Numerical Methods: Finite Difference Approach
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Lecture – 08
Compatibility, stability and convergence analysis

Welcome to the lecture series on numerical methods; finite difference approach, and in the previous lecture, we have discussed to this implicit and Crank Nicolson scheme and also for the different schemes compatibility and convergence analysis we have discussed.

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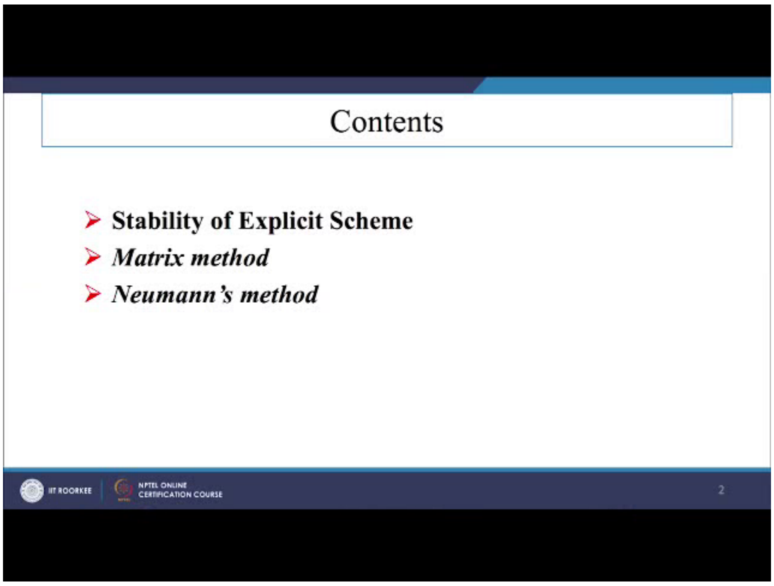

And in the present lecture, we will just go for this stability of the explicit schemes, how you can check the stability of this explicit schemes or implicit schemes by using 2 different methods, especially we are just finding this stability conditions. So, first one; it is called Matrix method and second one it is called Neumann's method. So, based on these 2 methods, we will just go for this checking of stability if a problem is given and if we are just applying this explicit approach, whether this system will provide a solution or not.

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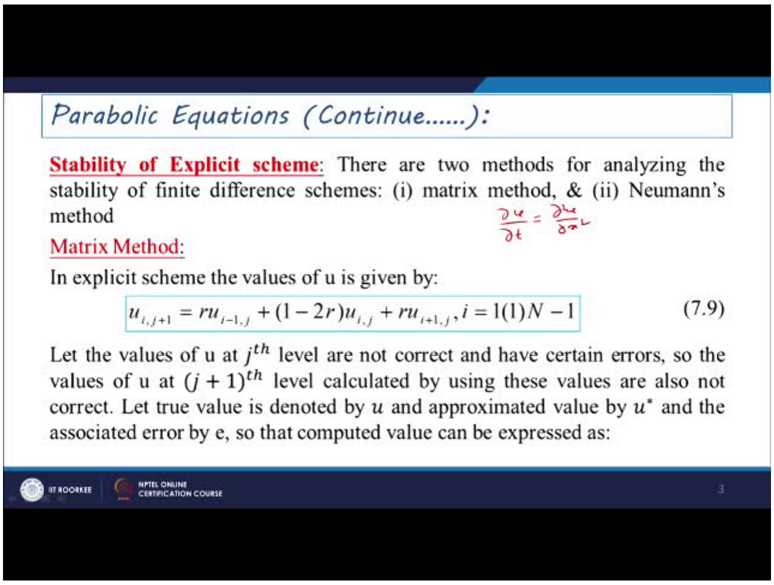

This means that solution it can provide, but whether that is feasible solution or not. So, for this, if you will just go for this stability of the explicit scheme, especially as I have told that 2 schemes, we will just apply; first one it is called Matrix method and second one it is called Neumann's method. And in the matrix method, if you will just consider this same equation as in the form of like del u by del t by this equals to del square u by del x square here. Then especially, we are just obtaining this discriticized scheme as in the form of like ui, j plus 1 is equals to r ui minus 1, j plus 1 minus 2 r ui, j plus r ui plus 1, j here. And where i is varying from 1 to N minus 1 and we are just computing these values at high j plus 1th level. So, that is why this values at i minus 1, j and ui, j and i plus 1, j it should be known to us here. Let the values of u at jth level are not correct suppose, since along the boundary always it is fixed.

But afterwards whenever we are just going for the computation then maybe it is not correct, accurate, or you can just find that there is some differences from this exact value or the approximated value at each of the levels. And have certain errors, so the values of u at j plus 1th level can be calculated by using these values are also not correct, since we are just considering this previous time step level values for this like next step level of calculations. So, that is why this error will get in increasing let the true value is generated by u and the approximated value is generated by u star and the associated error suppose by you here.

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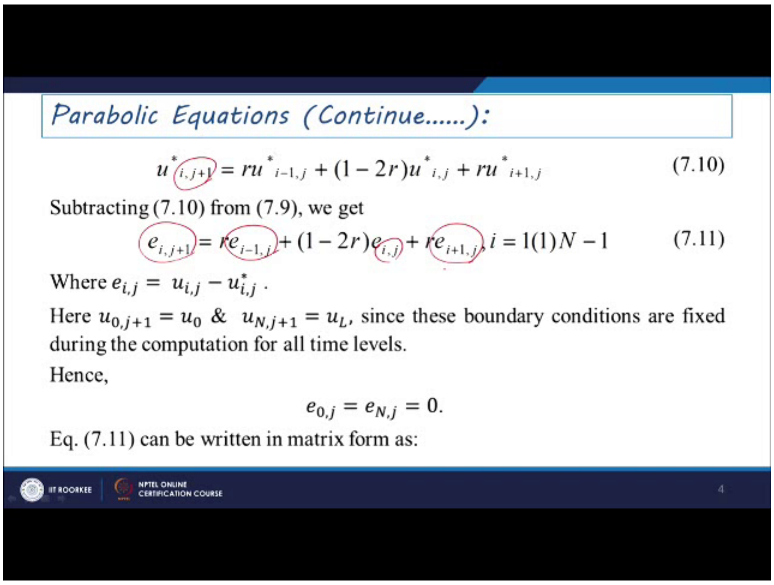

So, that the computed value can be expressed as in the form like u star i, j plus 1 this equals to r u star i minus 1, j plus 1 minus 2 r ui, j star plus r u star i plus 1, j.

Since, already we have explained here we are just considering one true value as u here, and the approximated value by u star you can just say that this is the discretisized solution and this can be the true solution means exact solution you can just consider or sometimes you can just say that this is the numerical calculated value. And if you will just subtract this true solution to the approximated solution here then we can just find at i, j plus 1 the point the error will be eij plus 1 and similarly this error propagated for each of these grid points are in the form of like ei minus 1 j, ei, j and ei plus 1, j here. Where i is varying from 1 to N minus 1, and we have known that along the boundaries the values are exact since it is just provided the exact solution along the boundary which should be satisfied at each time level of the calculation.

So, we cannot change or we cannot differ from this boundary condition values at any step of calculation. So, that is why we can just consider this e 0 j and eN, j equals to 0, since the error associated along the boundary is 0, since it is provided exactly the values the here above and if you will just write this error terms for each of this terms starting from 1 to N minus 1, then this error terms can be represented in a Matrix form as e 1, j plus 1 to eN minus 1, j plus 1, since e 0 is a fixed and it is a 0 there and eN is a 0 there itself and if you will just put these values here you can just find for the first state of calculation here this is just giving you the 0 value here. So, that is why they starting coefficient it will be 1 minus 2 r and the next coefficient is r there.

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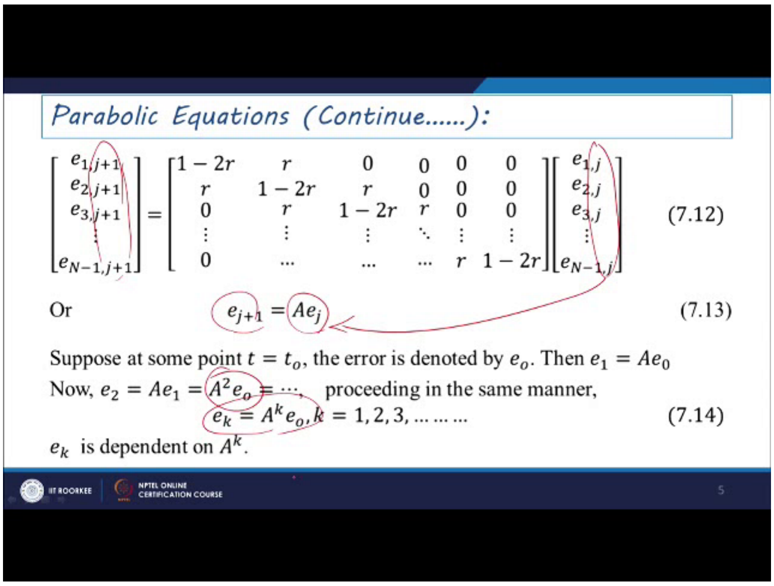

So, remaining after like boundary points you will have 3 coefficient that is r, 1 minus 2 r and r. So, that it is just written in this form here and your coefficients if you will just see that is just starting at first one is e 1, j since e 0 j is always 0. So, that is why u 1, j is the starting value and ending value it will be N minus 1, j; There itself. So, in a Matrix form if you will just represent this 1. So, ej plus 1 since all of this j plus 1 is a constant here, but the variation in the ith level that is at 1 2 3 up to n minus 1. So, that is why you can just write ej plus 1 this equals to Aej, since Aej also remain constant for this level here.

Suppose at some point t equals to t 0 the error is generated by e 0, then we can just write e 1, since A starting j equals to 0 suppose we can just write e 1 equals to A 0 here. Now e 2 can be written as Au 1. So, e recursively if you will just write e 1 is nothing but Ae 0. So, we can just write A square e 0 and if you will just proceeding in this manner we can just find that ek can be written as A to the power k e 0 here. And we can just find that ek is dependent on A to the power k here, since here this error associated with this each of these sequences that is associated inside.

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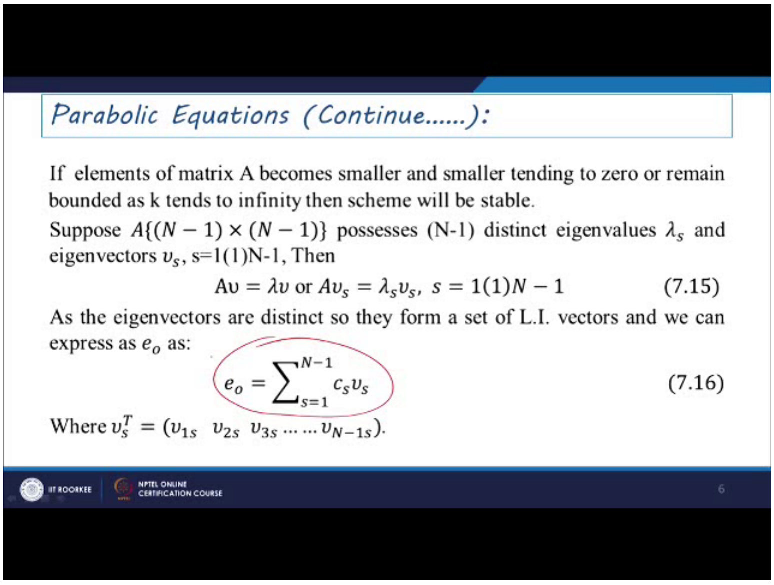

The matrix there itself if elements of the matrix A becomes smaller and smaller tending to 0 or remain bounded as k tends to infinity, then scheme will be stable. Suppose if you will just consider a is the matrix here since boundary values we are just restricted we have taken it out. So, we will can just form a matrix that is of order n minus 1 to N minus 1 for this is N minus 1 distinct eigen values suppose, this eigen value is a lambda s and the corresponding eigenvector is suppose Vs here, and S is varying from 1 to n minus 1. Then we can just write AV equals to lambda V, since this eigen values are distinct, so it can form this like linearly independent eigen vectors. So, that is why we can just a formulate e 0 can be expressed as a linear combination of all these linearly independent eigen vectors there, Which can be written as like Cs Vs here, where Vs can be written as V1s, V2s up to VN minus 1s here.

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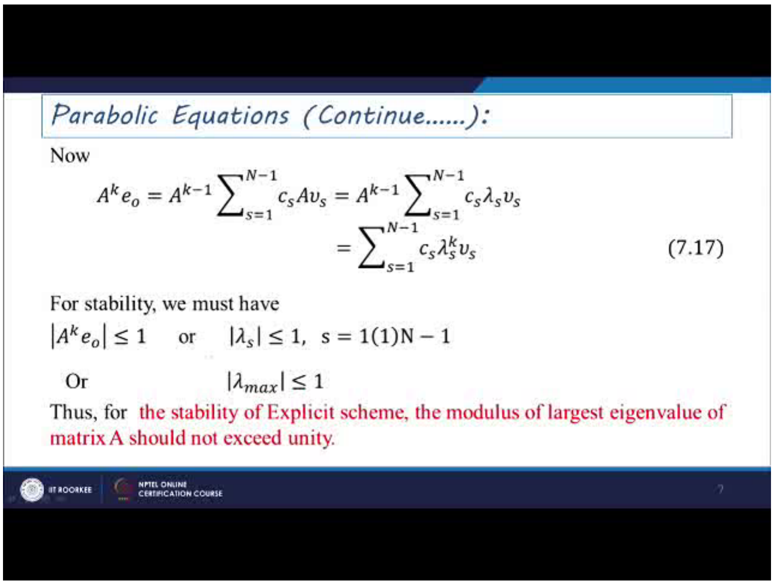

So, if we can just express e 0 in this form here. Then we can just write A to the power k e 0 as A to the power k minus 1 into Ae 0. So, Ae 0 it can be written as like AV, suppose e 0 is written as summation of s equals to 1 to N minus 1 Vs here. So, that is why if A can be taken to the inside the summation we can just rewrite this AVs as lambda s Vs here. Since in the previous slide we have just written that is as a Vs equals to lambda s Vs there. So, finally, we can just obtain after like Ak steps.

We can just obtain Cs lambda s to the power k into Vs there, and for the stability always we have to consider absolute value or we can just take this determinant of a to the power k is 0 it should be less than 1, or we can just consider lambda s to be less than 1 where, s is varying from 1 to N minus 1 or lambda maximum should be less than 1 thus for the stability of explicit scheme the modulus of largest eigen value of the matrix should not exceed unity.

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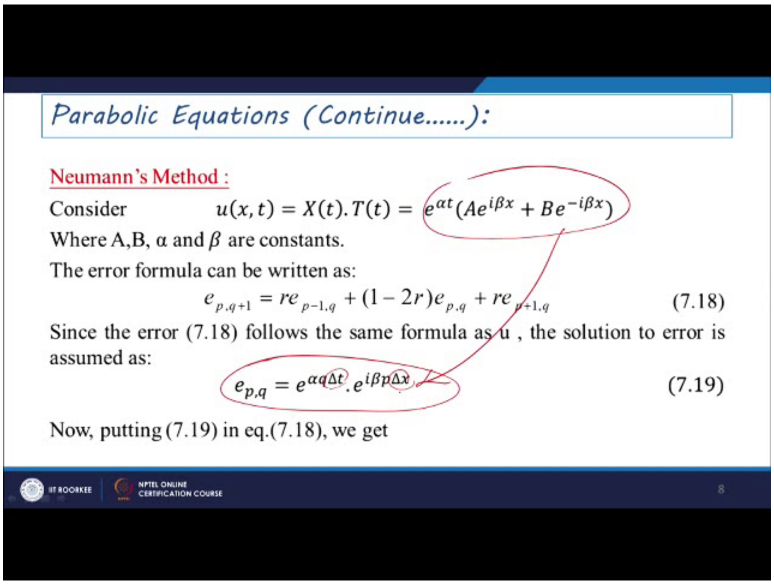

So, if you will just go for this Newman's method. So, especially that is a specifically they written in a variable separation form u of Xt can be written as in the form of a X of t and T of t, which can be written in a exponential form with this complex coefficients especially this is written in a generalized form as e to the power alpha t into A e to the power I beta x plus B e to the power minus I beta x, and we are just considering A, B, alpha and beta are constants there.

So, especially we are just representing this coefficient as a direct form here that is e of p, q that is the error associated with the each of these terms that can be written as e to the power alpha q del t, del t especially you can just say that that is the time difference error we are just getting and del x means it is the space difference error we are just operating there. And the error formula for this like a earlier problem if you will just discuss then that can be written as in the form of e of p, q plus 1 this is a r e p minus 1, q plus 1 minus 2 r e p, q plus r e p plus 1, q.

If you will just use e p, q as the error term associated with this p, q node, then we can just say that this can be propagated to p, q plus 1 as in the form like e to the power alpha q plus 1 del t into e to the power I beta p delta x, similarly it can be extended to rest of the terms that is as like first term if you will just see here that is p minus 1, q then p, q then p plus 1, q. So, which is written A in this form here.

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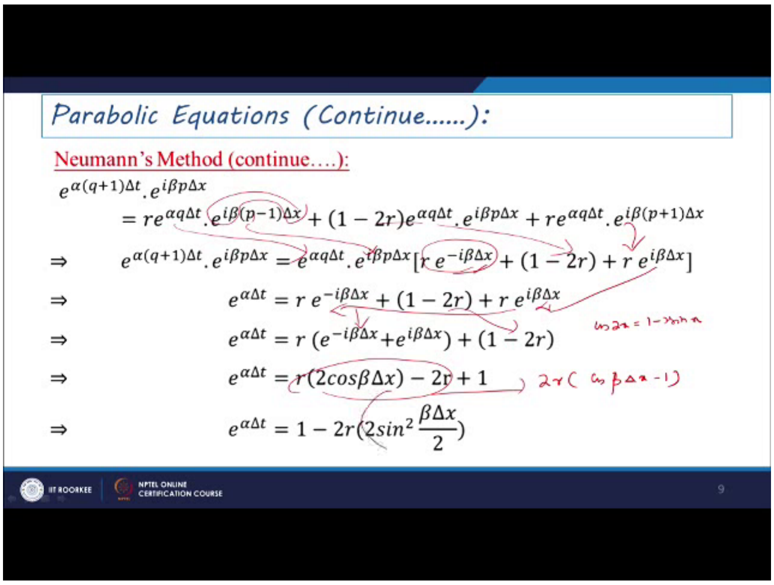

That is a r e to the power alpha q del t into e to the power I beta p minus 1 delta x plus 1 minus 2 r e to the power alpha q del t dot e to the power I beta p delta x plus r e to the power alpha q del t dot e to the power I beta p plus 1 delta x here.

So, if you will just compare these coefficients, especially if you will just see here that is e to the power alpha q del t since a we are just taking the common r e to the power minus I beta delta x. So, that is why if you will just take common minus I beta delta x from here. So, then we can just write this one as e to the power I beta p delta x from this term here. And this term we are just writing e to the power alpha q del t here. So, that is why inside this bracket it is just going like a r e to the power minus I beta delta x here and similarly if you will just write this a middle term. So, that coefficient will come as 1 minus 2 r and the last terms coefficient that will just come as r e to the power I beta delta x here.

So, if you will just neglect some of the terms at right hand side and left-hand side not neglecting that 1 we are just cancelling both the sides since these are the common terms there. So, we can just write e to the power alpha del t this equals to r e to the power minus I beta delta x, 1 minus 2 r plus r e to the power I beta delta x term and e to the power alpha del t can be written as r e to the power minus I beta delta x, since we are just writing these 2 terms in a combined form then this can be written as 1 minus 2 r here. So, then e to the power alpha del t this can be written as r since we have known that cos x can be written as e to the power minus ix plus e to the power ix by 2.

So, that is why it can be written in the form of r into 2 cos beta delta x minus 2 r plus 1 here and finally, we are just obtaining e to the power alpha r del t this equals to 1 minus 2 r since we are just obtaining here 2 r cos beta del t minus 2 r here. So, we can just take common from these 2 terms here, and we can just write this one as like 2 r into cos beta delta x minus 1 and which can be expressed as a since we know that like cos 2 x can be expressed as a 1 minus 2 sine square x.

So, in that form if you will just a transform then you can just find this one as 2 r 2 sine square beta delta x by 2 here, and in the Neumann's method if you will just go for this stability.

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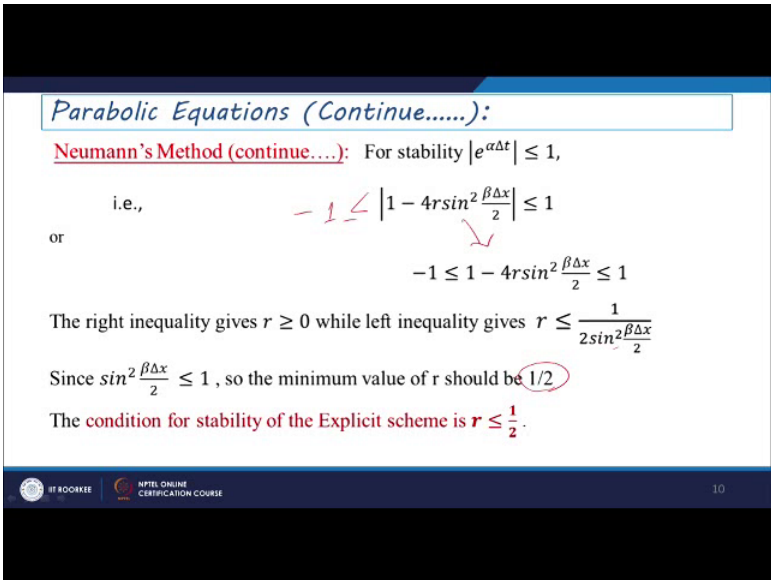

So, then we have to consider this e to the power alpha del t should be less or equal to 1. So, if you will just take this the left-hand side is less than 1; this means, that we can just write 1 minus 2 r 2 sine square beta delta x by 2 it should be less than one also. So, this implies that if you will just take the absolute value it should be less or equal to 1. So, then we can just write this should be minus 1 it can be written this side here. So, this is the relationship it can establish as the inequality form and the right inequality gives r greater or equal to 0, if you will just see while left inequality gives r is less or equal to 1 by 2 sine square beta delta x by 2 here, since if you will just see here sine square beta delta x by 2 is less equal to 1.

So, the minimum value of r should be half here, this implies that the condition for stability of the explicit scheme is r less or equal to half. So, this restriction we should have to follow up whenever we are just applying these explicit schemes; so for a practical example if you will just consider and if you will just apply these stability criteria, if it is not justified and how it is just coming to the picture that we will just discuss in this example.

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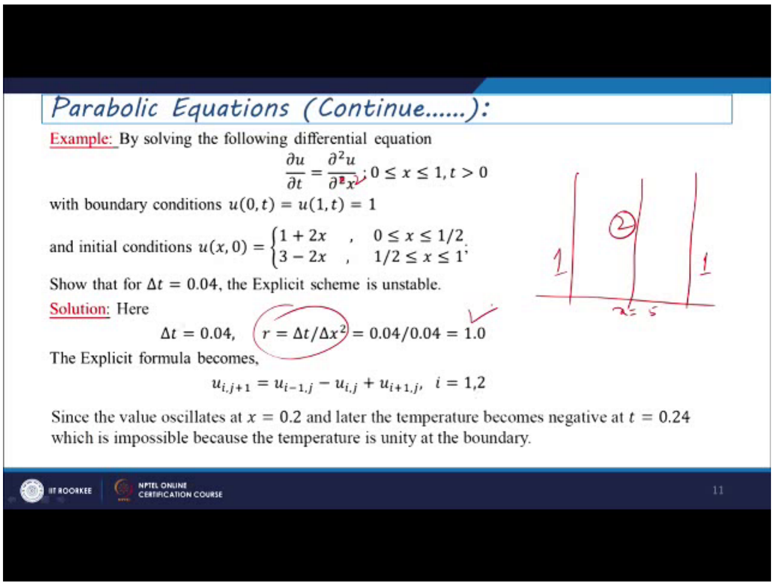

So, if you will just go for the solution of this a partial differential equation here. So, that is in the form of a del u by del t equals to del square u by del x square. So, especially you are just writing this is a del x square here, where x lies between 0 to 1 and t greater than 0 with boundary condition u of 0 t this equals to u of 1 t equals to 1 here. The boundaries it is prescribed at 0 and 1 as 1 there, and the initial conditions that the same symmetric condition we are just assuming here also that is at x equals to 0.5 this symmetricity will be maintained, that they will just take this condition as 2 here and del t is given as a 0.04 here, and the question is asked to solve this problem using Explicit scheme. So, we can just show that whether Explicit scheme will provide a solution or not in a feasible sense.

So, if you will just consider here delta t equals to 0.04, then we can just often r as a delta by del x square, which is written as 0.04 by 0.04 as 1.0 here, and the explicit formula for this scheme can be written in the form like ui, j plus 1 equals to ui minus 1, j minus ui, j plus ui plus 1, j where I is varying from 1 to 2 since the value oscillate at x equals to 0.2.

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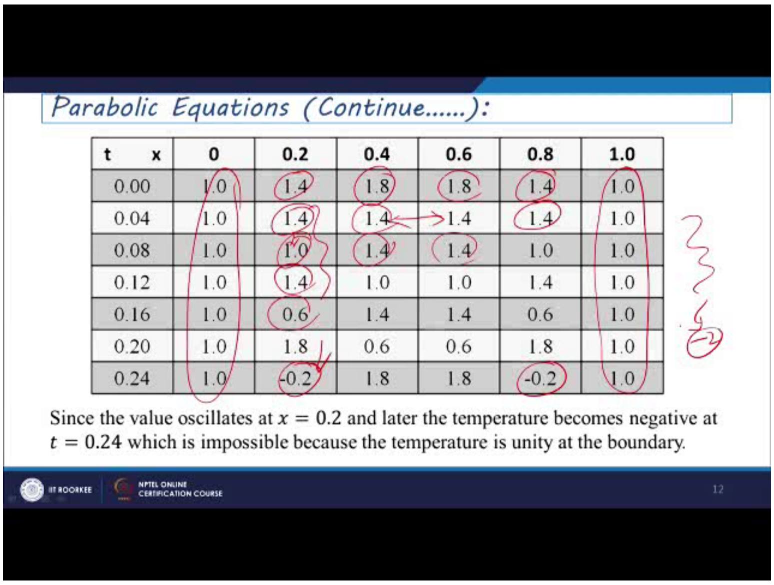

If you will just see this problem section here, then we can just find that whenever we are just applying this conditions here then we can just obtain along the boundaries we have like 1.0, 1.0, 1.0 all these boundary conditions are there and along this boundaries if you will just see this is a 1.0, 1.0, 1.0 it is just coming over there, but if we are just computing these values at 0.2th level it is just giving you 1.4, due to symmetricity the same value it is just appearing at a 0.8 level also and whatever it is just coming at 0.4 level the same value it is also coming at 1.8h level here. And if you are just proceeding the same value again we are just getting at like 0.2th level and the same value we can just obtain at 0.8th level also here and due to symmetricity 0.4 has the same value as of 0.6 here, but if we are just to proceeding further then you can just find that there is a oscillation it is just occurring here.

This means that it is just coming decreasing down again 1.0 it is just coming this satisfies the boundary condition again and 1.4 if you are just getting the symmetricity we are just getting along with 0.6 as 1.4 also, this means that the temperature it is just at the beginning it is a starting at 1.0, suddenly it is just getting increased and close to the boundaries then it is just getting decreased there. So, at 0.16 level if you will just see again this is getting decrease, and at 0.12 level if you will just see again it is just increasing. So, this represents oscillating profile here and whenever we are just proceeding towards like a after this 0.20 level we can just find that this is just going to negative level here, negative level means since we are just applying heat from the boundaries.

So, the core temperature cannot go down the temp like a negative sense, since if you are just was applying heat from the boundaries. So, always in a common sense if you will just take like a water pot suppose and if you are just supplying the heat along the boundaries the water will be getting heated it off and along the core section you can just find that this temperature will grows up it will never goes down there, but in this sense we are just getting that the temperature is going down and it is just a achieving the negative value after certain time steps.

So, it is not a signifying any physical sense that this stable solution we are just obtaining since at the present solution we can just visualize fast it is just oscillating and suddenly it is just approaching towards the negative value. Even if we if we the temperature is supplied or heat is supplied along the boundary. So, this physical scenario will never happen that in the core of this section you can have a negative value there itself. So, that is why it is just we have the statement that, since the value oscillates at x equals to 0.2 and later the temperature becomes negative at t equals to 0.24 which is impossible because the temperature is unity along the boundary here.

So, if you will just see our condition here also this means that, we are just neglecting one of this inequality in the first half; that means, that if we are just considering r is greater or equal to 0, when the left inequality becomes this one here, but if you will just consider like r less or equal to 0 here. So, we do not know what it will happen there over. So, that is why there is a restriction that whenever we will have like r less or equal to half only you can just find that the system is stable and we can have a feasible solution there.

Thank you for listen this lecture.