Multivariable Calculus
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Lecture – 10
Chain Rule-II

Hello friends. Welcome to lecture series on Multivariable Calculus. So, in the last lecture, we have seen what chain rule is and how can we apply chain rule on multivariable functions. Now, you will see some more properties of chain rule.

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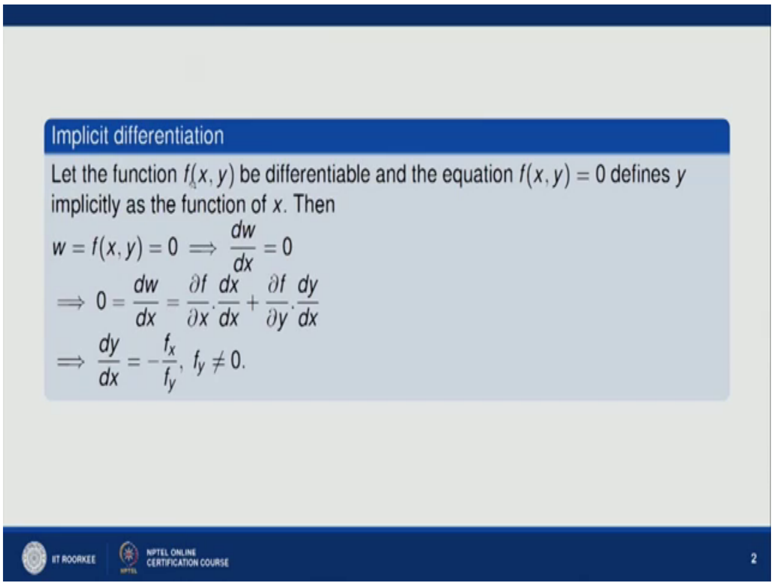

Now, let us suppose function x y function of 2 variables be differentiable and the equation f x y equal to 0 defines y implicitly as a function of x then. So, here we are taking w as a function of x and y.

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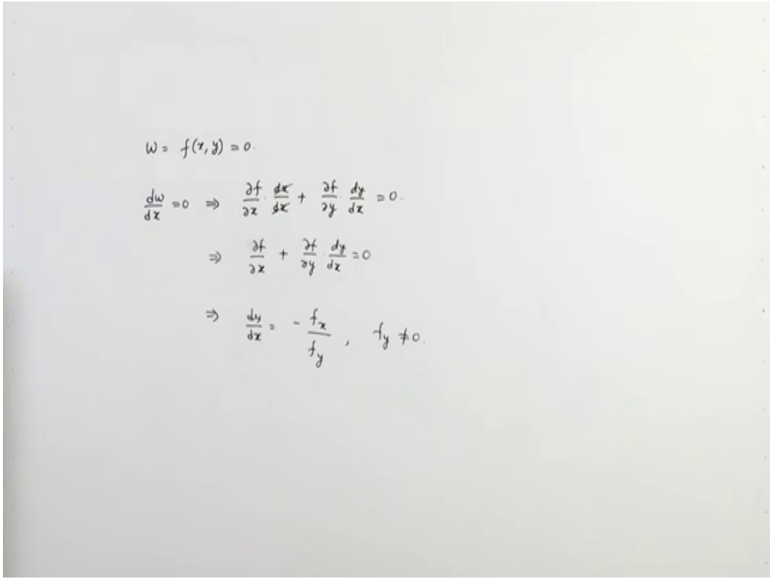

And which is equal to 0 where assuming that this function is a differentiable function and we can express y implicitly as a function of x.

Now, differentiate both side respect to x. So, what will obtain to? So, it is a again d w by d x will be 0 of course. So, this implies; now what will be d w by d x from here; now f is a function of x and y ok. So, we can apply chain rule here, it is del f by del x into d x by d x plus del f by del y into d y by d x equal to 0 because we are having 2 unknowns x and y here ok, we are having 2 unknowns x and y and we are differentiating respect to x. So, respect to x, they are differentiate ok.

So, d x; d x will cancel out and this is implies or it is 1. So, it is del f by del x plus del f by del y into d y by d x is equal to 0 and this implies d y by d x is equals to minus f x upon f y f y should not equal to 0. So, if you want to compute d y by d x directly if f is known. So, simply use simply, we can use this expression negative of f x upon f y that is differentiate partial respect to x divided by differentiate f partial respect to y ok. So, see some problems based on this.

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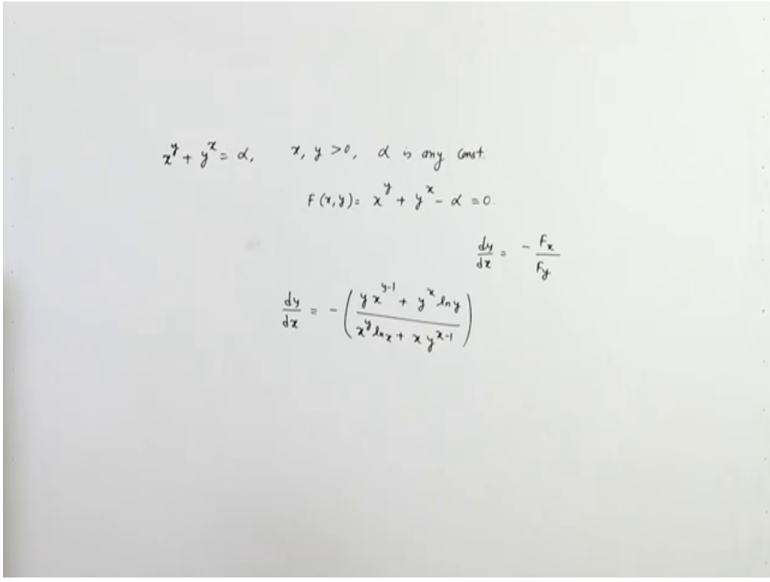

Suppose, f is suppose it is f raise to power y plus y raise to power x is equals to alpha where x and y are greater than 0 and alpha is any constant ok.

Now, you want to compute d y by d x for this equation you want to compute d y by d x. So, how can we do that? So, we can use this expression basically, we can let f of x y as x raise to power y plus y raise to power x minus alpha equal to 0. Now you know that d y by d x is nothing, but minus of f x upon f y provided f y should not equal to 0 of course, ok. So, from this expression, we can get back to find d y by d x which is negative of now f x; f x means differentiate partial respect to x. So, differentiate this expression partial respect to x; so keeping y constant.

So, it will y x raise to power y minus 1 plus y raise to power x l n y upon because you are treating y as constant now f y we are differentiating this expression partial respect to y keeping x constant. So, it will be x raise to power y l n x plus x y raise to power x minus 1. So, this will be d y by d x. So, simply using this expression we have solved this problem very easily we have find out d y by d x very easily using this formula ok.

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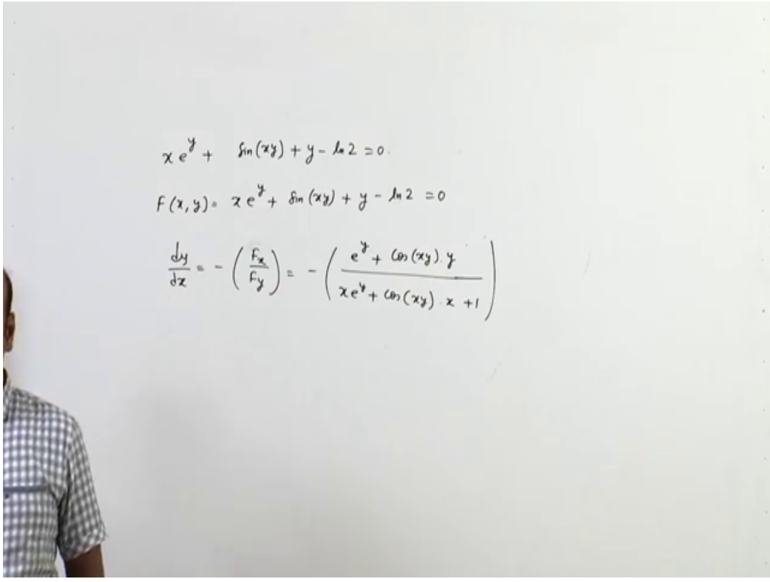

Now, the second problem; second problem is x e raise to power y plus it is sin x y plus y minus l n 2 equal to 0 and you want to compute again d y by d x. So, you can let f of x y as x e raise to power y plus sin x y plus y minus l n 2 is equal to 0 and d y by d x will be given as negative of f x upon f y which is equals to negative of now partial derivative of this f respect to x.

So, it is it is e raise to power y plus sin theta is cos theta and theta again that will be respect to x that is y plus 0 upon; now f y means partial derivative of this f respect to y. So, it is x e raise to power y plus cos x y into x and plus 1 of course, because derivative of this is plus 1 partial derivative this is plus 1. So, this will be d y by d x which we can compute easily using this result ok.

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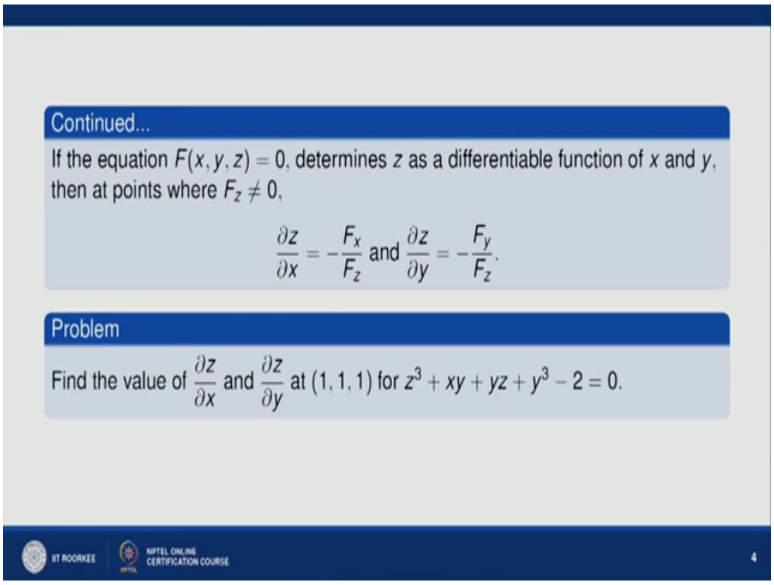

Now, now in a save 2 variable supposed a f is a function of 3 unknowns x y and z where z is a differentiable function of x and y and at points where f z is not equal to 0 we can easily say the del z by del x is given by this expression and del z by del y is given by this expression how we can obtain this expression.

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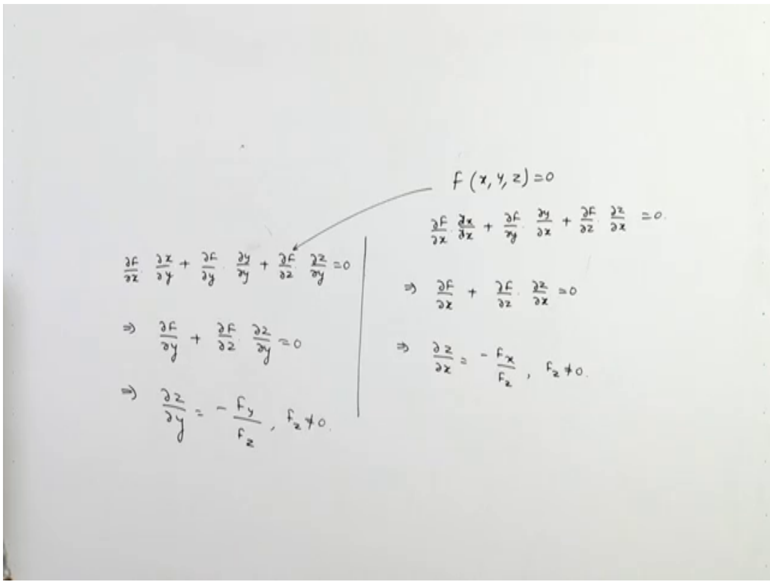

Now, here f is a function of x y z which is equals to 0. Now differentiate partial respect to x so; that means, you differentiate respect to x. So, that will be del f by del x into d x by d x plus del f by del y into del y by del x plus del f by del z into del z by del x equal to 0 you differentiate partial respect x you can write d here or del here.

Now, this implies del f by del x this is one now y and x are independent ok. So, del y by del x will be 0. So, this will be 0 plus del f by del z into del z by del x equal to 0 and this implies del z by del x will be minus f x upon f z provided f z is not equal to 0. So, that is how we can obtain del z by del x the first expression here in this PPT. Now if you want to compute del z by del y. So, similarly you can differentiate here partial respect to y. So, it will be del f by del y into del y by del x plus del f by ok, you have to differentiate partial respect to y. So, it is x it is x and here it is y ok, it is del f by del y into del y by del y plus del f by del z into del z by del y because your differentiating partial respect to y both the sides.

Now, again x and y are independent variables. So, this will be 0. So, and this is one. So, this implies del f by del y plus del f by del z into del z by del y will be 0 and this implies del z by del y will be minus f y upon f z provided f z is not equal to 0. So, that is how we can simply obtain del z by del y, suppose you have this problem find the value of del z by del x and del z by del y at 1, 1, 1 for this expression we can obtain these values directly also and we can use these formula also to find out del z by del x or del z by del y in this problem what is expression? Expression is equation is.

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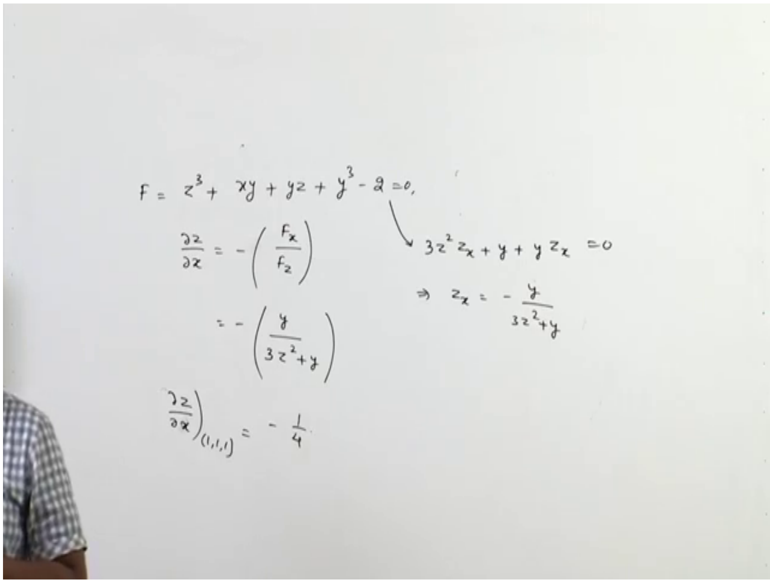

X cube plus it is x y plus y z plus y cube minus 2 equal to 0 say it is f and you want to compute say del z by del x.

So, it is negative of just now we have proved that it is minus of f x upon f z f z should not equal to 0. So, what will be f x differentiate this partial respect to x. So, it is it is y in the numerator upon respect to z, it is 3 z square plus y and at 1, 1, 1, it is at 1, 1, 1, it is minus 1 by 4, you can you can simplify this directly also differentiate this respect to x partially. So, x z is a function of x and y. So, it is 3 z square into z x plus from here it is y plus from here it is y into z x because y is a independent of x and is equal to 0.

So, this implies z x is minus y upon 3 z squared plus y the same expression similarly we can compute del z by del y also at point 1, 1, 1. Now come to some more problems based on chain rule.

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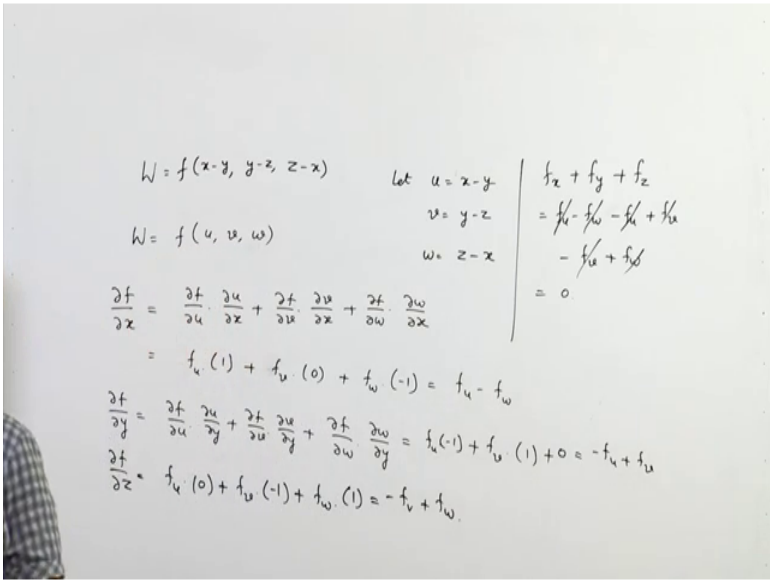

Say w is the function of x minus y y minus z z minus x is differentiable.

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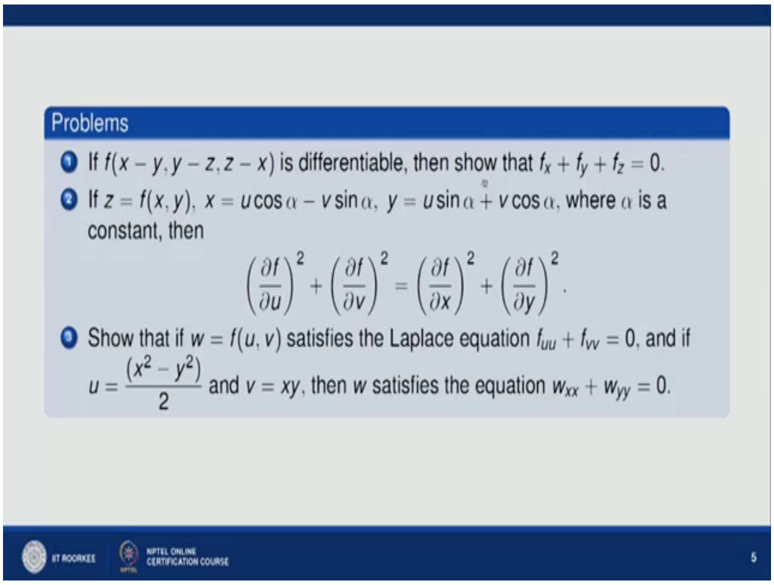

Then you have to show that f x plus f y plus f z equal to 0, this you have to prove. So, how can you prove this? So, let u equal to x minus y v equal to y minus z and w equal to z minus x. So, this w this is w in this also w we make it you make this as capital W. So, now, this capital W will be f of u v a small w now what will be del f by del x now this f is a function of u v and w and u and v w are again function of x y z. So, that will be del f by del u into del u by del x by chain rule del f by del v into del v by del x plus del f by del w into del w by del x and that will be f u into del u by del x from here is 1. So, it is one plus del f by del v is del v f v and del v by del x is 0 no x is here plus del f by del w is f w into del w by del x is minus 1. So, it is f u minus f w.

Now, similarly if you want if you compute del f by del y it is del f by del u into del u by del y plus del f by del v into del v by del y plus del f by del w into del w by del y and which is equals to it is f u into del u by del y is minus 1 plus it is f v del v by del y is one del y is 1 plus and del w by del y is 0. So, this is 0. So, this is minus f u plus f u.

Now if you compute del f by del z it is again del f by del u which is f u into del u by del z which is 0 plus f v into del v by del z which is minus 1 plus f w into del w by del z which is one. So, it is minus f v plus f w now if you take the sum if you take f x plus f y plus f z sum of these 3. So, it is f u minus f w plus it is minus minus f u plus f v and it is minus f v plus f w and this we cancel with this, this cancel with this, this cancel with this it is equal to 0. So, hence we have proved.

Hence, the first question, we have solved, now second problem suppose now the second problem is z in the function of x and y.

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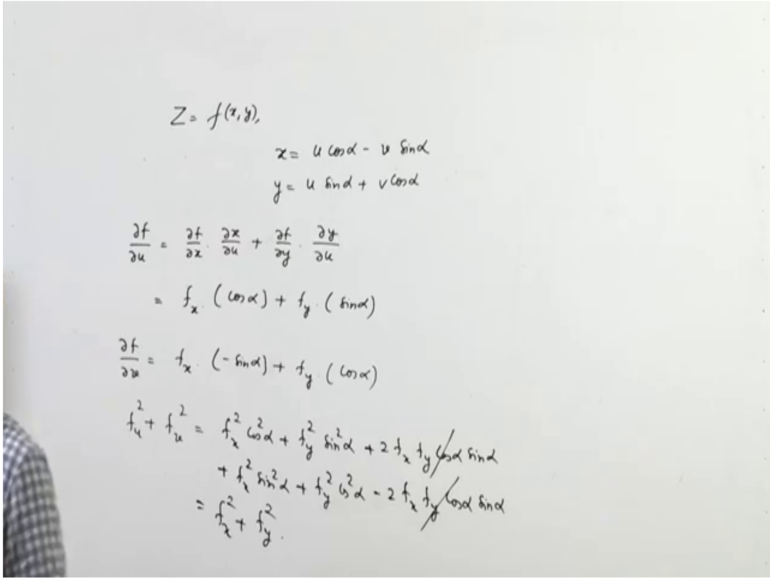

And x is equal to u cos alpha minus it is v sin alpha y is a function of again u and v this is u sin alpha plus v cos alpha has given the problem. So, alpha is a constant here where alpha is a constant.

So, x is a basically function of 2 unknowns u and v and why is again a function of 2 unknowns u and v then we have to show that del f by del u whole square plus del f by del v whole square is equals to del f by del x whole square plus del f by del y whole square. So, let us try to compute this. So, what is del f by del u it is del f by del x into del x by del u plus del f by del y into del y by del u by the chain rule. So, it is f x into del x by del u is cos alpha plus f y into del y by del u is sin alpha.

Now, del f by del v is again f x into del x by del v del x by del v is minus sin alpha plus f y into del y by del v is cos alpha. Now if you take f u square plus f u square this will be equals to a square and add both this equations. So, what will obtained this square this square is f x square cos square alpha plus f y square sin square alpha plus 2 f x f y cos alpha sin alpha plus f x square sin square alpha plus f y square cos square alpha plus. So, it is minus minus 2 f x f y cos alpha sin alpha.

So, basically I have square and add both this equations we obtain this expression this 2 cancel out now this is when you take f x square common it is sin square plus cos square which is one again it is when you take these 2 common it is cos square alpha plus sin square alpha which is again 1. So, it is equals to f x square plus f y square. So, we have prove this problem we have proved this equation and the next problem now which is of involving second order partial derivative let us try to solve this. So, that if w is equals to f of u and v satisfy the Laplace equation f u u plus f v v equal to 0 and if u is given by this and v is given by this then w also satisfy this expression I mean Laplace equation.

So, let us try to prove this.

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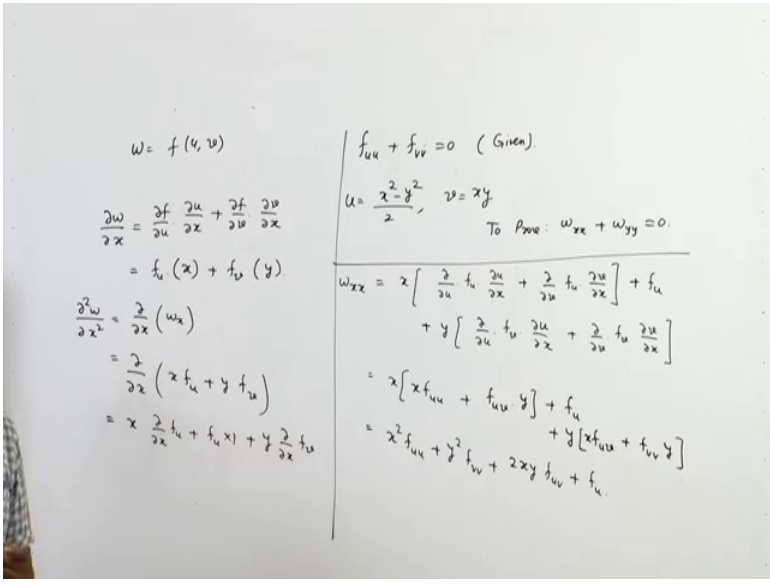

So, here w is the function of u and v an it is given to us that f satisfy Laplace equation that is f u u plus f v v equal to 0, it is given.

Now, u is the function of x and y which is given as x squared minus y square by 2 and v is again is a function of x and y which is x and 2 y we have to show that w also satisfy Laplace equation respect to x and y that is to prove we have to prove that w x x plus w y y is equal to 0. So, let us compute del w by del x first it is. So, del w by del x is by the chain rule it is del f by del u into del u by del x plus del f by del v into del v by del x which is equals to f u into del u by del x from here it is x plus f u into del v by del x from here it is y. Now, we have to compute del square w upon del x square which is which is this term similarly will we can compute del square w upon del y square now how can we compute this, this is del by del x of w x which is del by del x of w x is x of u plus y f v this expression ok.

Now, you differentiate here respect to x partially first as it is derivative of this respect to x plus second as it is derivative of this plus y is independent of x. So, y is common y comes outside and it is del by del x of f v now. So, w x x should be given as now it is now you have to apply chain rule for a f u because f u is a function of again x and y f is a function u and v and u and v is a function of x and y. So, you have to apply chain rule here its x into.

So, you can break it, here it is del by del u of f u into del u by del x del by del u of a f u into del u by del x del by del u of f u into del u by del x plus del by del v of f u into del v by del x because both involve function of x plus f u which is this term and plus y times again you have to apply chain rule here it is del by del u of f v into del v by del x plus del by del v of f v into del v by del x.

Student: Del u by del x.

It is del u by del x, it is x into now it is f u u now del u by del x from here is again x plus it is f u v into del u by del x on here it is y plus f u plus y into it is f u v del u by del x del u by del x is x plus f v v into del v by del x is y. So, this is this is what we are getting as del square w by del x square which is x square f u u plus y square f v v plus 2 x y f u v plus f u.

Now, similarly we can compute del square w by del y square. So, let us try to compute to that also.

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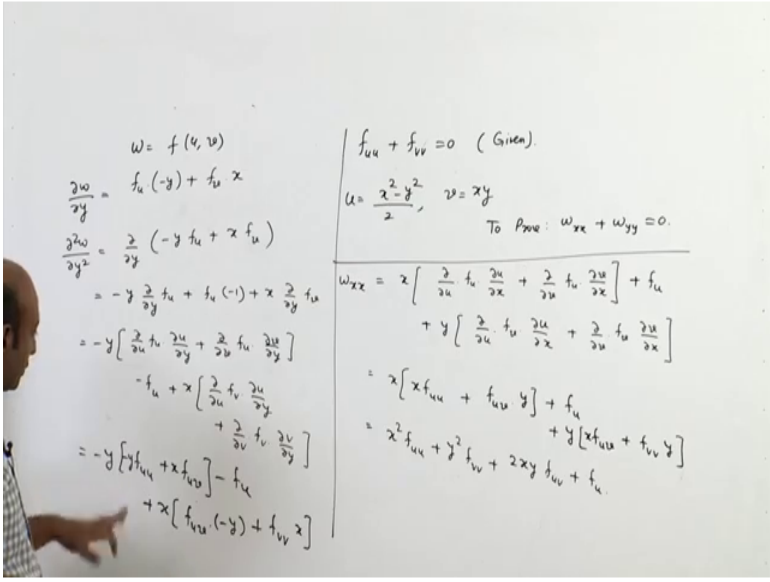

So, first del w by del y it is del f by del u into del u by del y which is minus y plus f v into del v by del y which is x. Now del square w by a del y square which is del y by del y of w y which is minus y f u plus x f v which is you apply for tool here it is minus of first as it is del by del y of a f u plus second as it is derivative of first plus x del by del y of f v ok. Now it is minus y times, now you have to apply chain rule here again it is del by del u of a f u into del u by del y plus del by del v of f u into del v by del y minus f u remain as it is f u plus x times again you apply chain rule here because f u is the function of x and y ok. So, how can you do that it is del by del u of f v into del u by del y plus del by del v of f v into del v by del y ok.

Now you can simplify minus y it is f u u del u by del y is minus y plus it is f u v del v by del y is x plus minus f u plus x times it is f u v del u by del y is minus y plus it is f v v and del v by del y is x now when you add the 2 terms I mean del square w upon del y square plus w f x when you add both the terms. So, from here you are getting ok.

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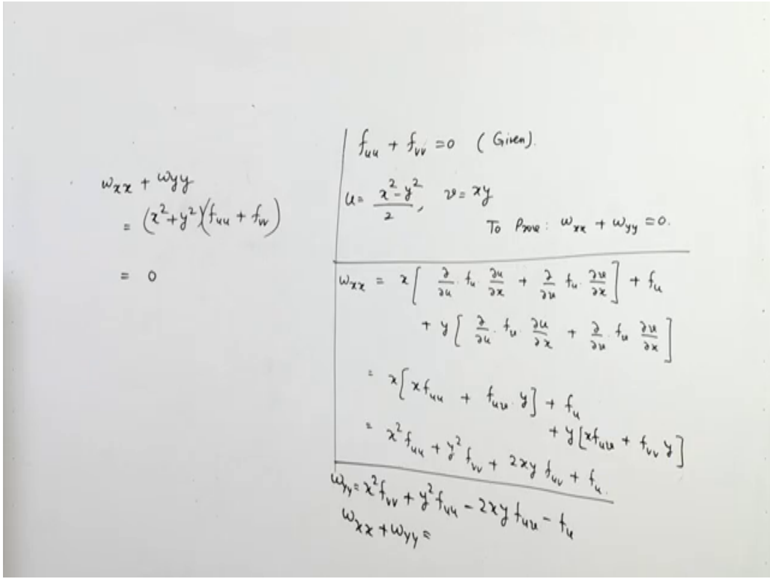

Let us simply this first this will be equal to this will be equal to it is it is x square f v v plus y square f u u plus it is minus 2 x y it is minus 2 x y f u v minus f u v. Now it is w y y ok, now when you add these 2 it is x square plus y square times f u u x square plus y square times f v v this cancel with this and this cancel with this. So, we are having w x x plus w y y as what we are having w x x plus w y y as x square plus y square times f u u plus f v v and this is equal to 0 which is given to us. So, it is zero. So, we have proved ok.

So, that is how we can apply chain rule for if we are involving second order partial terms partial derivatives now it is also important to mention in the problem that which variables are dependent and which variables are independent because sometimes, if it is not given to us. So, answers may be different for example, suppose you have to solve this problems for example, z is a function of w is and z is ok.

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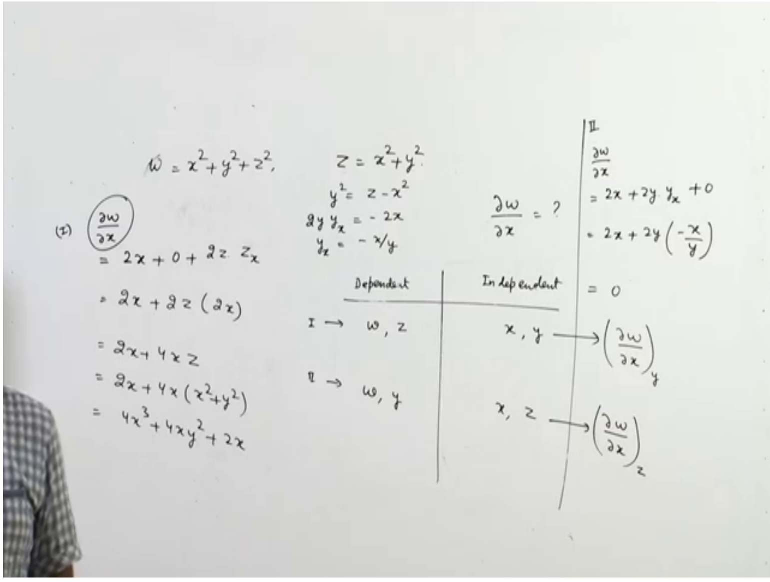

So, here it is not given that which variables are independent and which variable is dependent and we have to compute del w by del x now since del w by del x is to be computed. So, from this expression it is clear that x is an independent variable and w is a dependent variable it is clear that from this expression we can easily see that w is a dependent variable and x is a independent variable one thing is clear for other variables for other variables z and y z and y we are not knowing which variables are dependent and which variables are independent because we can also write y square as z minus x square. So, if you if you make this chart for example, dependent variables and independent variables from here from here w is dependent and x is independent it is clear ok.

Now, we have 2 possibilities, what are the possibilities may be possible that y is independent and z is dependent or we may have or we may have z as independent and y as dependent because we have one expression in x y and z. So, of course, one variable will be dependent and others 2 was independent definitely. So, so 2 variables will always be independent ok; so, these are 2 possibilities which we are having suppose we are of of for the first possibility, this first case, this is second case. Now for the first case, when we are taking w and z as a dependent variable and x and y as independent variable. So, what we have to compute we have to compute del w by del x for the first case.

So, del w by del x for the first case will be. Now we differentiate this expression partial respect to x keeping in mind that z is the dependent variable these are the dependent variable and these are independent variable now when we differentiate partial with respect to x it is 2 x plus now y is x are independent. So, partial derivative y respect to x is 0 plus now z is dependent on x and y. So, it is 2 z into z x and from here z x is 2 x. So, it is 2 x plus 2 z into 2 x from here. So, it is 2 x plus 4 x z and z is x square plus y square. So, it is 4 x cube plus 4 x y square plus 2 x.

Now, let us compute for a second case now first second case for the case 2 del w by del x now for second case we are taking w and y as dependent variable and x and z as independent variables ok. Now when you differentiate this partial respect to x keeping in mind that x and z are independent variable other are dependent. So, what will obtain it is 2 x plus 2 y into y x because now y is now y is dependent on x and z and deri partial derivative of z with respect to x will be 0 because x and z are independent variables. So, plus 0.

Now, from here how can we compute del z by del y. So, y square is z minus x square always first express a variable in terms of independent variables then compute partial derivatives ok. So, what will be what will be y x it is 2 y into y x which is minus 2 x. So, y x will be minus x upon y. So, when you substitute y x as minus x upon y here. So, this comes out to be zero. So, values are not same you see that when we take the first case value is del w by del x is this thing and when we take the second case values are this thing value is 0 values are not same. So, in the problem it must be clear that which variable we are treating as a dependent variable and which variable we are treating as an independent variables.

Now, for the first case when you are taking x and y as independent variable we have the notation and notation is for this for the first case for the first case notation is del w by del x with y this notation means we are taking x and y as independent variables and others are dependent variables for a second case we take the notation like this del w by del x with z this means we are taking x and z as dependent variables and other variables as independent variables this is a standard notation ok. So, based on this we have some problems suppose we have the first problem we want to solve it first problem says w is equals to x square plus y minus z plus sin t where x plus y equal to t ok.

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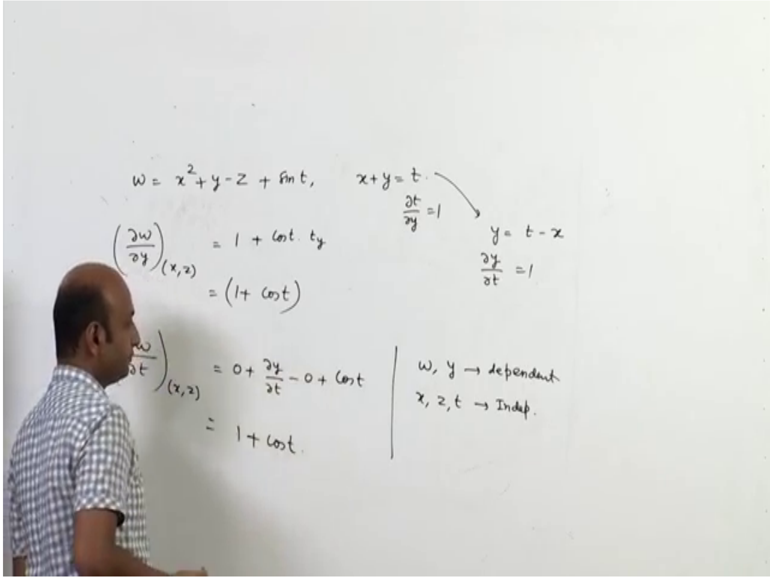

Suppose you want to solve the first part that is del w by del y with x and z. So, so what does it mean it means we are taking x y and z as independent variable and other 2 variables which are w and t as dependent variables now if you what do compute del w by del y it is with respect to y and we are taking x and y are independent. So, this is 0 this is one this is independent. So, it is 0 and it is cos t into t y t y means del t by del y because t is a dependent variable t depends on x y and z.

So, this is from here del t by del y as 1. So, it is 1 plus cos t. So, that is how, we can solve this problems suppose you want to compute del w by del t with x z fourth part of this problem del w by del t with x z this means we are taking x t and z as independent variables and w and y as dependent variables. So, how can we find this respect to 2 t you have to differentiate. So, this is. So, you are taking w and w and y has dependent variables and x z and t has independent variables now when you differentiate with respect to t. So, x and t are independent. So, it is 0 y is not independent of t. So, it is del y by del t minus z and t are independent. So, it is 0 plus t t is of course, cos t.

Now, how can you compute del t by del y by del t you first from this expression you write y in terms of independent variables then you compute del y by del t which is one. So, it is 1 plus cos t. So, in the same way we can solve all the parts of this problem.

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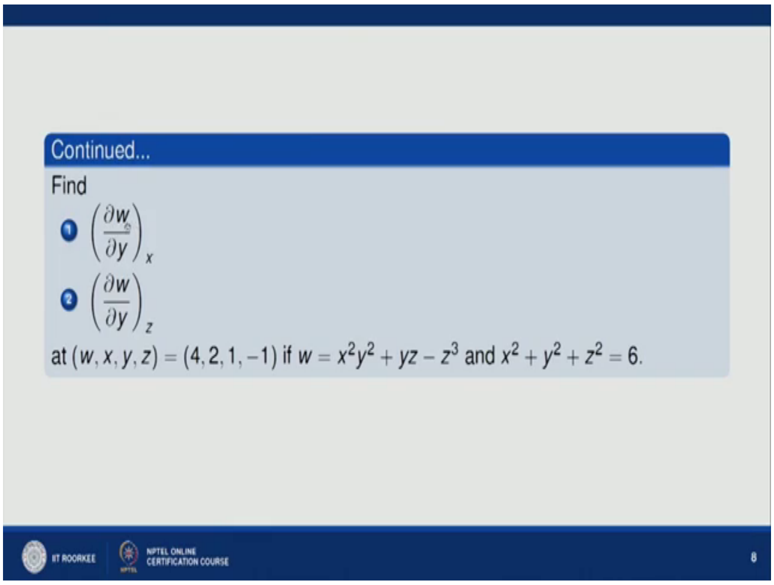

And similarly we can solve the next problem which is del w by del y with x; that means, we are treating x and y is independent and all other variables as dependent variables and similarly for the second part of the same problem.

Thank you very much.