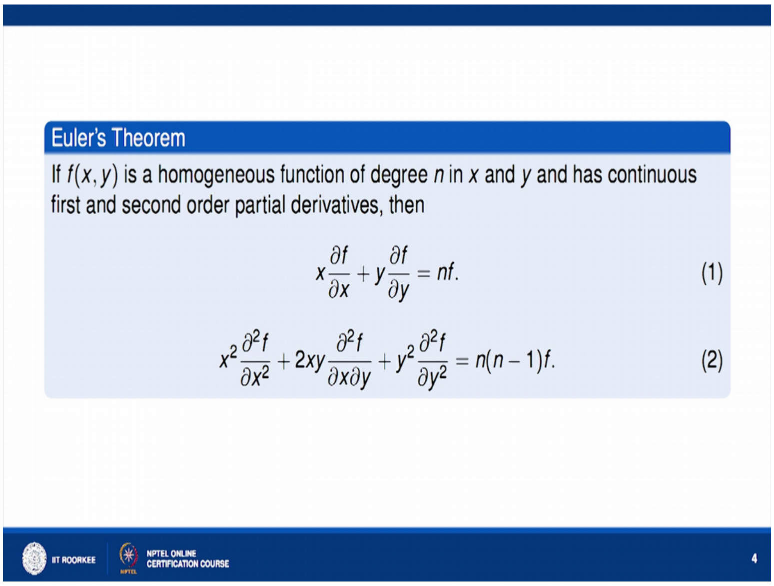

Now, comes to Euler's theorem, it states that if f is a homogeneous function of degree n in x and y and has continuous first and second order partial derivatives, then these two result hold. So, this is Euler's theorem. So, let us try to prove this theorem first.

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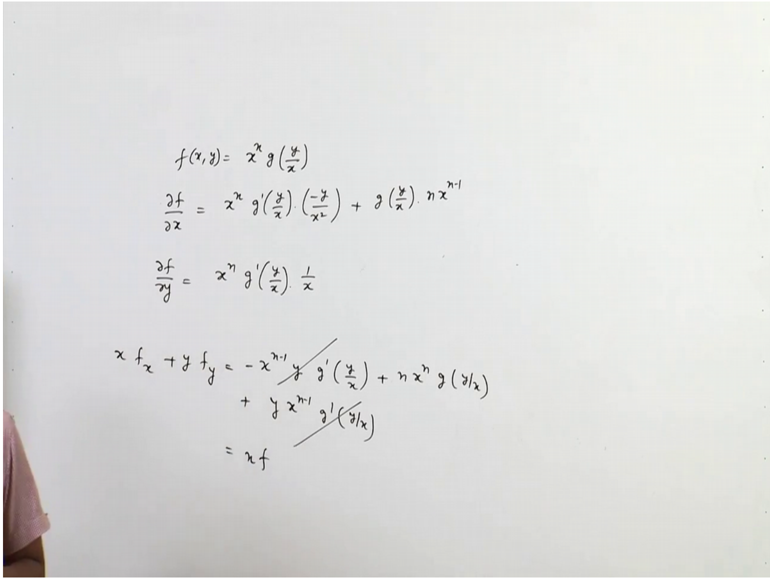

So, f is a homogeneous function it is given the statement. So, f can be writtn as x raised to power n, g of y by x. So, we are assuming f as a homogeneous function of degree n. Now, what will be del f upon del x, it will be x raise to power n first as it is derivative of second plus second as it is derivative of first. Similarly, what will be del f upon del y? It will be x raise to power n, g dash y by x and this again which is 1 by x.

Now, what will be x f x plus y f y, you simply multiply this by x and this by y and add them. What we obtain? We obtain it is x raised to power n minus 1, because it is x square minus 2 into x, it is x raise to power n minus 1 with negative sign times y, g dash of y upon x plus n, x raise to power n, g of y by x plus here when you multiply by y it is y, x raise to power n minus 1, g dash of y by x and these two terms cancel out. It is n times and this is nothing, but f only, so, it is n f. So, we can say that x f x plus y f y is n f. It is true only for homogeneous functions.

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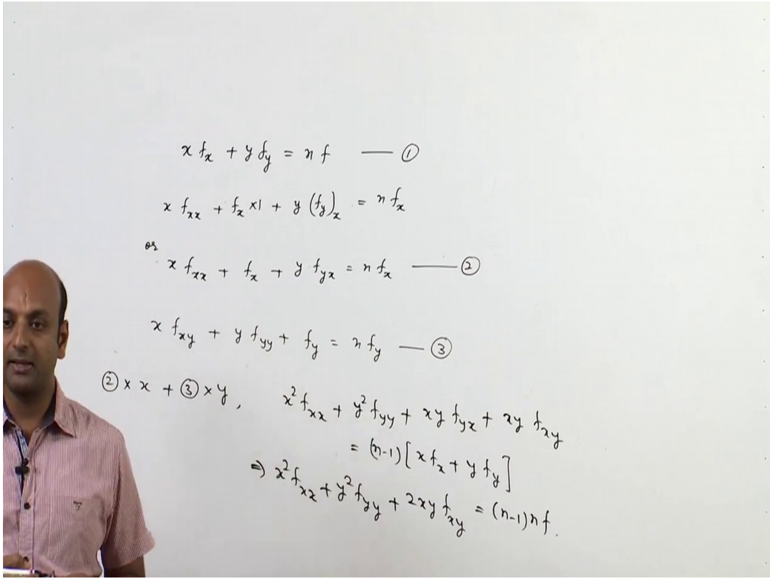

Now, the second result. Now, we have obtained that x f x, plus y f y is equals to n f, if f is a homogeneous function of degree n. Now, let us differentiate both the side partially respect to x. When we differentiate both side partially respect to x, what we obtain, it is first as it is derivative second plus second derivative first plus y into f y of x is equals to n of f x, which is or it is x f x plus f x plus y f y x is equals to n f x. So, this is, say second equation.

Now, again you differentiate equation 1, partially respect to y both sides. So, what you will obtain, it is x f x y plus y f y y plus f y is supposed to n f y. Now, you multiply equation – 2, by x and equation – 3, by y and add them. So, what you will obtain, it is x square f x x from here it is y square f y y plus it is x y times f y x plus it is when you multiply this by y it is x y times x y plus it is x f x plus y f y that we can put in the right hand side. So, what we will obtain, it is equals to n minus 1 times x f x plus y f y.

Now, since in the statement it is given to us that it has continuous first and second order partial derivatives. So, we can say that f x y is same as f y x both are equal, because it is having first and second order continuous partial derivatives. So, from here we can say that it is x square f x x plus y square f y y plus 2 x y we can call it f x y or f y is because both are same and it is equals to n minus 1 and this value is n f, so, it is n into f. So, in this way we obtain the second part of the theorem.

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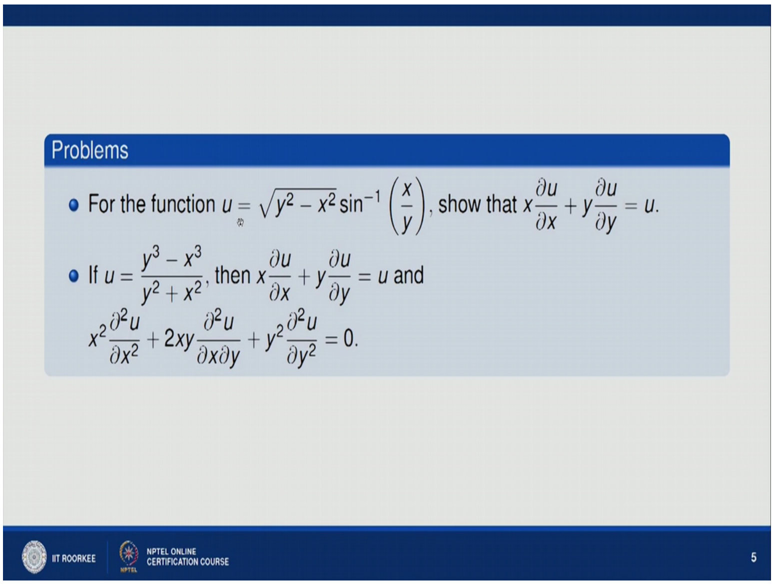

Now, say the other first problem first problem is u equal to under root y square minus x square sine inverse x by y. Now, sine inverse x by y is a homogeneous function of degree 0 and here it is also homogeneous function of degree 1, because when you take when you replace x by alpha x and y by alpha y, so, alpha square will come out from this inside bracket and under root we are having, it will be alpha; alpha will come out. So, basically, the degree of this problem this function is 1. So, by the Euler's theorem we can easily say that n u x sorry x u x plus y u y is equals to n u and n is 1. So, it is equal to u directly by the Euler's theorem. So, the first problem is over.

Now, come to a second problem. Now, for a second problem second problem is also homogeneous. You see, u equal to y cube minus x cube whole divided by y square plus x square. The degree of this function is 1. So, x u x plus y u y will be equal to n u and n is 1, so, it is u and for a second part, x square u x x plus 2 x y u x y plus y square u y y it is equals to n, n minus 1 into f and n is 1. So, when you substitute n equal to 1, so, this term will be 0. Hence, and the second part hence, the right hand side is 0. So, this problem is also over directly by Euler's theorem.

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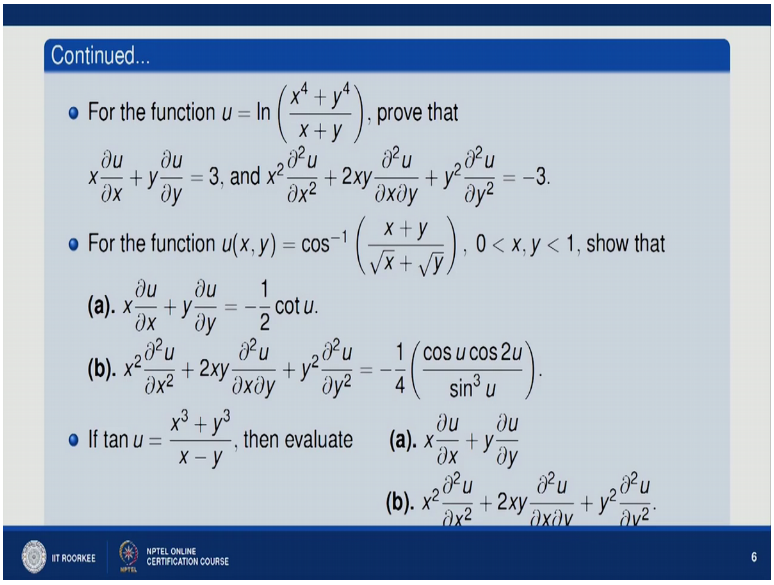

Now, let us come to these problems.

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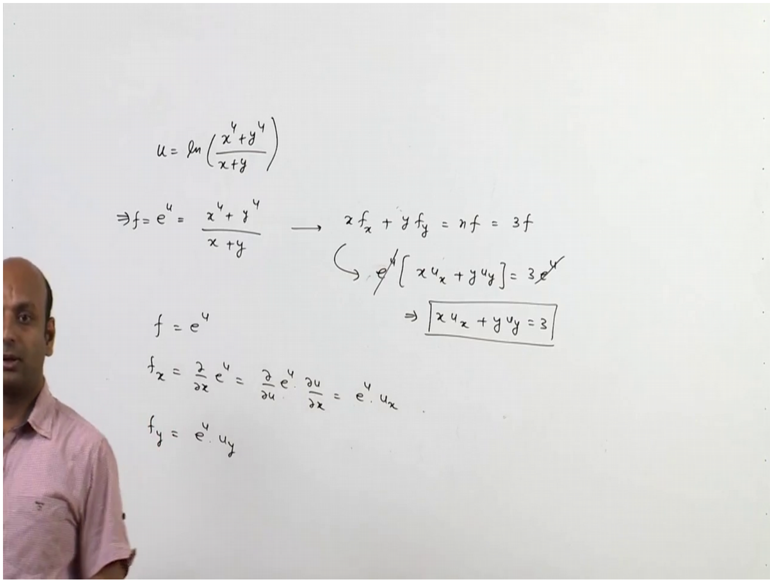

Now, the first problem is u is equal to l n, x raised to power 4 plus y raised to 4 upon x plus y. Now this function is not homogeneous function, because you can simply obtain because when you replace x by alpha x and y by alpha y, alpha is not coming out. So, it is not an homogeneous function and suppose you want to find out x u x plus y u y or x square u x x plus 2 x y u x y plus y square u y y, how can we obtain that?

Now, this implies e raise to power u is equals to x raised to power 4 plus y raise to power 4 upon x plus y, now this is homogeneous. Say, if this function is say f if e raise to power u is f, now it is an homogeneous function. So, for homogeneous function Euler's theorem is applicable. So, we can simply say that x f x plus y f y will be equals to n f and n is 3. So, it is 3 f and what is f? f is e raise to power u. So, what will be f x, it is del by del x of e raise to power u which is del by del u of e raise to power u into del u by del x and which is e raised to power u into u x and similarly, f y will be e raised to power u into u y.

So, when you substitute these values here, what you will obtain, it is x, e raise to power u will come out from both the terms, so, e raised to power u will come out it is x u x plus y u y is equals to 3 and f is e raised to power u, e raised to power u cancels out. So, this implies x u x plus y u y is equals to 3. So, that is how we obtain the first part of the problem.

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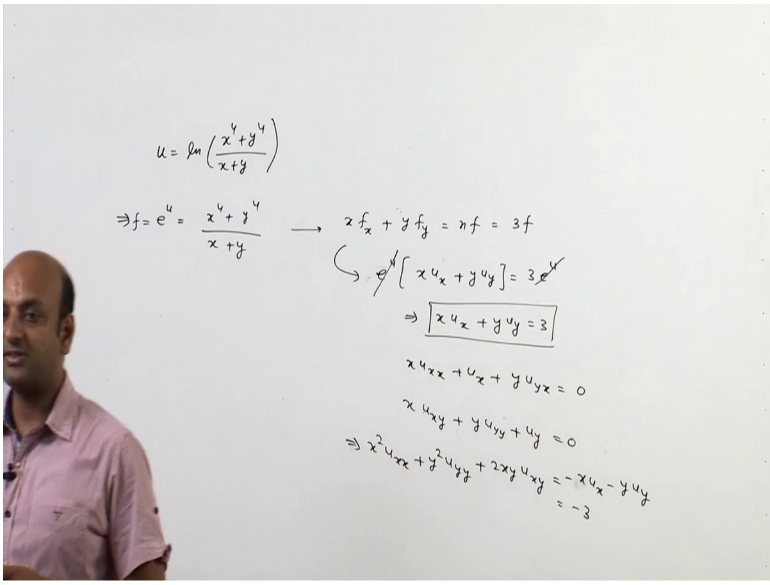

Now, second part can also be obtained, you differentiate partial respect to both the sides, what you will obtain? x u x x plus u x plus y u y x is equals to 0. Now, I differentiate partial respect to y both the sides. So, it is x u x y plus y u y y plus u y is equal to 0. Multiply this by x and this by y, add them, assuming these two are equal u x y and u y x are equal. So, this implies x square u x x plus y square u y y plus 2 x y u x y will be equals to it is x time u x plus y time u y that will go to right hand side, which is minus x u x minus y u y which is minus 3, because x u x plus y u y is 3. So, hence we can obtain that this value is minus 3.

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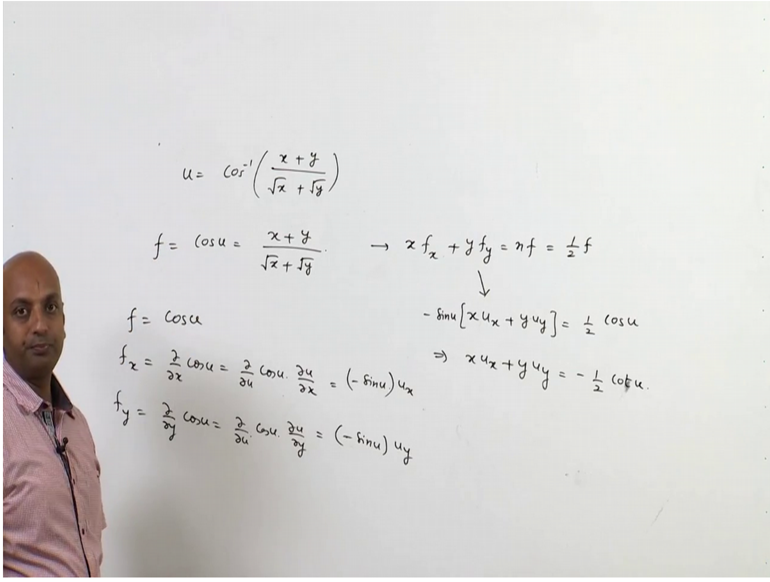

Now, come to second problem for a second problem also if you directly see the problem, directly the problem is not a homogeneous one. You see what is the function here function is u is equal to cos inverse x plus y upon it is under root x under root y for x y lying between 0 and 1. Now, this function as a whole is not an homogeneous function, but if you take f is equals to cos of u which is equals to x plus y upon under root x plus under root y, now, f is homogeneous function f is equal to this and it is a homogeneous function of degree 1 by 2. So, Euler’s theorem is applicable for f for the function f. So, what will be by Euler’s theorem, it is x f x plus y f y is equals to n f and n is 1 by 2 that is 1 by 2 times f.

Now, what is f? f is cos u. So, what will be f x again, it is del by del x of cos u which is del by del u of cos u into del u by del x and that will be minus sine u into u x. Again, what is f y? f y is del by del y of cos u which is del by del u of cos u into del u by del y which is again minus sine u into u y. Now, when you substitute these two values in this equation what you will obtain, this is x f x is minus sine u. So, minus sine u can come out it is x u x plus y u y will be equals to 1 by time 1 by 2 times f and f is cos u. So, we obtain that x u x plus y u y will be equals to minus 1 by 2 cot u. So, that is how we obtain the first part of the problem.

So, what I want to say basically, sometimes the problems are not homogeneous, but we can make it homogeneous by substituting f as some function of u. We can apply Euler's theorem for this f because this f is equal to this which is homogeneous and later on we find there we find f x and f y in terms of u and then we can simply find the values of x u x plus y u y. Again, suppose you want to find out x square u x x plus y square u y y plus 2 x y u x y for this problem.

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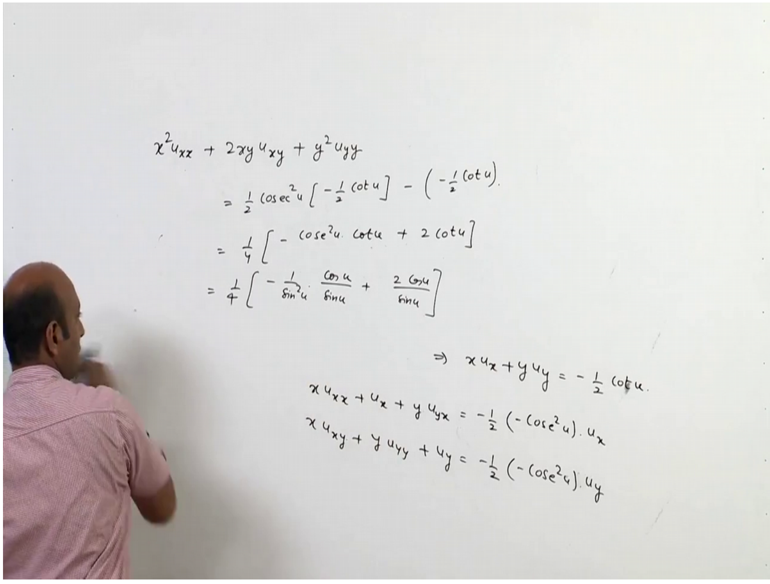

So, you differentiate both sides respect to x partially. So, what you will obtain x u x plus u x plus y u y x is equals to minus 1 by 2, now derivative of cot is minus cosec square u and u again that is u x. Now, do differentiate again this equation both sides respect to y partially. So, what you will obtain x u x y plus y u y y plus u y is equals to minus 1 by 2 into minus cosec square u into u y. Multiply first equation by x, second equation by y, multiply this equation by x this equation by y and add them.

So, what you will obtain, we obtain x square u x x plus 2 x y u x y plus y square u y y will be equal to it is 1 by 2 cosec square u will come out it is x u x plus y u y which is minus 1 by 2 cot u and this u x x u x and y u y will go to a right hand side which is negative of minus 1 by 2 cot u. So, what we have to show? We have to show this thing. So, you can take 1 by 4 common, when you take 1 by 4 common, it will be from this side we got minus cosec square u into cot u and it is plus 2 cot u and it is equals to 1 by 4 cosec square is 1 by a sine square cot u is cos upon sine plus 2 cos upon sine.

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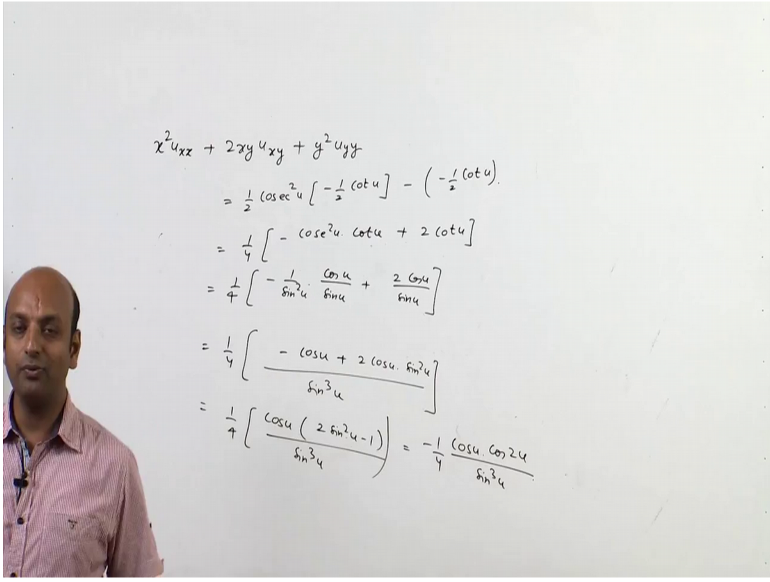

Now, you take the LCM and simplify, it is 1 by 4 you take the LCM sine cube u it is minus cos u plus 2 cos u times sine square u. So, what you have to show, cos u into cos 2 u. So, cos u can come out, it is 2 sin square u minus 1, and this is negative of cos 2 u. So, it is minus 1 by 4 cos u cos 2 u upon sin cube u. So, that is how we can prove this part of the problem.

Now, similarly we can go for the third problem here we can assume tan u as a f u. So, that f which is x cube plus y cube upon x minus y becomes an homogeneous function, we can apply Euler's theorem for f and later on we can take f u as tan u, find f x and f y substitute in these equations so that we can obtain the values of a and b part of this problem.

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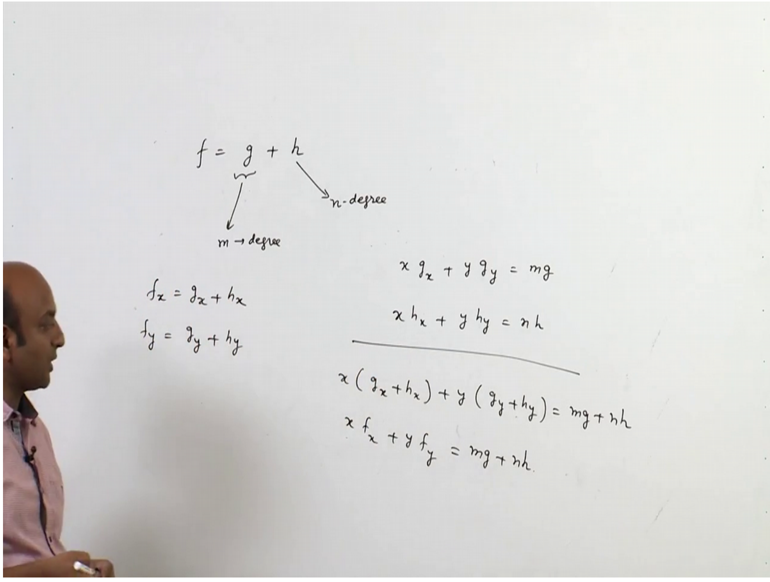

Now, we have one result for Euler's theorem let us discuss this result. Suppose, function f is a sum of two functions g and h, this is a homogeneous function of say degree n. This is also an homogeneous function of some other degree say m; what is given, oh, g is m and this is n. So, suppose this is m and this is n. So, suppose f is a function which is a sum of two functions g is a homogeneous function of degree m, h is a homogeneous function of degree n, now is f homogeneous? It can be homogeneous, if m is equal to n, if m is not equal to n so, f will not be an homogeneous function.

So, can we apply Euler's theorem for such type of problems? You see, g is an homogeneous function of degree m, so, for g, Euler's theorem applicable. So, we can say that x g x plus y g y is equals to m g, h is a homogeneous function of degree n. So, for h Euler's theorem is applicable. So, that will be x h x plus y h y is equals to n h, you add the two equations. When you add the two equations, it is x g x plus h x plus y g y plus h y and this equals to m g plus n h. Now, if f is equals to g plus h, so, what is f x it is g x plus h x and what is f y, it is g y plus h y. So, we can say that it is x f x plus y f y which is m g plus n h. So, this is a basically a consequence of Euler's theorem.

Suppose, a function can be expressed as sum of two homogeneous functions of different degree then x f x plus y g y, y f y is equals to m g plus n h. Here, degree of f is g is m and degree of h is n. Similarly, this is an homogeneous function.

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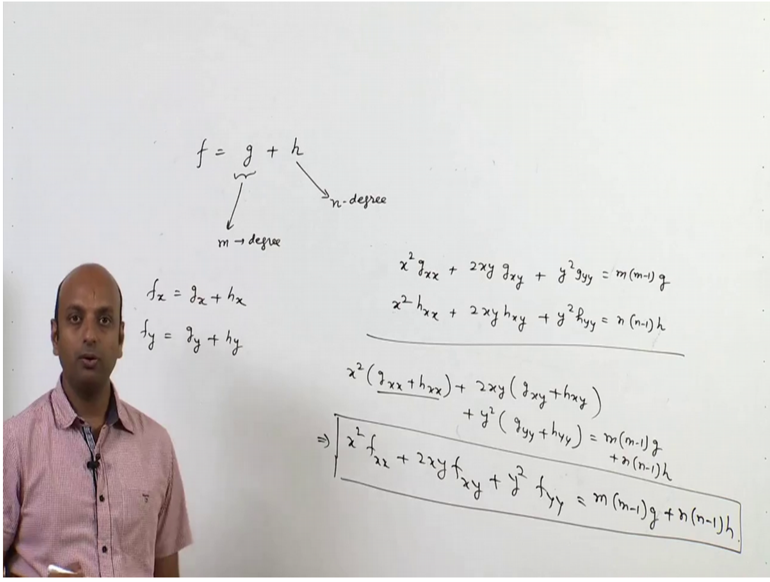

So, we can say that x square g x plus 2 x y g x y plus y square g y y is m minus 1 into g and again this is a homogeneous function of degree n. So, at x square h x x plus 2 x y h x y plus y square h y y will be equals to n, n minus 1 into h. Now, you again add the two equations. So, what you will obtain, it is x square g x x plus h x x plus 2 x y g x y plus h x y plus y square g y y plus h y y will be equals to m, m minus 1 into g plus n, n minus 1 into h.

Now, g x x plus h x x from here is f x x plus 2 x y it is f x y plus y square it is f y y, which is equals to m, m minus 1 into g plus n, n minus 1 into h. So, hence we have shown the second consequence of Euler’s theorem.

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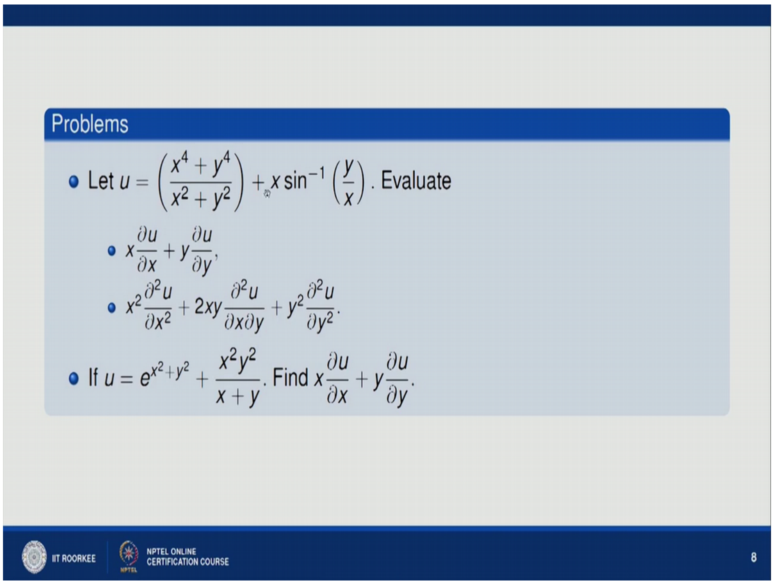

Now, suppose we have the first problem. So, in the first problem u is equals to this term plus this term, now this term is a homogeneous function of degree 2, the first term. The second term is also an homogeneous function of degree 1. So, both are homogeneous function having a different degree; it is a homogeneous function degree 2, it is a homogeneous function of degree 1. So, if you want to compute x u x plus y u y, we can directly apply the first consequence of Euler's theorem, that is, m g plus n h; where what is m here? m is 2 and g is, g is the first term plus n, n is 1, into second term.

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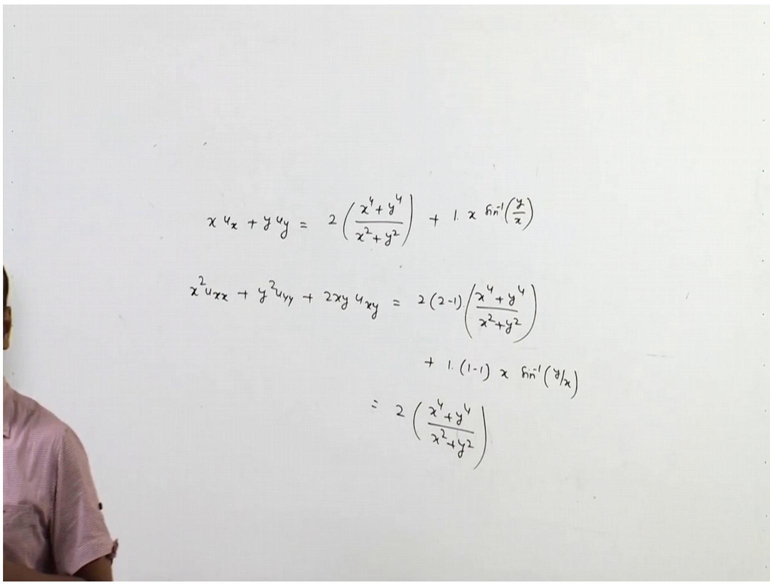

So, the answer of the first part will be first is x u x plus y u y will be the first term has a degree 2, so, it is 2 times g. g is the first term, is it ok; plus the second term has a degree 1, that is, n into h, n is 1 and h is x sine inverse y by x. So, this is the value of this expression.

Now, the value of second expression which is x square u x x plus y square u y y plus 2 x y u x y will be equal to m, m minus 1 into g plus n, n minus 1 into h, that is, simply 2 x raised to power 4 plus y raised to power 4 upon x square plus y square. So, that is how we can solve this problem.

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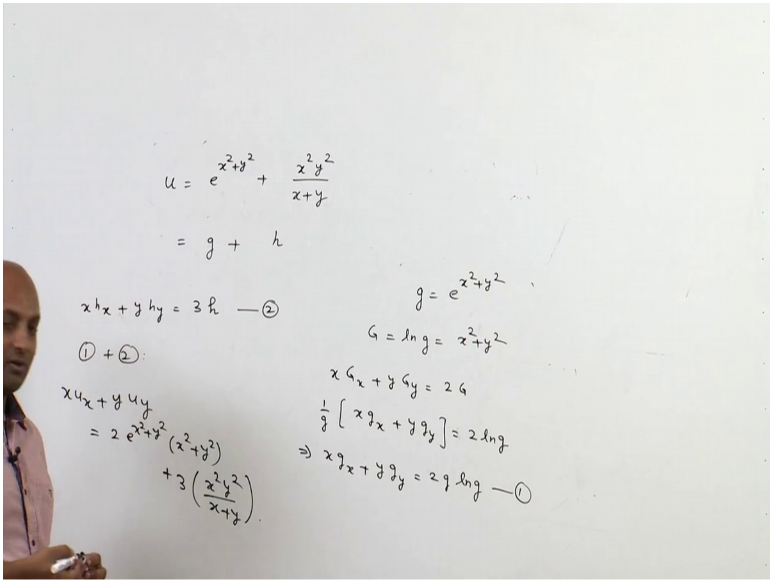

Now, suppose you want to solve second problem. In a second problem this is not homogeneous, however, this is homogeneous the second is u is equals to e raised to power x square plus y square plus it is x square y square upon x plus y, suppose it is g plus h; h is clearly homogeneous of degree 3, but g is not an homogeneous function. So, g is e raised to power x square plus y square. So, you can make l n of g is equals to x square plus y square, now this capital G is an homogeneous function of degree 2.

So, now we can apply the consequence of Euler's theorem on capital G and h. So, we can say that x G x plus y G y will be equals to 2 G and what will be G x from here, it is 1 by g times, 1 by g can come out x g x plus y g y will be equals to 2 l n g. So, this implies x g x plus y g y will be 2g l n g and for h it is x h x plus y h y will be equals to 3h. Now you add the two equations. It is 2, it is 1, you add the two equation. So, what you will obtain, it is x times g x plus h x which is u x, y times g y plus h y which is u y is equal to this plus this and hence x u x plus y u y which is equal to 2 times x g is e raise to power x square by y square log g is x square by y square this side plus 3h, h is. So, that is how we can find out the value of this expression.

So, thank you very much.