Multivariable Calculus
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Lecture - 11
Change of Variables

Hello friends. So, welcome to lecture series on multivariable calculus. So, we have already discussed what chain rule are? And how we can apply chain rule on several variable functions? Now change of variables ok. So, what change of variables are? Let us; see, suppose z is a function of x and y.

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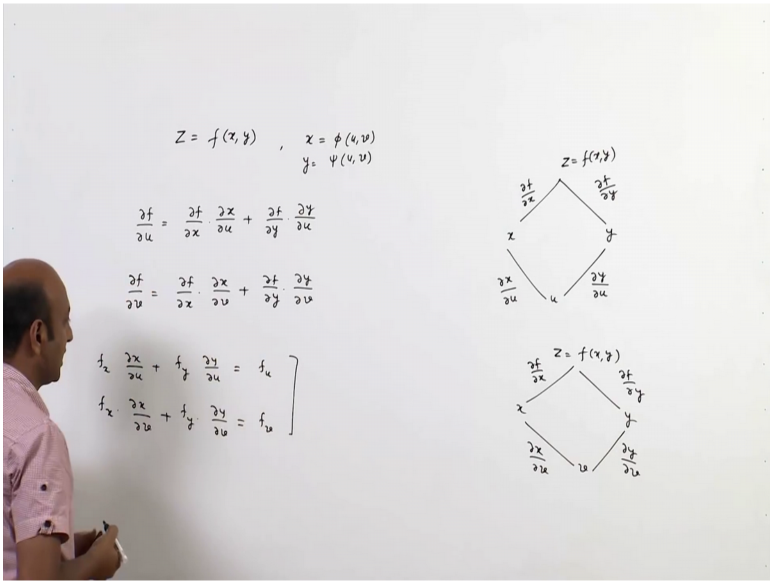

Again suppose z is a function of 2 variables x and y, x and y are independent variable and z is a dependent variable. Now x and y suppose x is the function of u and v again and y is again a function of u and v.

So, we already know, that if you want to compute say del f by del u by the chain rule we can write del f by del x into del x by del u plus del f by del v into del v by del y into del y by del u. Because f is a function of x and y, and x and y both are the function of u ok. Similarly, del f by del v will be del f by del x into del x by del v plus del f by del y into del y by del v. This we can also understand by t diagram, as I already discussed, z is a function of x and y and x and y both are the functions of u.

So, it is del f by del x it is del x by del u here it is del f by del y here it is del y by del u. So, del f by del u will be this into this plus this into this which is this term ok. And similarly, if z is a function of x and y, and you want to compute del f by del v. So, it is del f by del x, it is del x by del v, it is del f by del y, it is del by del v. So, del f by del v will be this into this plus this into this, that is this term ok.

Now, basically let us compute del f by del x and del f by del y from these 2 equations ok. What are the equations we are having here? It is f x into del x by del u plus f y into del y by del u is equals to f u, this is the first equation ok. The second equation is f x into del x by del v plus f y into del y by del v is equal to f v. So, these are 2 equations we are having. Now basically these are the 2 system of equations ok.

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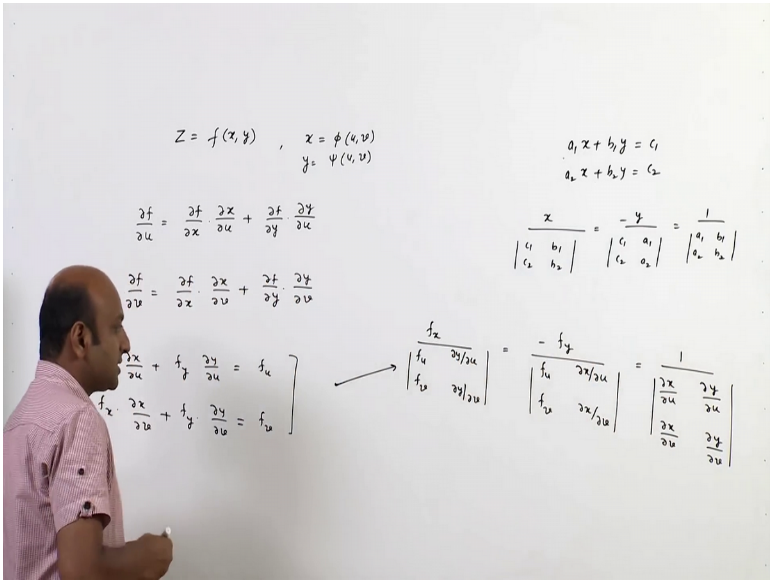

Let us try to compute f x and f y in terms of f u and f v ok. So, how can we compute? We already know Cramer rule it is a 1 x plus b 1 y plus is equal to c 1 and a 2 x plus b 2 y is equal to c 2, then it is x upon, it is c 1 c 2 b 1 b 2 minus y upon determinant c 1 c 2 a 1 a 2 and 1 upon determinant a 1 a 2 b 1 b 2. So, this is a simple Cramer rule by which we can find the values of x and y.

Now, suppose, because these are the unknown's f x and f y are the unknowns ok. So, they may be treated as x and y. And this is a 1, this is a 2, this is c 1 this is a 2, b 2 and c 2. So, let us compute f x and f y from these 2 equations. So, similarly from these 2 equations it is f x upon determinant of it is f u f v c 1 c 2, and then b 1 b 2. It is these 2;

that is del y by del u upon del y by del v, which is equals to minus f y upon determinant of c 1, c 2. Again c 1 c 2 which is f u f v ok. And then a 1, a 2, a 1 is this, a 2 is this that is del x by del u and del x by del v and is equals to 1 upon determinant of a 1 a 2 b 1 b 2; that is, del x by determinant of del x by del u, del x by del v, and then it is del y by del u and del y by del v ok.

Now, this term this determinant is basically this determinant, this determinant del x upon del u, del y upon del u, del x upon del v del y upon del v is also denoted by del of x y upon del of u v.

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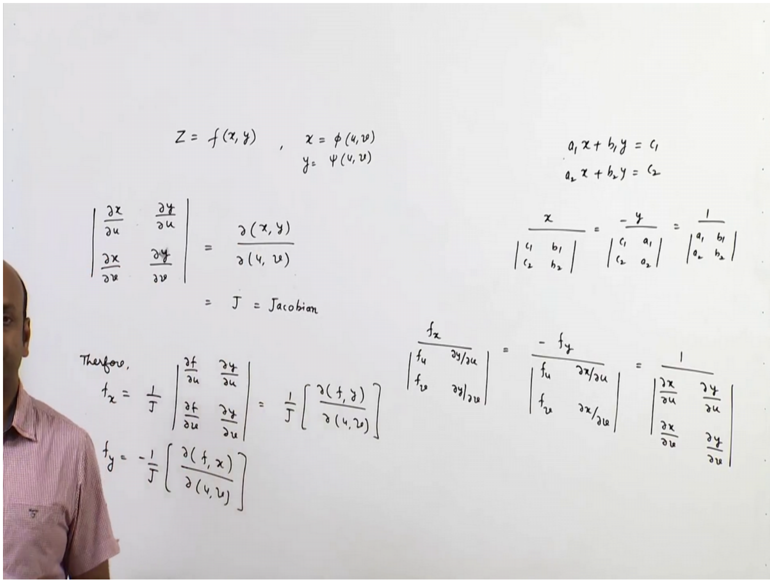

Which we call as denoted by J and it is called Jacobean. Jacobian of x and y ok. This is denoted by j, this term which is this determinant, and this is called we call it Jacobean ok. So, we can write f x as from this equation, f x can written as therefore, f x will be 1 upon J times. Because when you compare this with this term; it is 1 upon J, J is this term ok, 1 upon J and this will this will go to a numerator. So, this will be determinant of del f upon del u, del y upon del u, del f upon del v, del y upon del v; which is 1 upon J times, it is del of f y upon del of u v.

Ok, that means, del f upon del u del y upon del u del f upon del v and del y upon del v ok. Similarly, f y will be minus 1 upon J this will go to a numerator, and this will be del of f x upon del of u v. Which is del f upon del u, del f upon del v, del x upon del u del x upon del v, this term. So, that is how we can compute f x and f y if we know the values of del f upon del u or del f upon del v. So, this is basically change of variables. So, that is what this given the PPT, that is del f upon del x is 1 upon J times del of f y upon del of u v.

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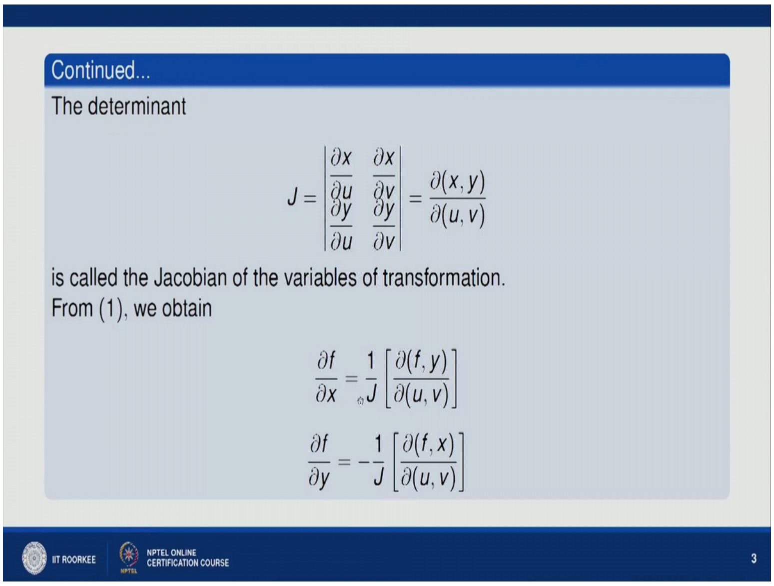

Similarly, f x upon f y will be minus 1 upon J times del of f x upon del of u comma v ok.

Now, let us try to solve some problems based on this. Then we will see some other properties of Jacobean. So, first problem is here z is a function of x and y and x is r cos theta.

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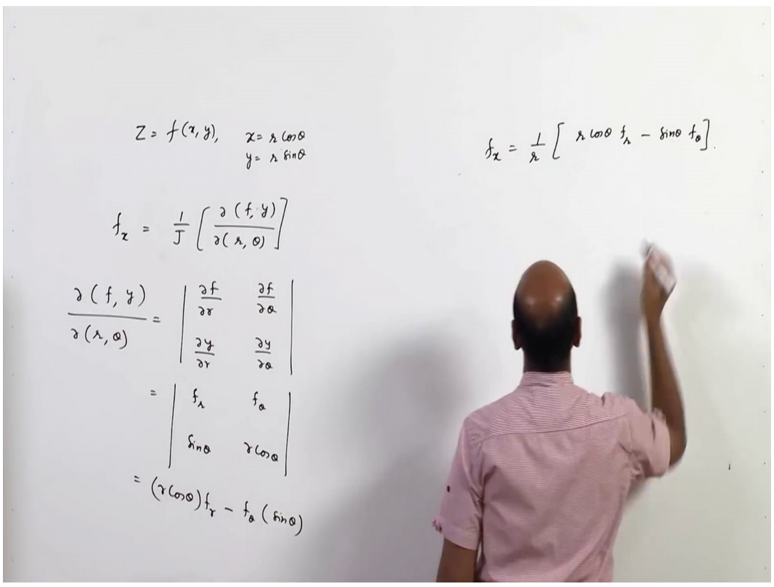

And y is r sin theta. So, we have to prove that f x square plus f y square is equal to f r square plus 1 by r square times f theta square ok. So, let us try to prove this; so, f x as we already know that f x is 1 upon J times del of f y upon del of r theta. Because instead of u and v, here we are having r in theta u is r and v is theta ok.

So, first let us compute J. J is Jacobean is del of x y upon del of r theta. It is determinant of del x upon del r del x upon del theta, del y upon del r, del y upon del theta, which is equal to; now what is del x upon del r from here? It is cos theta. Del x upon del theta it is minus r sin theta. Del y upon del r is sin theta. And del y by del theta is r cos theta. When you take the determinant, it is r times cos square theta plus sin square theta, which is r into 1 which is r ok.

Now, what is del of f y upon del of r theta, it is determinant of del f upon del r del f upon del theta del y upon del r del y upon del theta, which is equal to determinant of it remain as it is f r, f theta remain as it is. And del y by del r from here is from here is sin theta, and del y by del theta is r cos theta. So, the determinant will be r cos theta into f r minus f theta into sin theta. So, what will be f x then. So, f x will be 1 by r times because J is r ok. And this value is this thing and times it is r cos theta f r minus sin theta into f theta. So, this is f x.

Now, similarly how can you find f y.

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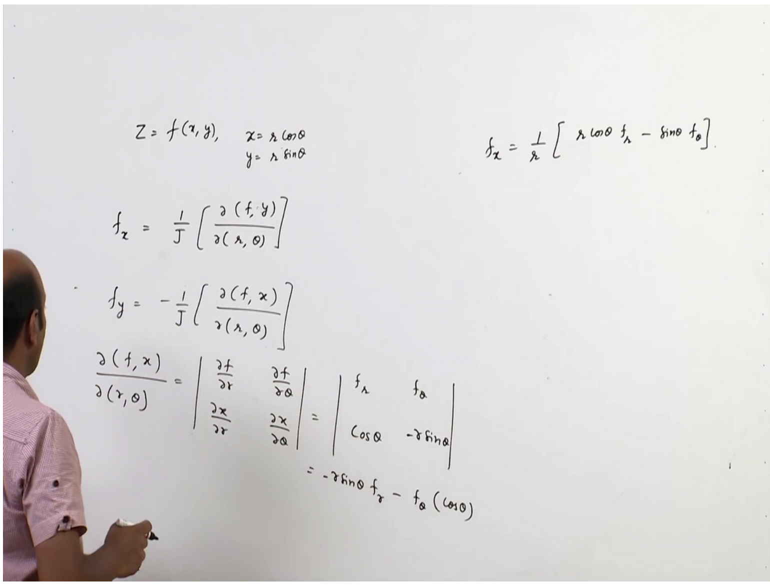

Now, f y will be minus 1 by J times del of f x upon del of r theta. So now, let us compute this we already know J is r. So, let us compute this term this, this term is del of f x upon del of r theta, which is determinant of del f upon del r del f upon del theta, del x upon del r, and del x upon del theta. And it is equal to it is f r, it is f theta del x upon del r from here, it is cos theta and del x upon del theta is minus r sin theta. So, this value will be minus r sin theta f r and minus f theta into cos theta.

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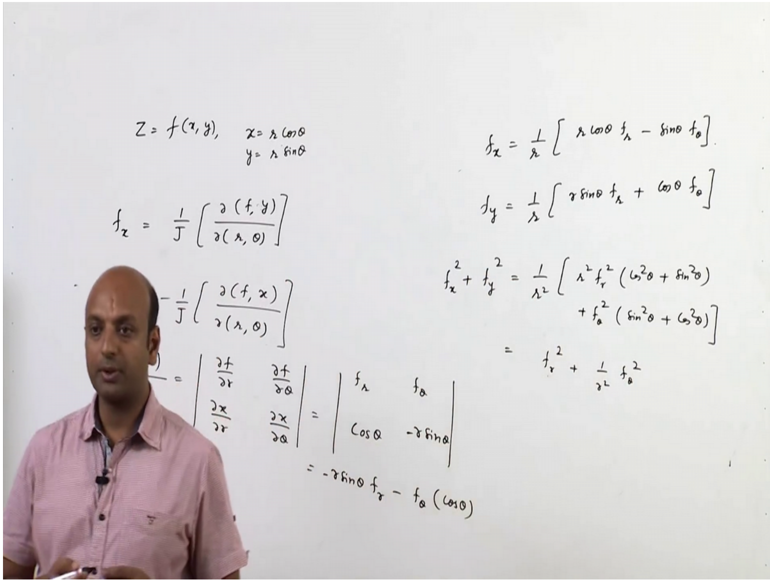

So, so, what will be f y, then f y will be so, negative will be cancelled from this negative. So, it will be 1 by r times r sin theta into f r plus cos theta into f theta. So, what will be f x square plus f y square, this we have to compute ok. We have to compute f x square plus f y square. So, square and add both the equations. So, when we square and add we get 1 by r square times. It is a square a minus b whole square it is a plus b whole square.

So, it will be r square times f r square will be common it is cos square theta plus sin square theta from this side. From this side it is f theta square and it is sin square theta plus cos square theta. And 2 times this into this, and 2 time this into this will be cancel out. Because they are having opposite sign. So, what we are getting from here? R square also cancel out it is f r square plus 1 by r square times f theta square. So, hence we have proved, hence we have proved this problem.

So, that is how using change of variables also, we can show such type of problems. We can directly use chain rule also no problem. But we can also show we can also try to solve this problems using change of variables. In the same problem, I want to emphasize one more property which is you see, you see a when we deal with single variable functions say y equal to f x. Then we can always right d y by d x is equal to reciprocal of d x by d y. That is always true for single variable functions, but it may not be true for partial derivatives. Let us see what is del x by del r from here.

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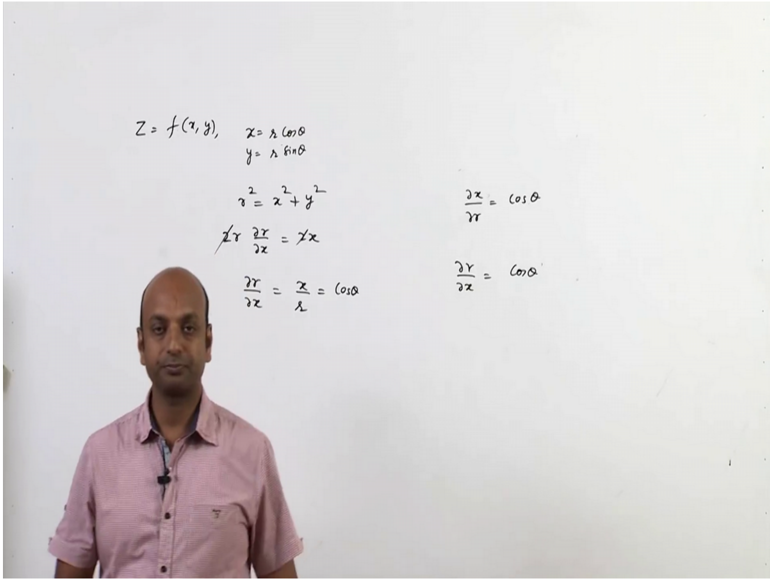

it is cos theta ok.

Now, if you want to compute del r by del x, what it will be. Now we cannot compute del r upon del x from this equation. We first have to express r in terms of independent variables. Now for r independent variables are x and y. So, what will be equation of r in terms of x and y it is r square equal to x square plus y square ok. Not from this question, because this r equal to x sec theta, but sec theta, but this sec theta ok.

We cannot differentiate from here. Partially because, we have to express r in terms of independent variables ok, x and y are independent not x n theta ok. So, what will what will be del r from del x from here. It is 2 r del r by del x is equals to 2 x 2 2 cancels out. So, del r by del x is equals to x by r, and x by r from here it is cos theta. So, it is also cos theta. So, we have seen that del r by del x is not reciprocal of del r by del x.

Now, let us come to second problem. Second problem is here z is function of x and y.

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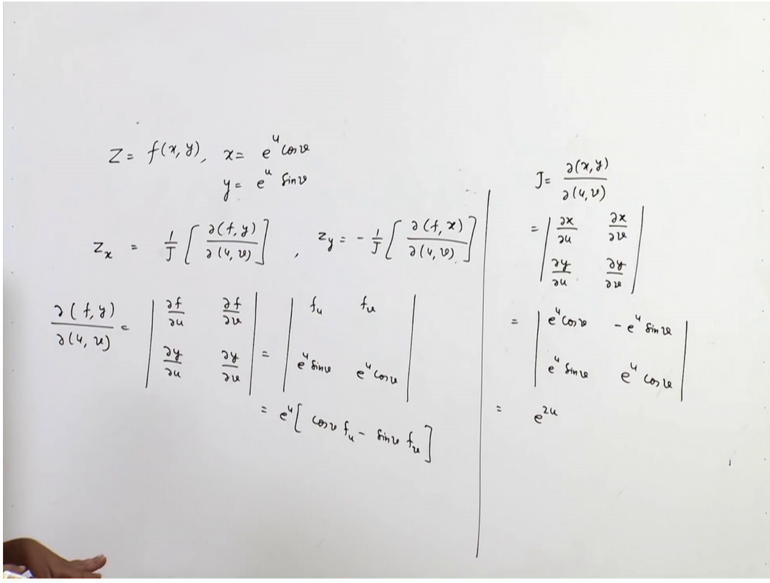

and x is x is e raise to power u cos v and y is again e raise to power u. It is sin v. And then we have to show that z x square plus z y square is equal to this expression ok. So, how can we prove this is it, this is it ok. So, first we compute z x, z x z x or f x both are same. So, how to compute this is 1 by J times del of f y upon del of u v. Here independent variables are u and v. And what will be z y z y will be minus 1 by J times del of f x upon del of u v.

So, first we compute J. So, what will be J? J will be del of x y upon del of u v, which is equal determinant of del x upon del u del x upon del v del y upon del u and del y upon del v. And this further equal to del x upon del u from here is e raise to power u cos v, it is minus e raise to power u sin v, it is e raise to power u sin v, and it is e raise to power u cos v. And it is equals to e raise to power 2 u. Because sin a square plus cos square is 1.

Now, let us compute this term. This term is del of f y upon del of u v, which is equal to determinant of del f upon del u, del f upon del v, del y upon del u del y upon del v. Which is equals to determinant of f u, f v. This term is e raise to power u sin v and this is e raise to power u cos v, and this is e raise to power u will can come out. And it is cos v f u minus sin v f v sin v f v ok. So, this is this term. Now what will be del of; now what will be this term ? So, this we have to remember that J is e raise to power to u, now let us compute del f x upon del u v it is equals to again f u f v.

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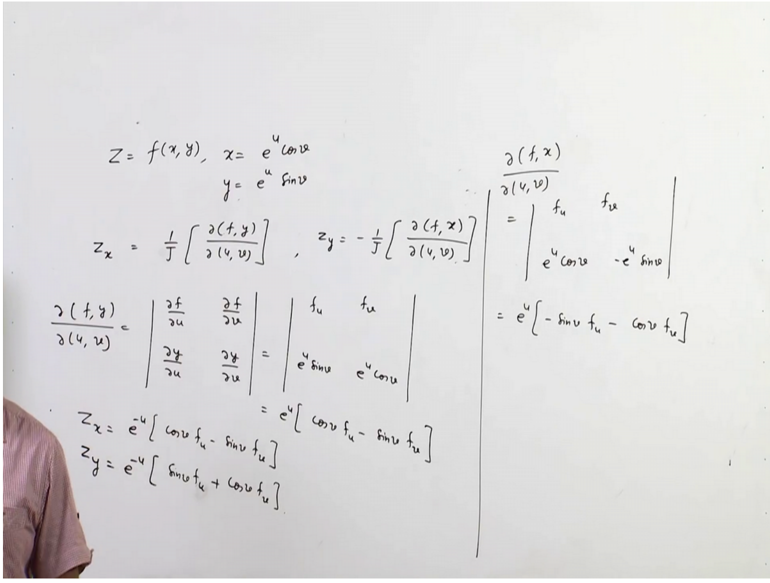

Then del x by del u del x by del u from here is e raise to power u cos v and del x by del v is e minus e raise to power u sin v. And determinant will be e raise to power u can come out minus sin v f u minus cos v f v ok.

So, what will be z x? So, z x will be 1 upon J times this quantity. So, 1 upon J time this quantity, J is e raise to power 2 u. So, it is e raise to power minus u times cos v f u minus sin v f v. And what will be z y? Again, e raised to power minus u times, this negative will cancel out. So, it is sin v, f v oh f u plus cos v f v. Now you simply square and add both the equations, when you square and add both the equations J x square plus z y square, it is e raise to power minus 2 u will come out. It is f u square because, this square plus this square is 1. It is f v square because this square plus this square is one and the product term will cancel out because there having a opposite sin. So, we have proved the result.

Now, we have some properties of Jacobean. So, let us try to show property these properties one by one. So, the first property is if J is del of u v upon del of x y.

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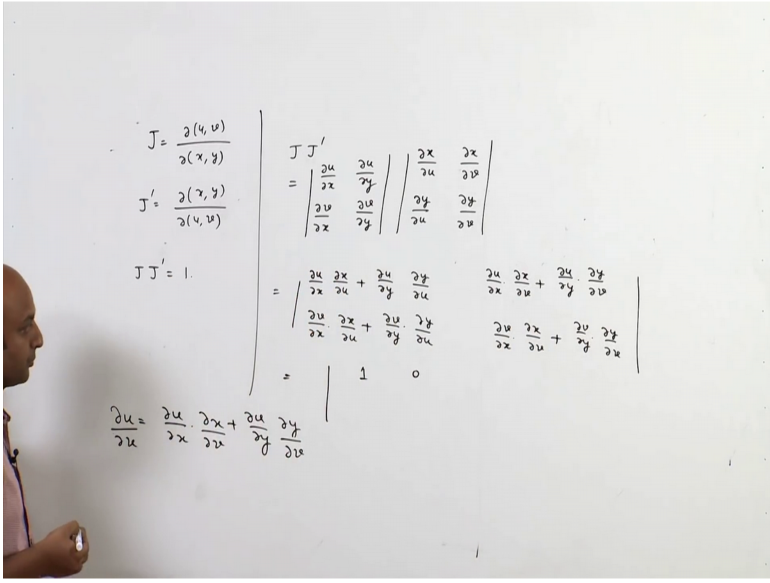

And J dash is del of x y upon del of u v, then J into J dash is always 1. Then J into J dash is always 1. This is the first property, and the proof is very easy. So, what is J into J dash. J into J dash is this into this ok. Now this is determinant of del u by del x into del u by del y del v by del x into del v by del y. And this determinant J dash is this thing which is del x upon del u del x upon del v, then del y upon del u, and then del y upon del v ok. This is one determinant, and this is second determinant and the product of these 2 will be J into J dash.

Now, determinant of a matrix A and determinant of matrix B will be determinant of A into B, we already know this result. So, determinant of this into determinant of this will be determinant of this matrix into this matrix ok. Take this as a matrix the inside 2 cross 2 rectangular we are calling it a matrix, And this is a matrix so, determinant of A into determinant of B determinant of A into B. So now, we are computing A into B. So, this row this column which is del u by del x into del x by del u, plus del u by del y into del y by del u. This is the first term. The second term is, del u by del x into del x by del v plus del u by del y into del y by del v.

Similarly, the next 2 terms are this row this column. So, that that will be del v by del x into del x by del u plus del v by del y into del y by del u. And then this is del v by del x into del x by del v plus del v by del y into del y by del v . Now what is what is del u? Del u we can easily write it is del u by del x into del x plus del u by del y into del y ok.

Because u is a function of x and y ok. We can easily write this expression now divide the entire expression by del u ok. When you when you divide the entire expression by del u. So, we obtain this term, and this upon this is 1. So, this term is the first term is 1.

Now, in the same expression, let us divide the entire expression by del v. Now u and v are independent. So, del u by del v is 0 ok. So, this is the value of this term is 0; that means, the value of this term is 0.

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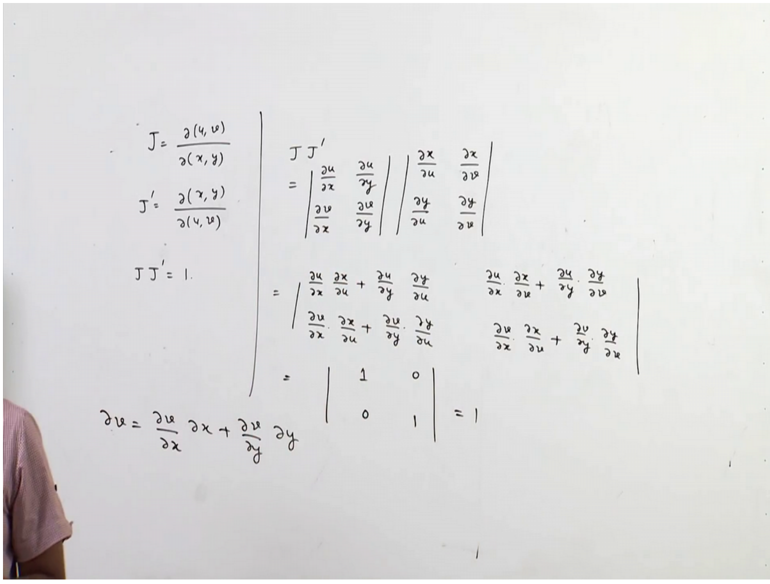

Similarly when you when you find del v it is del v by del x into del x plus del v by del y into del y. You divide the entire expression by del u, when you divide the entire expression by del u. So, del u and v are independent. So, this will be 0, hence the value of this expression will be 0. And when you divide the entire expression by del v. So, del v by del v is 1. So, the value of this expression will be 1. So, determinant is 1. So, hence we have shown that J into J dash is always 1.

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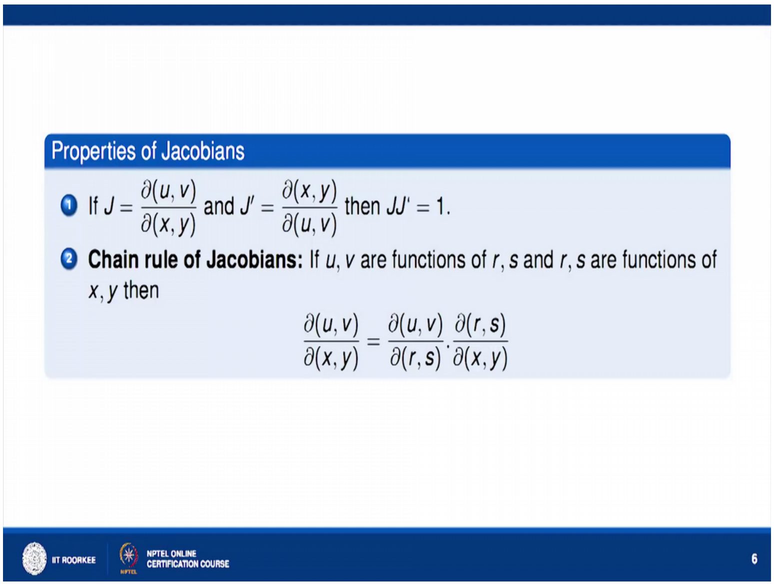

Now, second property is chain rule type property of Jacobean. What it is del of u v upon del of x y?

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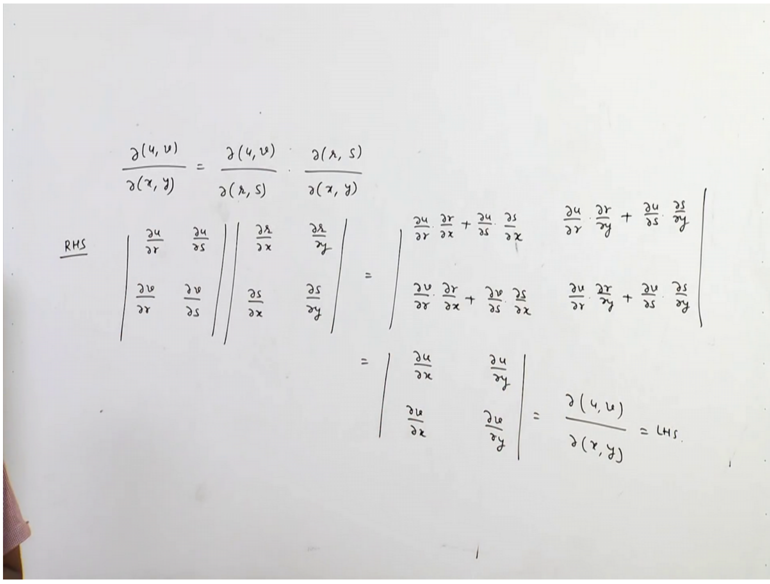

Can be written as del of u v upon del of r s into del of r s upon del of x y. So, this is basically chain rule properties of Jacobean ok. Another proof of this is also easy. Let us try to prove it, let us take the right-hand side. Right hand side is what? It is del u by del r in del u by del s, del v by del r and del v by del s. And determinant of this is del r by del x, del r by del y del s by del x, and del s by del y. Again, the determinant product of this determinant is equal to determinant of this row this column which is del u by del r into del r by del x. Plus, del u by del x into del s by del x.

Then this row this column which is del u by del r. This row this column into del r by del y, plus del u by del s into del s by del y. Now this row this column, which is del v by del r into del r by del x plus del v by del s into del s by del x. Then this row this column the last one, that is del v by del r into del r by del y plus del v by del s into del s by del y. Now the first term, first term is basically del u by del x if you carefully see ah, this term this term basically u is a function of r n s ok.

So, when you apply chain rule. So, it is del u by del r into del r by del x plus del u by del s into s by del x. So, it is basically this term is basically del u by del x. Similarly, this term is basically del u by del y again this is basically del v by del x and this is del v by del y. So, it is nothing but del of u v upon del of x y which is left hand side. So, we have proved. So, this is chain rule of Jacobeans.

Now, let us solve some problems based on this.

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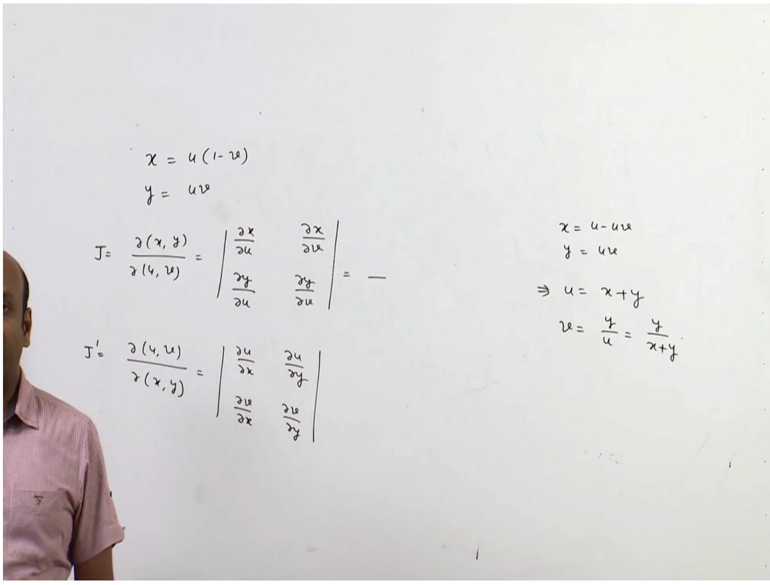

The first property so, we can easily verify this property, this is x and y is u v. If you compute say if you take J as del of x y upon del of u v. So, it is del x upon del u del x upon del v, it is del y upon del u and del y upon del v. So, that we can easily compute ok. Now if you compute J dash which is reciprocal of this that is del of u v upon del of x y. It is del u upon del x, del u upon del y, del v upon del x and del v upon del y. Now first you have to express u and v in terms of x and y. Then only we can compute J dash. For J we are having no problem, because J because x and y are already in terms of u and v. So, for finding this expression there is no problem, but for this you have to first express u and v in terms of x and y.

So, what is x? X is u minus u v and y is u v. So, when you add these 2, this implies u is equals to x plus y. So, u in terms of x and y is x plus y. And what will be v v is y upon u. So, that is y upon x plus y ok. So, we can compute del u by del x from here del u by del y, del v by del x, del v by del y from this expression ok. Compute J and J dash and we can easily verify that J into J dash will come out to be 1 ok.

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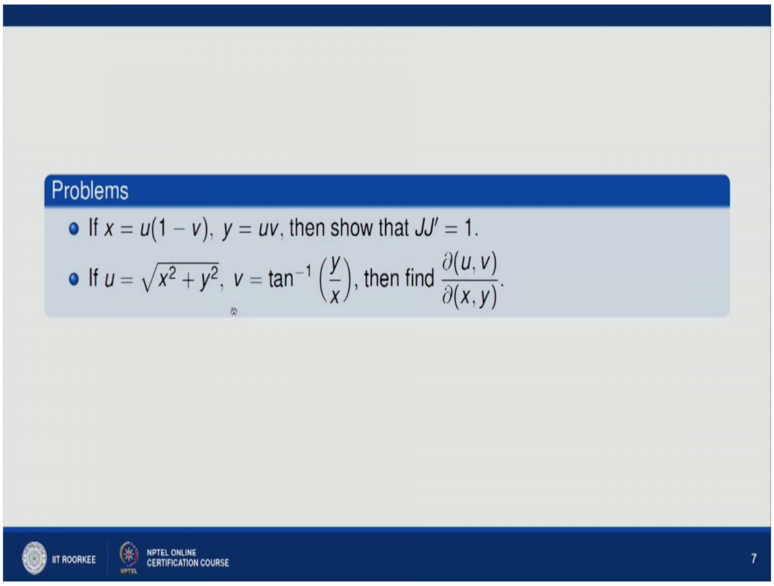

Now, the next problem. Ah let us try to solve the next problem now.

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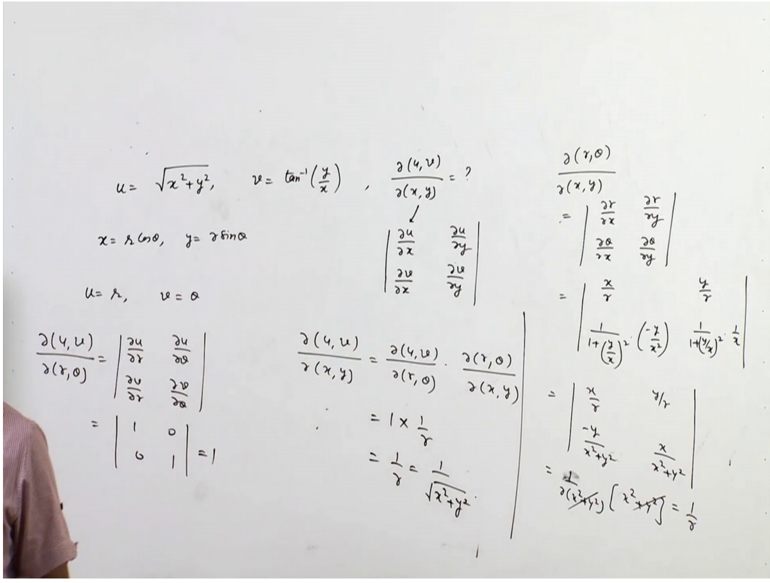

Next problem is u is under root x square plus y square and v is tan inverse y by x. And you have to compute del of u v upon del of x y. So, this we can compute directly also. This is nothing but determinant of del u by del x, del u by del y del v by del x and del v by del y. This we can compute directly also. But we can compute this using chain rule also, how? So, let x equal to r cos theta, and y equal to r sin theta. So, what will be u, u will be r and v will be theta. And what will be del of u v upon del of x y, it will be del of u v upon del of r theta into del of r theta into del of x y. By the chain rule property of Jacobeans.

Now, what is del u v upon del r theta, del u v upon del r theta is determinant of del u upon del r del u upon del theta del v upon del r and del v upon del theta, which is because u equal to r and v equal to theta, which is 1 0 0 1, and it is 1 ok. Now del of del of r theta upon del of x y is equals to determinant of del r upon del x del r upon del y del theta upon del x and del theta upon del y. Which is equal to, which is equals to del r upon del x. Now you have to express r in theta in terms of x and y ok. So, what is r? R is r square is x square by y square. And what is theta, theta is tan inverse y by x.

Basically, we already know that this value this Jacobean is basically r. Let us compute this thing ok. We have a result that del of r theta point del of x y is basically I think 1 by r let us see ok. So, what will be del r upon del x from here it is x by r ok. What is del r by del y from here it is y by r. What is del y by del theta by del x from here it is 1 upon 1 plus y by x whole square into and it is by with respect to x that is minus y by x square . And del theta by del y is 1 upon 1 plus y by x whole square into 1 by x ok. So, that is x by r y by r and it is minus y by x square plus y square, and it is x upon x square plus y square ok. And this is equal to basically x square basically, 1 by r x square y square can come out. And it is x square plus y square, that cancel out and it is 1 by r yeah.

So, it is 1 by r basically it is a result basically that del of r theta upon del of x y is 1 by r. We can use it directly also. So, so now, what will be this expression? This expression it is one it is 1 by r. So, it is 1 by r and r is under root x square plus y square. So, we can solve such problems using chain rule also ok. Now, similarly if we have function of 3 unknowns. Then also we can convert using change of variable.

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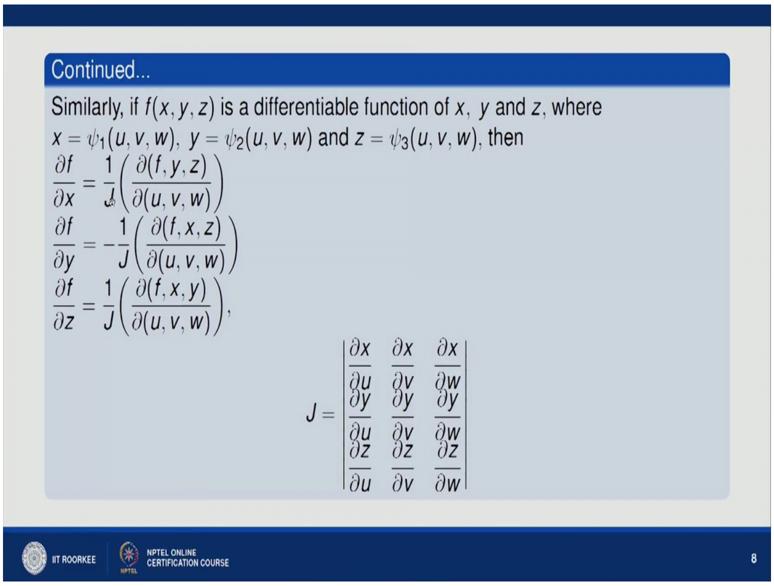

So, f x we can obtain as 1 by J times del of f y z upon del of u v w. Similarly, del f upon del y and del f upon del z; where J is simply del of x y z upon del of u v w.

Now, let us let us try to solve the first problem is easy you can easily solve the first problem. Let us try to solve the second problem, now the second problem u is x square plus y square plus z square.

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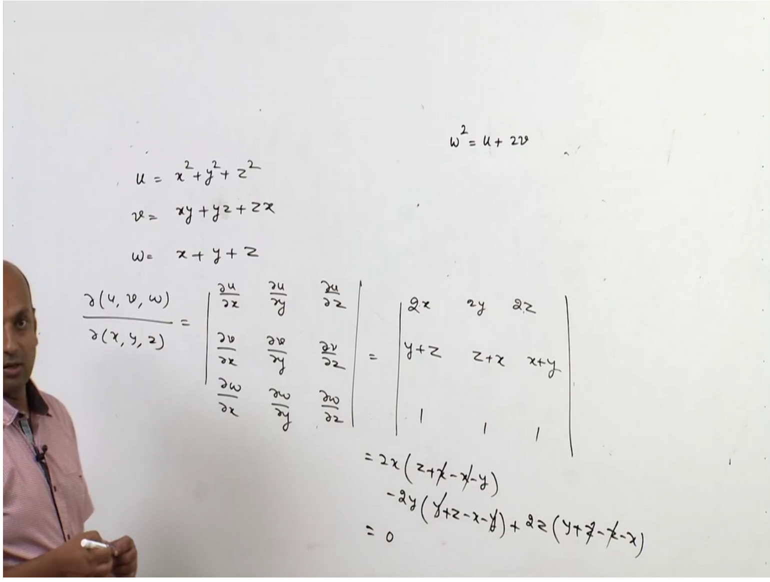

And v is x x y plus y z plus z x, w is x plus y plus z. And you have to compute del of u v w upon del of x y z. So, what will be this Jacobean this is del of u upon del of x del u upon del y del u upon del z, del v upon del x del v upon del y del v upon del z. Again, del w upon del x del w upon del y del w upon del z, and it is equal to what is del u upon del x ? 2 x 2 y 2 z, it is del del v upon del x it is y plus z. It is 2 it is z plus x it is x plus y, and it is 1 1 1 ok.

Now, when you simplify this. So, we get 2 x times z plus x minus x minus y minus 2 y times y plus z minus x minus y, and it is plus 2 z times y plus z minus z minus x. So, when you simplify this. This will cancel out. When you simplify this, you will get 0. So, Jacobean of this is 0.

Now Jacobean comes out to be 0, what does it mean? It means that these relations of u v and w are not are not independent, they are dependent ok. And what is the relation? Relation you can easily verify. You very easily see that w square is equal to basically u plus 2 v. So, these are relation. That is why Jacobean is coming out to be 0 ok. So, this is all about change of variables.

So, thank you very much.