Multivariable Calculus
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Lecture - 01
Functions of several variables

Hello friends. Welcome to lecture series on Multivariable Calculus. So, the first lecture deals with functions of several variables, that what do you mean by function of 2 variable or more than 2 variables? How can you find domain and range of those functions Now in our eleventh or 12 standard, we were already deal with what do you mean by function of single variable. Let us recall those definitions and then we will we will come to functions of several variables.

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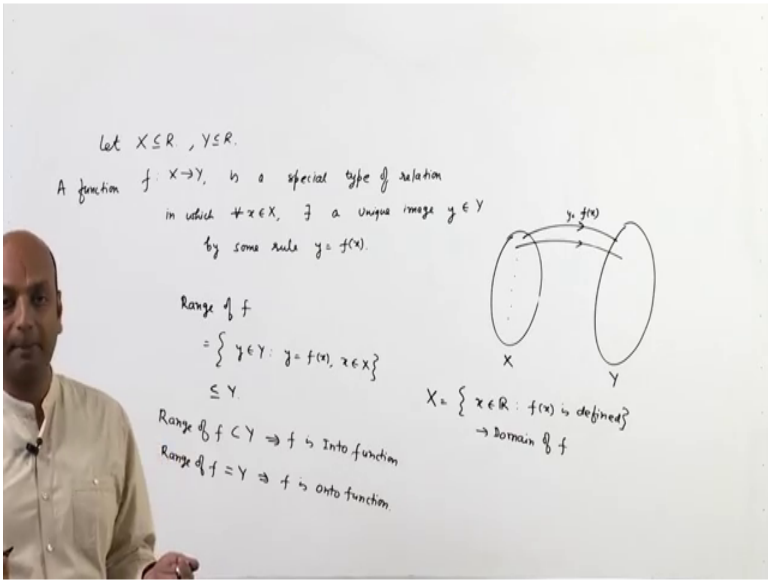

Now, let X as subset of R. Then f from X to R or X to Y suppose Y is also subset of R, then f from X to Y is called a function basically a function, what a function basically means. A function f from X to Y is a special type of relation in which for every x in X, there exist a unique image y in Y by some rule say Y is equals to f X.

So basically, what do you mean by functional the function of a single variable. It means, it is a special type of relation in which you take any x any element x in X, there will always exists, a unique image y in Y by some rule say Y equals to f X. So, you have basically this is suppose X and say this is suppose some Y, this is a function from X to Y.

We have several elements, a small x in capital X, some elements in x and they will always exist some y in Y such that Y equals f X.

Now, this X where function is defined ok, this we call as domain of the function ok. This X is called Domain of the function. So, X is all those x belongs to R, where f X is defined ; this we call as Domain of f. Now what do you mean by range f? Now, now what do you mean by f functions defined in simple speaking, we say that functions defined. If it is not a complex number and the denominator start equal to 0 ok, that we say it is a functions defined and all those x is function defined, we call that set as Domain of the function.

Now, the image of a domain of the function on Y, that set is called Range of the function. So, what how is define range of a function. So, Range of f is simply all those y belongs to Y such that y is equals to f x and x belongs to capital X. Collection of all those y where such that y equals to f x and x belongs to capital x is called range of the function.

And it is quite natural that this range of a function is always a subset of capital Y. If it is a proper subset of capital Y, we call such function as into function ok. If range is, if range of f is proper subset of y, then this is called f is called Into mapping or Into function ok; Into function. And if range of f is equal to Y, then f is called Onto mapping or Onto function. We have also defined one to one mapping or one to one function or many to one function. How can you find that? If you any 2 distinct x in X and the image concerned to that distinct x all also district. For every p f for every distinct pair x minus x to in capital X, we say that the mapping is one to one.

So, how can I define one to one mapping?

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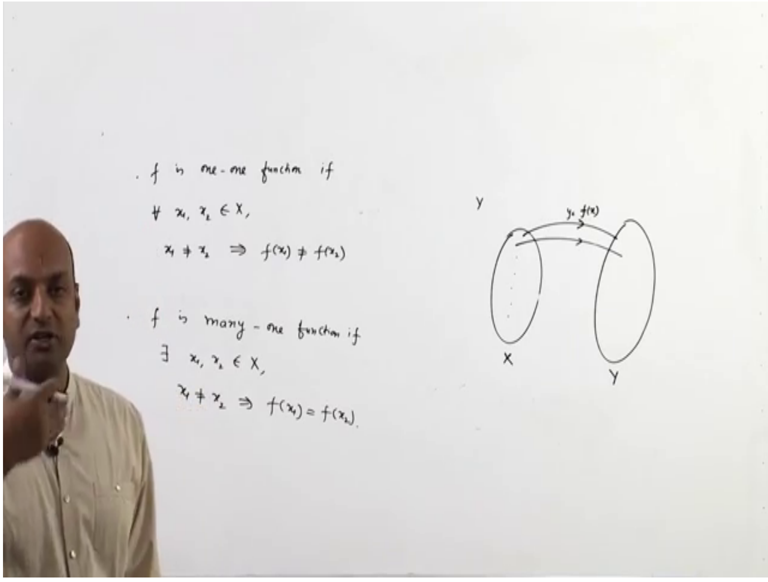

One to one means, f is set be one to one function or mapping if for every x 1, x 2 in X ; x 1 not equal to x 2 implies f x 1 not equal to f x 2 ; that means, for any 2 distinct pair in capital X the corresponding image in Y are also different. So, such mappings are called one to one mapping. And if there exist a distinct pair in capital X, such that the image corresponding to those distinct that distinct pair is same, then that mapping is called many to one mapping. So, f is. So, f is called many to one mapping, many one mapping or many one function, if there exist x 1, x 2 in X such that x 1 not equal to x 2 implies f x 1 is equals to f x 2.

So, such mappings are called many to one mapping. So, this all this we have already studied in our 10 plus 2 standard that what are many to one mapping. What are one mapping when a function said to be into and onto.

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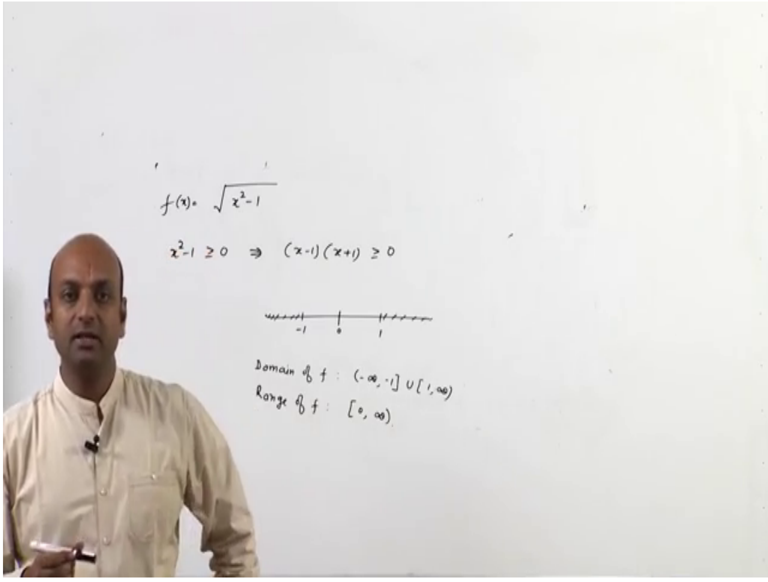

Now suppose, they are the single variable function f x is equals to a simple example say under root of say x square minus 1 ok. Now want to find out the domain and range of this function. So, this function will not be a complex number if, if x square minus 1 is greater than equal to 0. So, this quantity should be greater than equal to 0, then this function will not be a complex number.

So, that is how you can be defined domain of a function. So, this implies x minus x plus 1 and x plus 1 should be greater than equal to 0. So, if you take this is minus 1 and plus 1 ok, this is 0 when x is minus 1 and this is 0 when x is plus 1. You take any point in between say x equal to 0. When you put x equal to 0 here, it is minus 1; when you put x equal to 0 here, it is plus 1; plus 1 into minus 1 is not greater than equal to 0. So, shade will not be in between minus 1 to plus 1. Shade will be from this site and from this site. That is the domain of f will be minus infinity to minus 1 union 1 to infinity. And what is the range of the function?

Now, this function f will never be negative because it is in under, square root. So, this will never be negative. So, range will always be positive; that means it is from 0 to infinity. So in this way, we can you defined Range and Domain of a function. So, this a simple illustration of a single variable function. Now we come to several variable functions. Now come to function of 2 or more than 2 variables. Now first is 2 variable function f x y.

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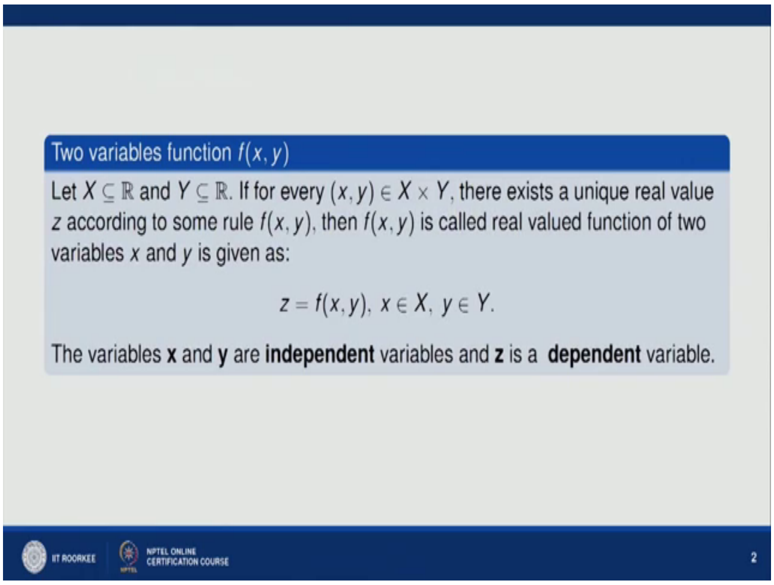

So let X, a subset of R and Y, a subset of R. If for every x y belongs to X cross Y, there exist a unique real numbers z according to some rule f x y, then f x y is called real valued function of 2 variables x and y ok. Now instead of one variable, we have having 2 variables x and y ok.

The same definition we extend to 2 variable functions; that means, if for every x y in X cross Y, there exist or unique image z according to some rules say z equals to f x y, then we say that it is a function of 2 variable x and y. Now here we are involving 2 variables, 2 input variables and those input variables are called independent variable and z which is a function of x and y which is equal to f x y that z is called dependent variable.

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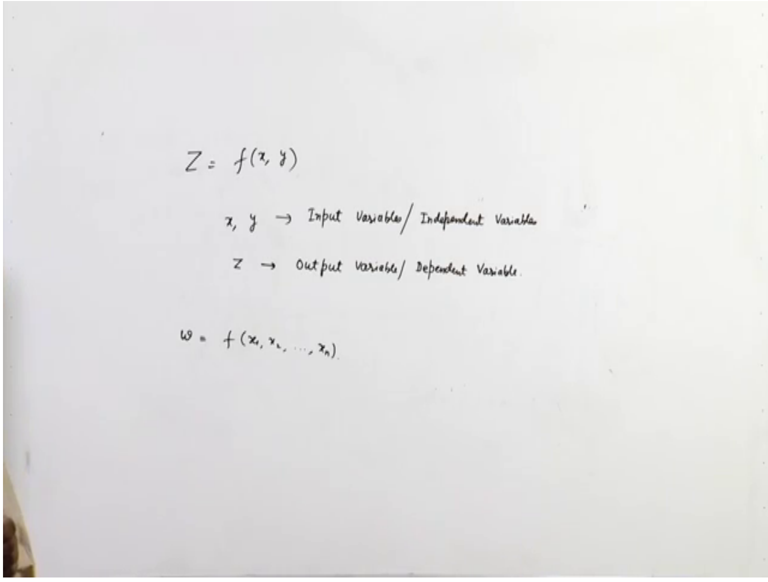

So, basically if we have this function say Z equals to f x y, so this input variables this are also called Input variables or Independent variables and this Z which depends on x and y because if you change the values of x and y according the value of Z will change ok. This Z is called Output variable or Dependent variable. Now here we are having only 2 variables x and y; similarly we may have more than 2 variables say n number of variables. So, we can also defined w is equals to f x 1, x 2 up to x n. This is a function of n variables ok.

So, in the next slide, we are having function of n variables.

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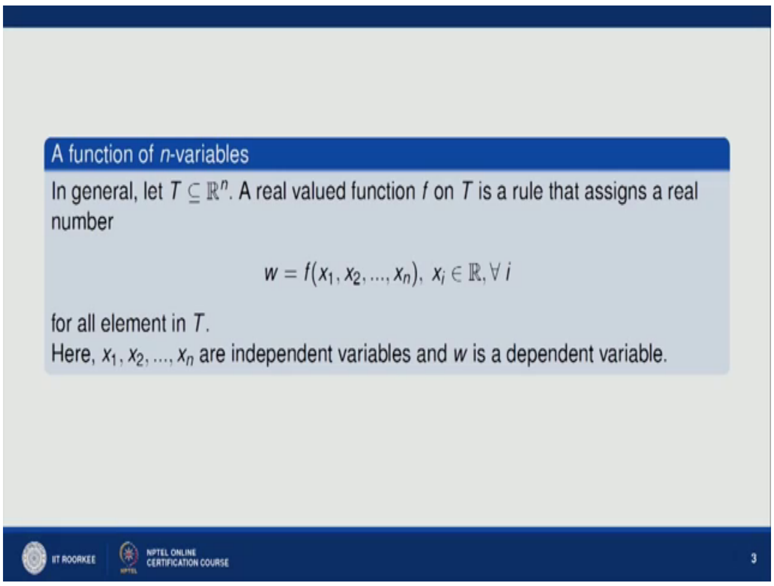

Let T be a subset of R n of real valued function f on T is a rule that assigns a real number w equal to f of x 1, x 2 up to x n where each x i belongs to R. Now here these x i's these x 1, x 2 upto x n, these are the input variables which we are also called independent variables. And this w which depends on x 1, x 2 upto x n is called output variable or dependent variable ok.

Now, how can you define domain and range of 2 variable functions and similarly we can extend the concept for n variables functions. In a similar way as we did for 2 variable functions, we can we defined domain and range for 2 variable functions as well. You see we are having a 2 variable functions Z equals to f x y. Here x and y are independent variables and Z is the dependent variable. Now domains of this function f are all those x y belongs to R 2 ok.

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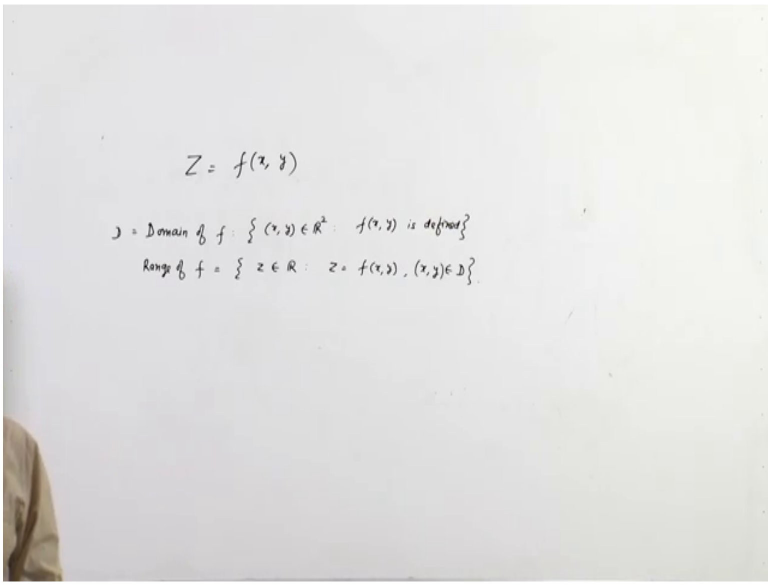

All those x y belongs to R 2 such that f x y is defined. Collection of all those x y in x y plane basically, where f x y is defined we call that set as a domain of the function f. Now the range of a function f, range of f are all those z belongs to R such that z equal to f x y and x y belongs to domain. Say this domain is represented by D. So, belongs to D.

So, collection of all those z, collection of all those z such that z equal to f x y where x y belongs to D is called range of the function f. So, that is how we can define domain and range of 2 variable function; similarly we can define domain and range for and variable functions also.

Now, let us discuss 2 examples based on this.

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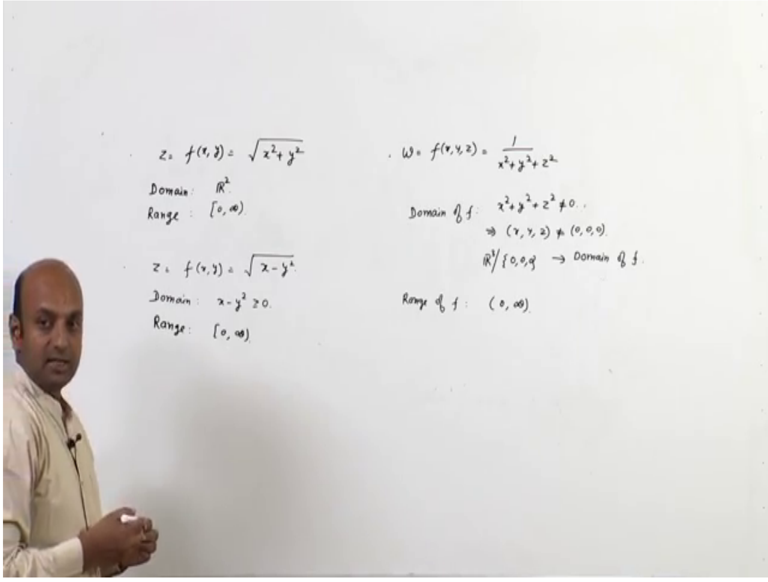

The first example is suppose f x y is equals to is equal to under root x square plus y square ok. This is z equal to f x y. Now what is the domain of this function? Now domain of this function is, this quantity should be greater than equal to 0 and since x square and plus x square and y square are nonnegative quantities and some will also be non-negative.

So this, this is always non-negative for any X Y in R. So, we say the domain is entire R 2 ok. You take, you take any element x in R 2 x square plus y square is always greater than equal to 0. So, this will con consist of domain of a function and range will be definitely this is under root square root quantity, this will never be negative. So, the range will be from 0 to infinity. 0 will be closed because when x and y both are 0, f will be 0. So, that will included in the range of the function ok.

Similarly, the second example suppose, suppose z equals to f x y is equal to under root of it is x minus y square. So, what is the domain of this function? Domain will be x minus y whole square. All those x y all those x y in R 2, such that x minus y square is non-negative will consist of domain of this function f. And range will be of course 0 to infinity, it will never be negative ok. Now come to 3 variable function suppose, suppose function is w is equals to f x y z is equal to one upon x square plus y square plus z square. Now for this problem of course denominator should not be equal to 0. For domain, for domain of this f x square plus y square plus z square is not equal to 0. So, because if denominator 0, this function will not defined. So, we say that domain of the function are all those x y z such that the denominator is not equal to 0. And this is 0 only one x y z all are 0.

So, this implies x y z should not equal to 0, 0, 0. And therefore, therefore, domain are all those R 3, R 3 means collection of x y z excluding origin, excluding origin will be the domain of f. And what will be range of f? Range of f will be. Now this quantity will never be 0 because 0 because it will be 0 only when this tends to infinity ok. So, this is never be 0. So, it will be 0 to infinity, open interval 0 to infinity. And it is always positive because 0 to infinity is always positive. So, it is from 0 to infinity.

Now, the next example is to find domain and range is w is equals to y z log x.

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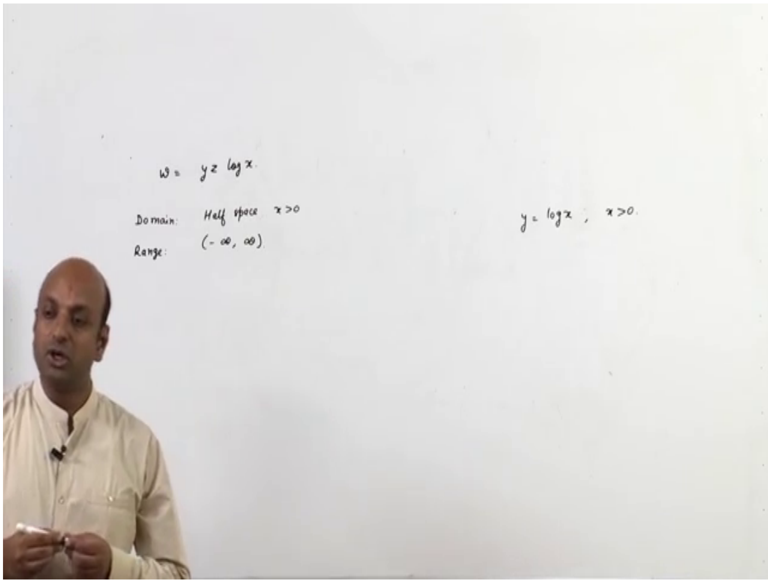

Now we know that, we know that if we have a function y equal to log x say. So, this log x is defined when x is greater than 0 ok. So, this y and z there is no problem because this is a multiplication of 2 simple variables y into z. Now this log x is defined only when x greater than 0. So, this will give Half region or Half space. We can say when x is greater than 0, this is basically Half space. We should say Half space because it is in all x y z where x is greater than 0. So, this means Half space ok. And the range will, now this y z x may anything, y z is may be anything. So, it is from minus infinity to plus infinity. So, this will be the range of this function. So, these are few illustrations that how can we find domain and range of several variable functions.

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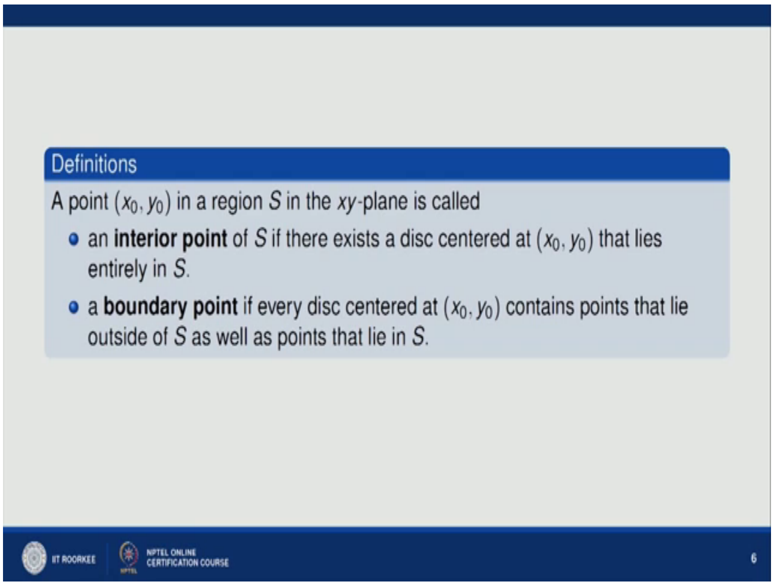

Now, come to few more definitions. Now first is interior point. Now point x naught y naught in the region S in the x y plane is called an interior point of S, if there exist a disc centred at x naught y naught that lies entirely in S. Now what does it mean? Let us discuss by an example.

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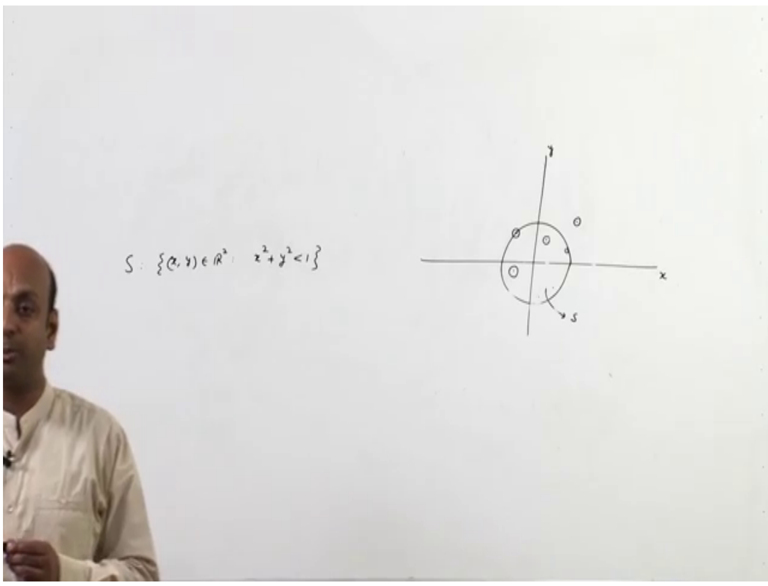

Now consider this region S, all those x y in R 2, such that x square plus y square is lesser than 1. So, what this region is? So, this is x axis and this is y axis and this is the basically a unit circle are region inside this circle ok. So, this is 1. So, this is basically the circle. So, this is all those, all those points which lies inside the circle easiness ok. So, S is all those all those x y which are inside this circle.

Now, you take any point inside this region ok. So, this is S basically. Now you take any points inside this S, say this point take, take this point ok. Now there will always exist that disc centred at this point. Disc means circle. You can say, you get you can always draw disc centred at this point such that this disc always lies totally inside this region. So, this point we can call as this point as integer point.

You see if you take this point, if you take a point here say, you take a point here, you draw circle on a disc centred at this point, there will always exist, there will always exist at this centred at this point such that the disc lies totally inside this region. So, we again say that this point is an interior point if you take a point very close the boundary, it still there will exist a disk centre at this point radius may be very small, but they will always exists a disc centred at this point such that the disc lies totally inside this region. So, this point is also an interior point. So that means, all the points that it have this region has, all the points are integers. You take any point, you take any point all the points are integer points.

Now, if you take a point in the boundary suppose; if you take a point on the boundary, now if you take if you draw a disc centred at this point, then this disk no matter whatever radius you take, this disc will never been entirely contained in the region because there are some point which lies outside this region. So, this point cannot be an interior point. If you take a point outside this S, say here and you draw any disc centred at this point, may be may be a larger disc ok, but the disc centred at this point will never be contained totally inside this region. So, this point again is not an integer point ok. So, all those points which lies inside this region S are the interior points.

Now, we come to a second definition, that is Boundary point. Now point x naught y naught in the region S in the x y plane is called a Boundary point, if every disc centred at x naught y naught contains points that lie outside of S as well as point that lie in S. Now let us discuss this by an example, again by an example. Suppose, suppose instead of this, you are having now this region ; that means, boundary of a unit circle. So, what does it mean?

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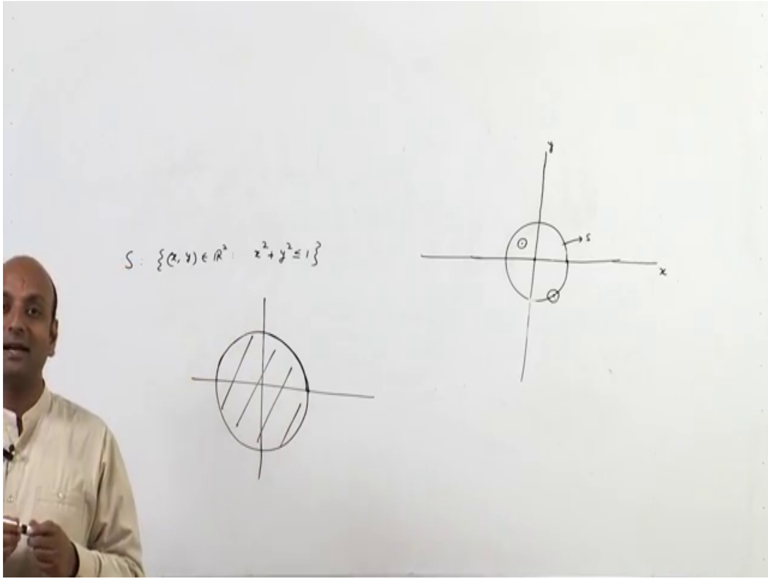

This boundary, this is S, only the boundary ok. Now you take any point on the boundary of S, say you take a point here now. You take, you take any disc, no matter how small, how large radius you are taking. You take any disc centred at this point. Draw a disc centred at this point. You will always find some points outside this region and you will always find some points on the region. I mean they will, I mean the intersection, the intersection of this disc, this disc on the complement of S and intersection of this disc on S never be empty. There will be a point, there will always exist some point that lies outside this disc and lie that lies on the disc. So, this point is an boundary point.

However, if you take a points in the inside region say here, if you take a point here, so there will exist a disc whose intersection with the outside S is ah, whose intersection with inside S is an empty. So, this point is not a boundary point. You, you take a, you take a suppose, you take this region S less than or equal to S. So, this S contains all the point that lies on the boundary and all the point that lies inside the region S ok.

Now if you take a point, if you take a point say here ok, say here, now there will exists a disc centred at this point intersection of this disc with this S is non-empty by intersection of this disc with outside S is empty. So, this point is not a boundary point because for every disc, for every disc no matter how small the radius you are taking. The intersection of this disc with a region S and outside of S should not empty, that is definition of boundary point. So, all the points that lies on the boundary are the boundary points of S ok.

Now based on this, we can decide whether region is open or close. How can we do that? How can we say this? Let us see.

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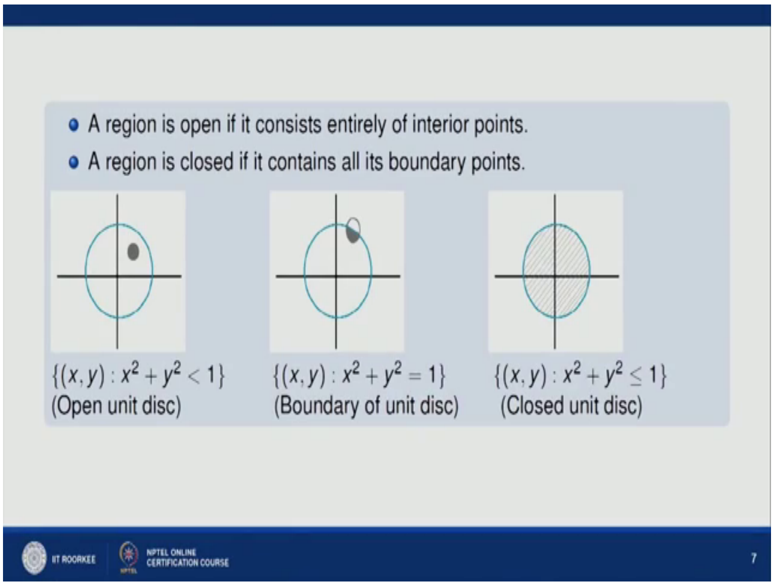

A region is open if it can take consist entirely of it is interior points, if it consists of entirely of it is interior points, then we say it is the region is open. For example, if you see on the first figure here on the slide, then this figure is an unit disk without having boundary and it consists of all it is interior points. All the points that are in that, that are inside the region, all are interior points. So, this region we can say as an open region.

now region is closed if it contain all it is boundary points. You see, if you take third example here, this is ah, this contain boundary points as well as, as well as interior points. It contains all. It is boundary points you see. The boundaries include it and contain all it is boundary points. So, this region is closed.

So, in this way, we can def, if we can say that they were whether region in R 2 plane, I mean in x y plane whether it is open or whether it is closed. If it consist entirely of a interior points, we say the reason is open and if it contain all it is boundary points, we call it is close region. Now there may be a region which is neither open or closed. For example, if you take if you focus on the first figure and in this figure, you include only those boundary points that lies, that lies on the upper half of the circle ok; that means, that means, you are, you are, you are talking on only the this region. This is S and on the boundary only these points are included, not on the lower side of the circle ok. On the boundary only these points included.

Now, it does not contain all it is interior points because these it is these also interior point but it is not it, this set does not contain those points which is not closed. And these are interior points, so it is not open also because if it is open, then it cons, than it consists entirely interior points. So, there may be some regions which are neither open or closed; there may be region which are open and there may be region which are only closed. Now a region may be bounded or unbounded also. How can decide that?

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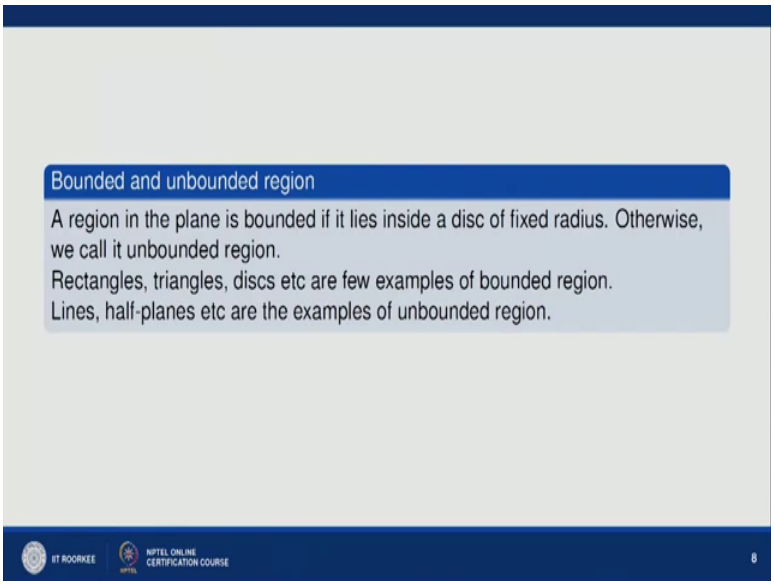

Now region in the plain is bounded if it lies inside the disk of fixed radius otherwise we call it unbounded region. Say you have this line segment ok.

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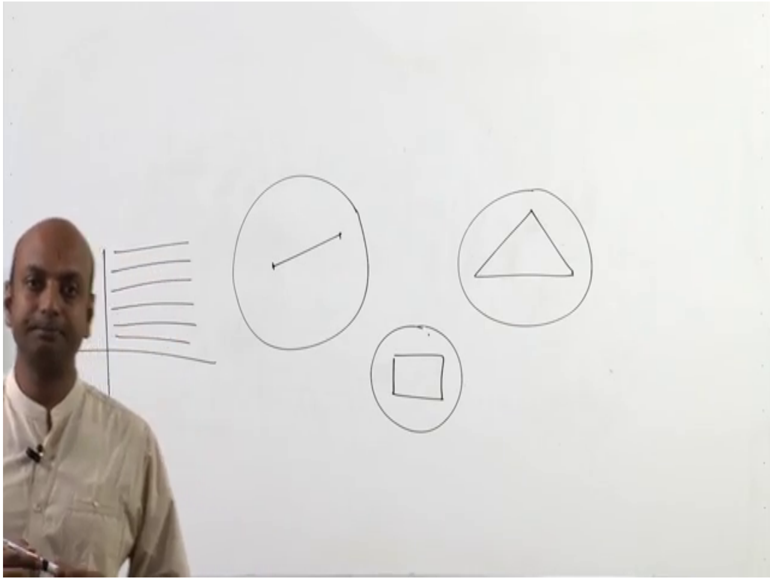

now you can always find the circle of fixed radius that covers this line segment, I mean, inside which this line segment lies. You can always find the circle of it is radius. So, we can say that this line segment is a bounded region. You can say ah, you can you can take a triangle also. If you take a triangle, you can always find a circle of fixed radius such that this triangle lies inside the circle.

So, we can say that the triangle is bounded region. You can take a rectangle again, you can find a circle of fix radius such that the rectangle lies the totally inside the region. So, we can say that this rectangle is a bounded region. Now suppose, suppose you are taking R 2 plane, the entire R 2 plane, I mean the first quadrant of the x y plane, suppose you are taking this region, now you will you get, never find circle of fixed radius such that this entire first oct, first quadrant of x y plane lies entirely in that circle. So, this region is unbounded region. So, a line, half planes, etcetera are the few examples of unbounded regions. Now contour lines. now contour lines now. What do you mean by a contour lines? So, let us understand this by an example.

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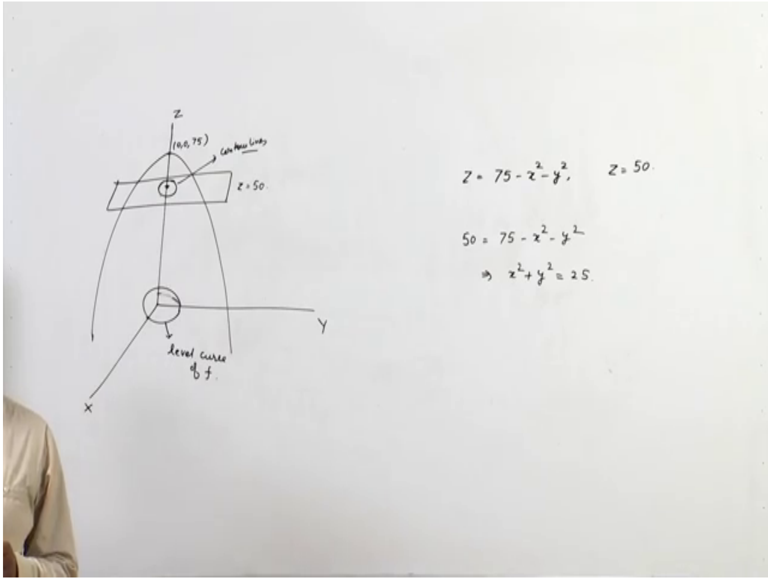

So, what contour lines are? Now you take this example. Now here z is 75 minus x square minus y square and z is 50. Now if you draw, have to draw this figure. So, this is x, y and z. Now when x, y both are 0, z is a 75. So, when x y both are 0, z will be 75 here.

For any other values of x and y, z will always be less than 75 because this is positive x, square is positive y square positive negative signs are here. So, for any other values of x and y, z is always less than 75. So, it is it is a parabolite of something like this. Now we have a plane z equal to 50. We have a plane it is 75, it is 0, 0, 75. Here suppose we have z equal to 50, now z equal to 50 is a plane, is a is a plane in the x y plane where x and y may be anything but z is always 50 ok.

So, it gives a plane of this type. It is z equal to 50. Now when z is equal to 50, when you substitute z, z equal to 50 here, so it is 50 equal to 75 minus x square minus y square. So, this implies x square plus y square is equals to 25. So, it is gives the circle basically, x square plus y square equal to 25. Now at z equal to 50, at z equal to 50. If you draw circle of radius 5 and centre origin, if you draw a circle of radius of this type radius 5 and center origin, so this circle is basically this curve basically, we can say this curve is called Contour lines.

This curve is basically called Contour lines. Now this, this on the x y plane, this circle on the express basically is a circle. So, this circle on the x y plane is basically called level curve of f when z equal to 50. So, this is how you can define a contour lines and contour lines basically you have you have a surface z equal to f x y, a plane that equal to say intersect that surface and gives the curve. That curve basically a contour line and the and that that contour line on the that that curve on the x y plane basically called Level curve. So, here also we will say the same thing.

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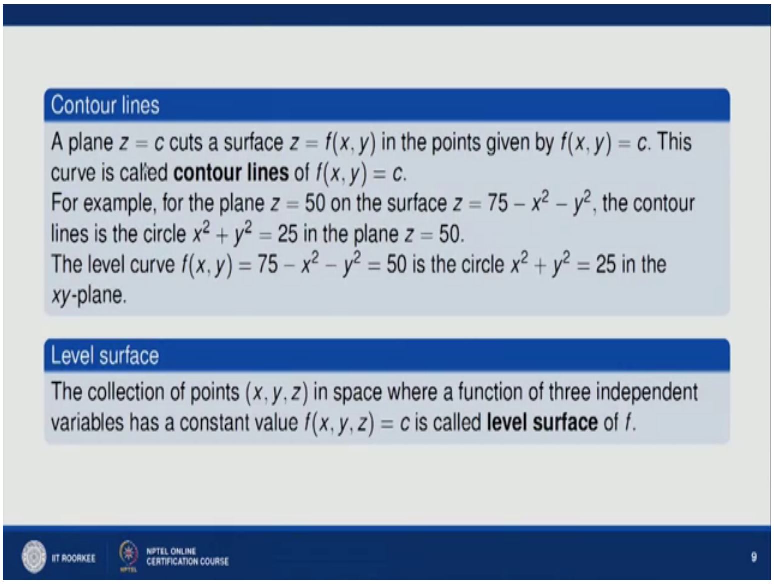

A plane z equal to c intersect a surface, this in the points given by f x y equal to c, this curve is called contour lines of f x y equal to c.

Now, here it is a level curve. Here it is a level curve because we have a curve here. Now if you have a up ah, here only 2, 2 independent variables are there. If instead 2 independence variable, we are having 3 independent variables, so this will give a level surface. So, the collection of all points x, y, z in space where a function of 3 independent variables have constant value of x, y, z equal to c is called label surface of f. So, these are took 2 problems. Let us discuss these problems quickly.

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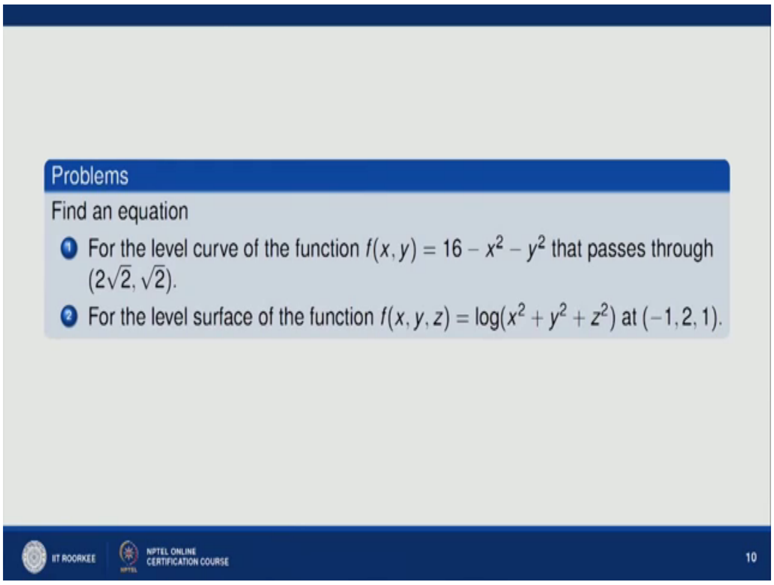

First is find the equation of for a level curve the function this that passes through this.

So, what is first example?

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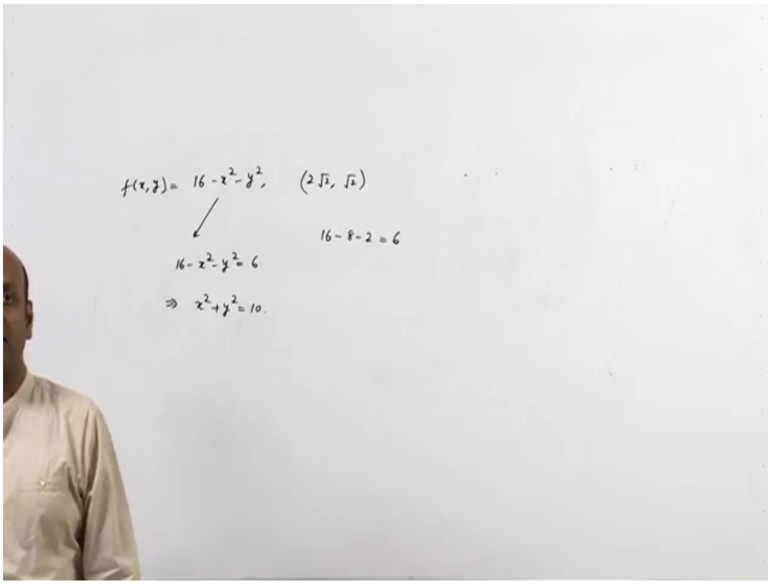

This is f x, y is equal to 16 minus x square minus y square and point is 2 under root under root 2. Now when you substitute under root 2, under root 2 and 2 under root 2 under root 2 and another 2 here, so we will obtain it is 16 minus it is 8 minus 2 that is 10 that is 6 sorry that is 6. So, so, the level curve of this will be 16 minus x square minus y square equal to 6 and implies x square plus y square equals to very simple, you simply substitute because it passes through this point. It passes through this point means this is equal to 6 and when you substitute equal to 6 here, so this will give a level curve of f at this point. Similarly we can find level surface of the function, this at minus 1, 2 comma. So, that is how we can we define function of several variables domain and range of a function of 2 or more variables. Domain here is the region and that region may be closed or may be open, may not be closed may not be open or may bound it or may be unbound it so.

Thank you very much.