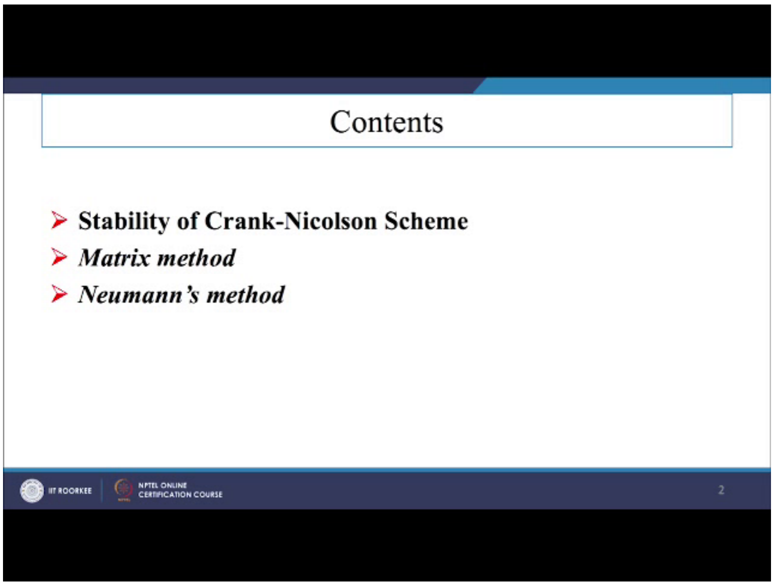
Numerical Methods: Finite Difference Approach
Dr. Ameeya Kumar Nayak
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 09
Stability of Crank-Nicolson scheme

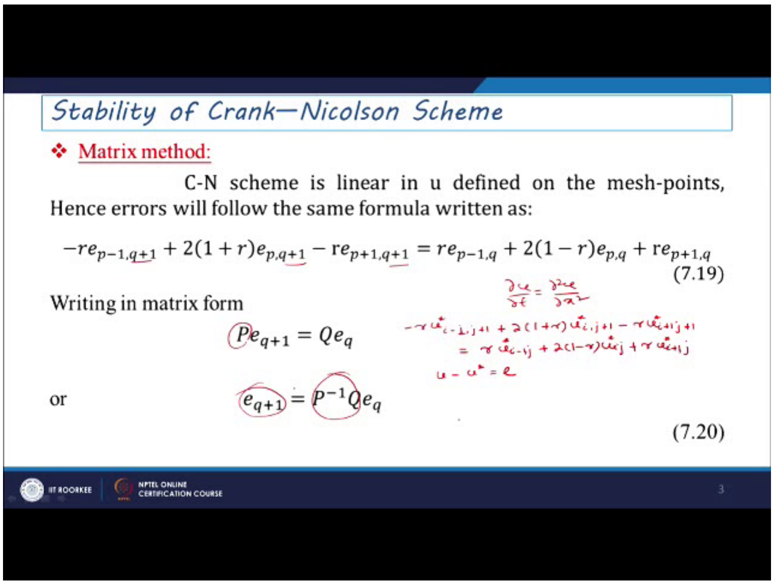
Welcome to the lecture series on numerical methods a finite difference approach. In the last lecture, we have discussed explicit scheme and the implicit methods and their stability and convergences.

(Refer Slide Time: 00:32)



And in the present lecture we will discuss about this stability of Crank Nicolson scheme with both these methods like Matrix method and Newman's method.

(Refer Slide Time: 00:42)



So, whenever we are just going for this Crank Nicolson scheme especially if you will just consider this equations, that is in the form of like del u by del t this equals to del square u by del x square. Then we can just obtain this discretisized scheme as in the form of like minus r u i minus 1 j plus 1 plus 2 into 1 plus r u i j plus 1 minus r u i plus 1 j plus 1 this equals to r ui minus 1 j plus 2 into 1 minus r ui j plus r u i plus 1 j.

So, if you will just consider like u as the true solution and u star is an approximated solution that I have just discussed in the last lecture and if we can just take the differences like u minus u star. So, that will just give you this error terms here, especially as I have told you that if you will just consider this coefficient. So, that is in the form of like minus r u i minus 1 j plus 1 plus 2 into 1 plus r u i j plus 1 minus r u i plus 1 j plus 1 this equals to r u i minus 1 j plus 2 into 1 minus r uij plus r u i plus 1 j here.

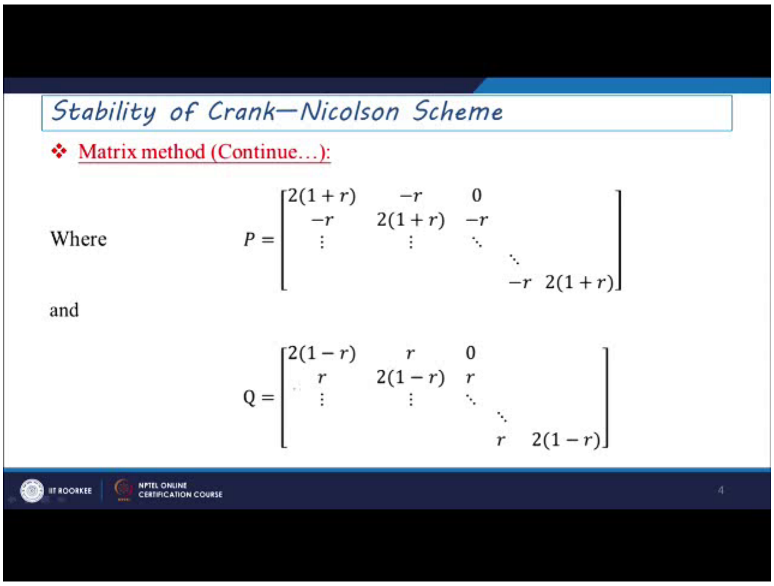
And if you will just consider this approximated solution that is in the form of like u star here and if you will just take the difference from this earlier equation that is in the form of like u minus u star in each of these coefficients and that is just signified as e here and each of these coefficients that will be remained in that position there. So, then we can just write these a error terms as in the form of like minus r e p minus 1 q plus 1 plus 2 into 1 plus r e p q plus 1 minus r e p plus 1 q plus 1 this equals to r e p minus 1 q 2 into 1 minus r epq plus r ep plus q here.

Since we have just replaced here p in terms of for like i and q in terms of for j here; so, that is why this error terms are represented in this form here. So, if you will just express this complete system of equations that is, if this p will vary from like i equals to 1 to n minus 1 and j will vary from 1 to n minus 1 there, then we can have a n minus 1 system of equations and we can just formulate a matrix that is of order like n minus 1 cross n minus 1 here.

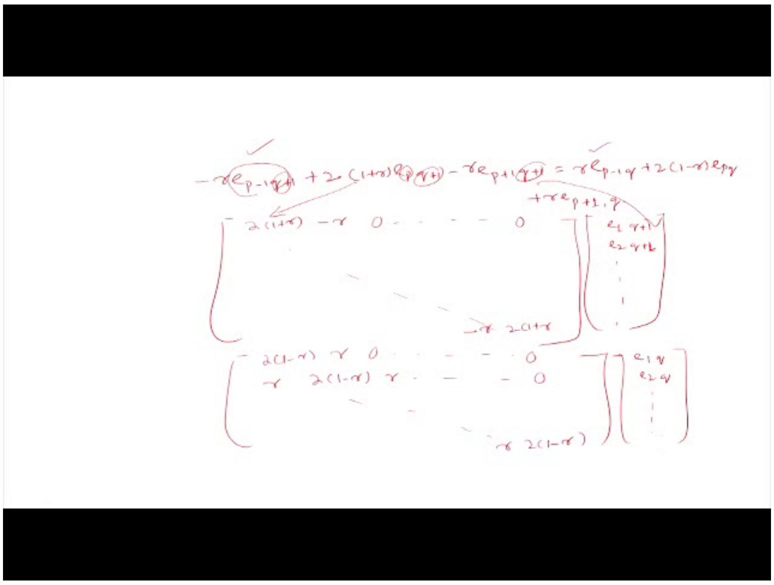
So, if you will just express this one in matrix form then we can just write it as p u eq plus 1 since if you will just see here. So, this q plus 1 coefficients remains same in each of the terms on the left-hand side. So, that is why we can just express this complete coefficients that is in the form of a P here and these coefficients that is in the form of eq plus 1, but especially we can just signify that one as like e 1 like 1 e sorry q plus 1 e 2 q plus 1 up to e q plus 1 q plus 1 there.

So, that is why if we can just express this one in a separated form that we want to keep this variable in the left-hand side only and all of this matrix into the right-hand side then we can just write this complete system as P inverse Q into u q here.

(Refer Slide Time: 04:37)



(Refer Slide Time: 04:54)



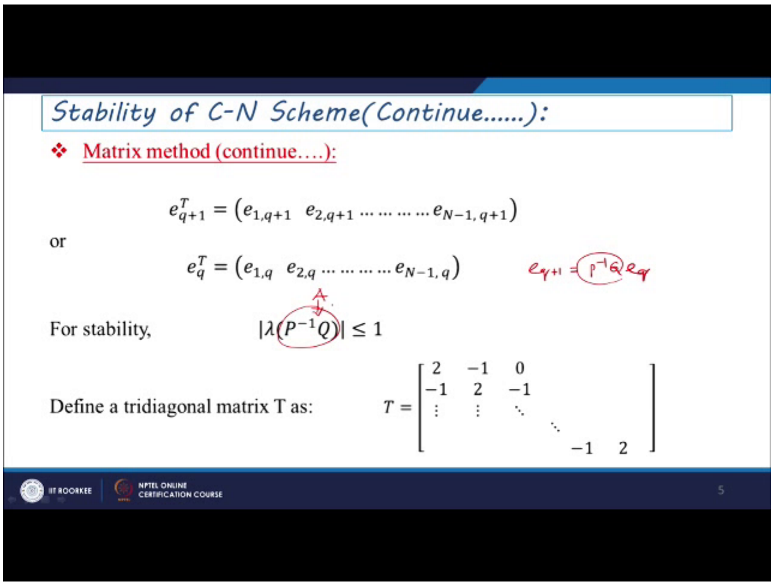
So, if you will just see here these coefficients of P that can be written as like 2 into 1 plus r minus r 0 if you will just see here that these coefficients especially this equations are represented as in the form of minus r e p minus 1 q plus 1 plus 2 1 plus r e p q plus 1 minus r e p plus 1 q plus 1 this equals to r e p minus 1 q plus 2 into 1 minus r epq plus r e p plus 1 q here and if you will just compatible form if you will just write here so first term this will just take this value from the boundary condition itself and it is especially 0 as I have discussed in the last lecture.

Since this boundary conditions are fixed along these boundaries and which cannot be changed with the solution process. So, that is why this error is always 0 along the boundaries so that is why this first term it will just to take the 0 value there itself and if you will just put the first coefficient here that is nothing but the first coefficient will be p here.

So, if you will start from 1 here then we can just write first coefficient as 0, so that is why this first coefficient will come over here as 2 into 1 plus r and second coefficient as minus r here and the third itself it will start as a 0 here itself. And similarly, if you will just write all the terms then in the last term we can just write this one as like minus r 2 into 1 plus r here, and this coefficients it can be multiplied that is in the form of like e 1 q plus 1 like e 2 q plus 2. So, likewise if you will just go; so, sorry this is a q plus 1 her so all remains constant. So, especially if you will just see here since a q plus 1 is fixed at each of the terms so that is why we are just writing this one as q plus 1 here and if you will just write this a right-hand side terms you can just find that first point if you will just see p equals to 1 this will also take this boundary condition here.

So, that is why this coefficient will be 0. So, they starting coefficient it will just come in the form of like here as a 2 into 1 minus r here, and next coefficient you can just see that this will just come as r here and rest of this elements as a 0 here, and immediate next element if you will just see that will just come as r 2 into 1 minus r. So, likewise it will just continue and in the final term we can just get it as like r 2 into 1 minus r here and the coefficient if you will just see this can be written in the form of since q is fixed here we can just write this one as like e 1 q e 2 q up 2 up to e of n minus 1 q if you will just write this one.

(Refer Slide Time: 07:58)

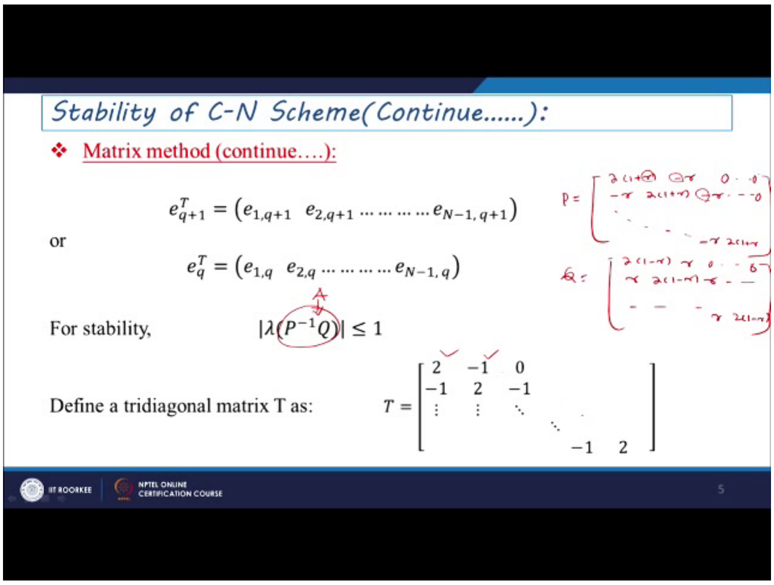


So, in the final form if you will just see this equation just takes this form like e of q plus 1 T this equals to e 1 q plus 1 e 2 q plus 1 up to e of N minus 1 q plus 1 there. So, the that is why this is just forming a system of matrix or a system of equations which can be transformed to a matrix that is in the order of n minus 1 into n minus 1 here and if you will just see this right hand side term that is nothing but e 1 q as I have explained e 2 q up to en minus 1 q here and if we will just write this one as in the form of like since our matrix is written as in the form of like eq plus 1 this equals to P inverse Q eq here.

So, that is a e q plus 1 that is nothing but P inverse Q eq here and for stability we already have discussed in the implicit approach that we should have to write this P inverse Q as in the matrix form since always we will if you we will have like distinct Eigen values then we can just write it in a linear combination from there and each of this Eigen value s it can be represented like a into vs as we have written this vectors has a vs there. So, avs equals to lambda as vs there and in each of this multiplication of matrices it can be transformed to lambda square lambda q likewise it can just go to lambda to the power n minus 1, and that is why if you will just take this stability of these schemes then we can just write this one as the multiplication since here this matrix is formed as bare itself we have just written in terms of a that is why I am just writing this one as a here a is nothing but P inverse Q for this system of equations here and if you will just write for this stability of this scheme here.

So, then we can just write lambda into P inverse Q it should be less or equal to 1 and to get this formulation in a generalized form if you will just see here. So, then in each of these coefficients if you will just visualize here P is a associated this terms like. So, P is associated this terms like P equals to it is defined as a 2 into 1 plus r minus r 0 second term

(Refer Slide Time: 10:15)

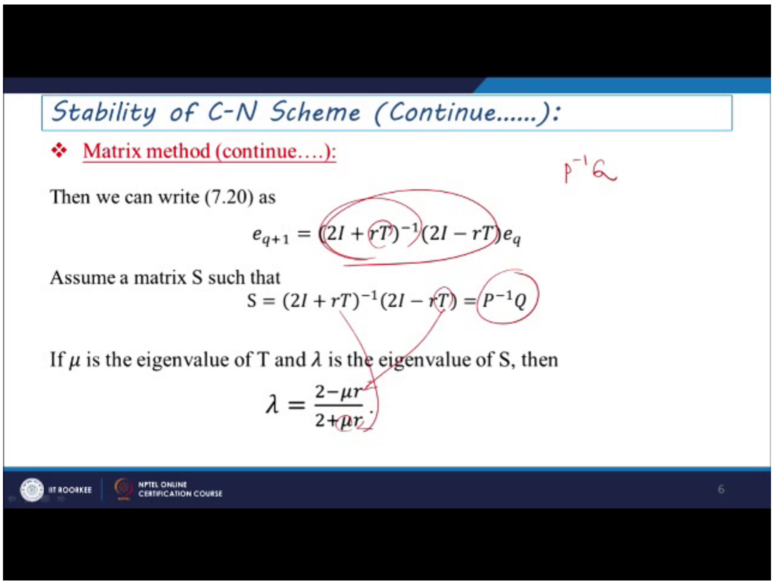


If you will just see it is minus r 2 into 1 plus r minus r 0 rest of the terms we are just considering.

So, likewise if the last term it is just coming as a minus r 2 into 1 plus r here and similarly, if you will just see here. So, q term it is also involving this terms like 2 into 1 minus r r 0 and then the next term r 2 into 1 minus r r, it is just continuing. So, likewise is a last term it is just written as r 2 into 1 minus r. So, if you will just take these diagonal elements here. So, each of this r coefficient that involves a multiplier term as to there and the immediate next term if you will just see this also contains like minus 1 minus 1 so likewise it is just going.

So, that is why we want to write this r coefficients in a separate matrix form and here we have defined that coefficients as in the form of T here. So, that is why if you will just see here r coefficient that is just taking 2 here then this minus r coefficient that is minus 1 here again if you will just see the second row this is just taken as minus 1 then 2 minus 1. So, likewise we have just written as the T as a matrix here.

(Refer Slide Time: 11:52)



So, if you will just write this e of q plus 1 as in the form of since we have just written this matrix that is in the form of like P inverse Q, then we can just formulate that one as since t is the matrix which takes the coefficients of r there itself. So, it can be written in the form of rT here and the rest of this elements if you will just see that just to take these coefficients that it is just expressed in the form of like 2 at the diagonal positions there itself. So, we can just express that one as a 2 into a identity matrix there. So, that is why this complete system it can be written in the form of like 2I plus rT inverse since we are just writing there that is like P inverse into Q there.

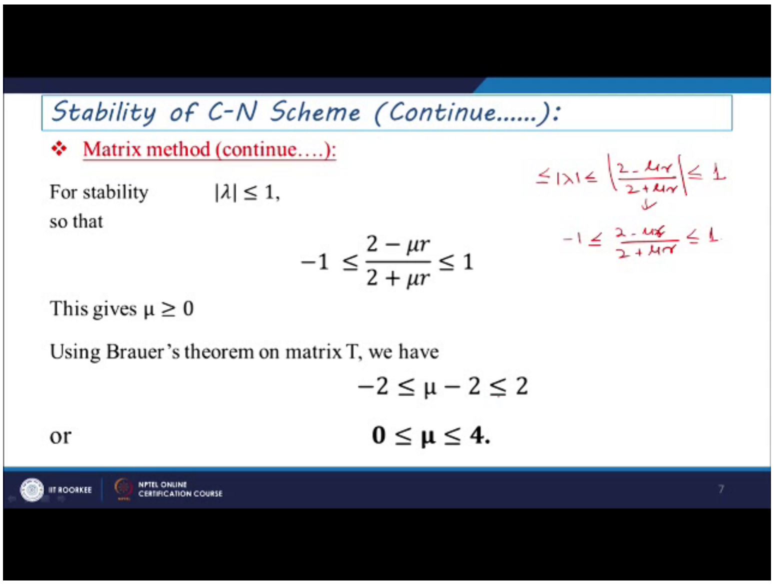
So, that is why these diagonal elements only we are just representing here. So, directly it can take this as in the inverse form here that as 2 I plus rT whole inverse and 2I minus rT into eq here. So, if you can just see this one clearly it can be signified from this matrix sense if you will just see both these matrices here. So, if we want to find this Eigen values of this matrix. So, especially we have to signify this as a complete matrix from here then we can just subtract the lambda values from each of these diagonal elements there itself.

So, to do this; this one so if you will just assume that S is a matrix which is defined is in the form of like 2I plus rT whole inverse into 2I minus rT that is nothing but P inverse Q here. So, we can just obtain this Eigen value S if we can just write it in a determinant form. So, if suppose mu is the Eigen value of T and lambda is the Eigen value of S here then this relationship between T and S can be written as like lambda equals to 2 minus mu r by 2 plus mu r.

Since if you will just see here this above system here we are just writing here S as the system that is nothing but P inverse Q and if you will just consider here 2 since 2I means this is just taking 2 into all of these Eigen value s sorry identity matrix there. So, that is why in each of these diagonal entries if you will just subtract this minus lambda. So, it can be represented as minus 2 lambda minus 2 lambda up to the last value as minus 2 lambda there, and if you will just see this other coefficient here that can be written as like the rT term rT term means if you will just see this is also taking this elements that is in the form of like a diagonal elements tri diagonal elements just it is taking the coefficients.

So, that is why it can be written in the form of like 2 minus mu r since we are just writing this inverse terms there and that is why it can just take to the down here. So, that that is why this is just written as a 2 plus mu r. Since a mu is Eigen value associated with a t value here and in the upper side we can just write this is 2I minus rT. So that is why so T coefficient that is just replaced by this mu here. So, that is why this Eigen value of s can be represented in the form of 2 minus mu r by 2 plus mu r and for stability already we have known that we have to consider that absolute value of lambda it should be less or equal to 1.

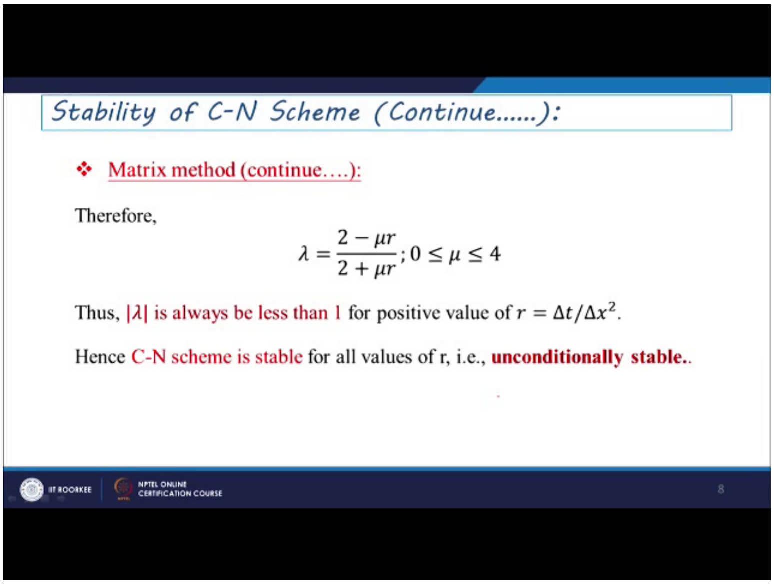
(Refer Slide Time: 15:31)



So, if you will just consider this lambda should be less or equal to 1 here, then we have this inequality like lambda this is less or equal to 2 minus mu r by 2 plus mu r here. So, that is why we are just writing this one as in the form of like lambdas would be less or equal to if you will just consider both the sides there itself so since it is just considered as 1 here. So, we can just write this absolute value it is also less than 1 there. So, if we are just writing in this form there then we can just write this one as minus 1 it should be less or equal to 2 minus mu r by sorry 2 plus mu r this should be less or equal to 1 there and which gives same the value like mu minus 2 it should be lies between minus 2 and 2 there.

Since mu is greater than 0, if you will just use Brauer's theorem that is especially applicable for this spectral radius that takes the largest Eigen value in magnitude sense so that is why we are just considering this mu it should be lies between 0 and 4 there.

(Refer Slide Time: 16:42)

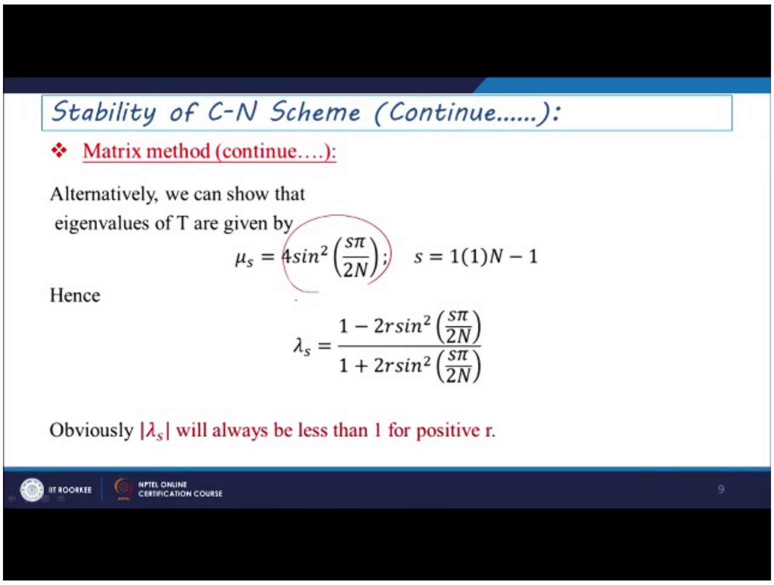


And if you will just continue this calculation then we can just find that lambda equals to 2 minus mu r by 2 plus mu r for mu lies between 0 and 4. And especially we can just visualize that if modulus of lambda is the less than 2 minus mu r by 2 plus mu r.

So, then we can just consider modulus of lambda value is always less than 1 whenever we will just consider this any positive value of r equals to del t by del x square hence Crank Nicolson scheme is a stable for all values of r that is nothing but it is unconditionally stable. Especially if you will just go for like our earlier discussion we you can just find that explicit scheme whatever it is just applicable it is just stable whenever your r value is less than half and we have just explained that explained by considering one example that if we are just considering like, explicit scheme for this problem like del u by del t equal to del square u by del x square. So, after certain step it is just showing some unrealistic physical behavior.

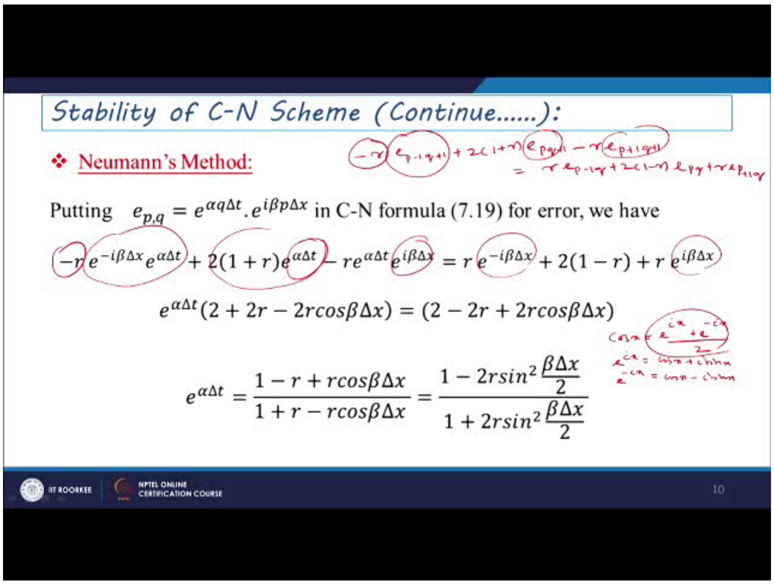
So, that is why it is best to use like Crank Nicolson scheme instead of explicit scheme, since explicit this scheme puts some restriction that it can be applicable whenever you will have this r should be less than half, but this does not have any restriction for Crank Nicolson scheme. So, that is why we can just move with any type of like del x or del t condition to deal this differential equation partial differential equations. So, alternatively we can also show that if we can just fine directly this Eigen value s by using this Bauer Fike theorem or like Gershgorin theorem.

(Refer Slide Time: 18:40)



So, you can just obtain this Eigen value that as in the form of like 4 sin square by s pi by 2N. Even if for that condition if you will just check this lambda s value. So, that will just come as like 1 minus 2 r sin square s pi by 2N by 1 plus 2 r sin square s pi by 2N since it is just defined in the form of like 1 minus 2 r mu by 1 plus 2 r mu there so; obviously, if you will just consider this lambda S is less than 1 for all positive values. So, we can just to get this condition also. If you will just go for like a Newman's method here. So, Newman's method already we have discussed that we have to consider this error coefficients as in the form of e to the power alpha q del t into e to the power i beta p delta x.

(Refer Slide Time: 19:24)



And if you will just put this one in the error formulation since our coefficients are just it is just taking in the form of like minus r. So, if you will just see our earlier formulation you can just visualize that these coefficients are taking is in the form of like minus r e p minus 1 q plus 1 plus 2 into 1 plus r e pq plus 1 minus r e p plus 1 q plus 1 this is just taken as like r e p minus 1 q plus 2 into 1 minus r epq plus r e p plus 1 q here.

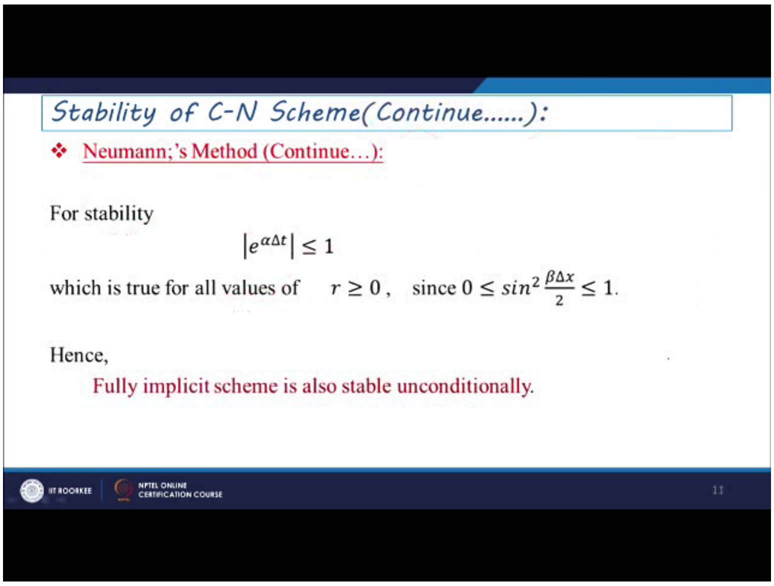
And if you will just put all this coefficient terms here that can be written in the form of since minus r is present there so minus r it can be written and e of p minus 1 q plus 1. So, directly if you will just put these values here it can be written in the form of like e to the power so some manipulation in the last lecture we have done and finally, we have just often that one as e to the power minus i beta delta x e to the power alpha del t since some of these coefficients we can just take from this left hand side and some of the coefficients from the right hand side, as the r coefficients and it can just get cancelled it out. So, that is why this complete formulation it is just taking as e to the power minus i beta delta x e to the power alpha del t for this e of p minus 1 q plus 1.

And if you will just take these terms for like e of p q plus 1; so, it is just taking in the form of like e to the power alpha del t here and similarly if you will just take this coefficient for e of p plus 1 q plus 1 after canceling this left-hand side coefficients and right-hand side coefficients some of the coefficients of r, then this term will be reduced to e to the power i beta delta x. Similarly, the same coefficients you can just visualize in the right-hand side also. And especially if you will just write it in a compact form then we can just take common e to the power alpha del t in the left hand side and it can be written in the form of like 2 plus 2 r minus 2 r cos beta delta x this equals to 2 minus 2 r plus 2 r cos beta delta x here, since already you have known that cos x it can be written in the form of like e to the power x plus e to the power minus x by 2.

So, that is why it is just represented in this form here. If we can just express this terms as in the form of e 2 the power alpha del t as like 1 minus r plus r cos beta delta x divided by 1 plus r minus r cos beta delta x, since a 2 both sides it can be taken common and it can be cancelled it out. So, this can be written as like 1 minus 2 r sin square beta delta x by 2 by 1 plus 2 r sin square beta delta x by 2. Especially if you will just see here. So, usually cos x it is just written in the form of like e to the power ix plus e to the power minus ix by 2 here.

So, this just clarifies the sense that whenever we are just expressing this e to the power ix it is also written as cos x plus i sin x and in that form you can just write e to the power minus ix that as cos x minus i sin x. So, from there itself or directly if you just add it up then you can just obtain this formula.

(Refer Slide Time: 23:04)



So, if we will just go for this a further movement of this Newman's method we can just consider the stability condition that is nothing but modulus of e to the power alpha del t it should be less or equal to 1 which is true for all values of r greater or equal to 0 here.

Since we are just considering here e to the power alpha del t that is nothing but if you will just see that is nothing but absolute value of this one here and if this is less than one which is true for all values of r greater or equal to 0, since if you will just consider this condition that is in the form of like 1 minus are 2 r square sin square beta delta x by 2 it should be lies between 0 and one there itself. Hence, in fully implicit scheme it is always we can just say that it is unconditionally stable here.

Thank you for listen this lecture.