Numerical Methods: Finite Difference Approach
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Lecture - 07
Implicit method

Welcome to the lecture series on numerical methods finite difference approach in the last lecture we have discussed this explicit scheme and based on that we have solved one problem and in the present lecture we will start about implicit scheme it is compatibility and afterwards we can just go for this convergence analysis and all other things. So, if you just to go for the implicit method.

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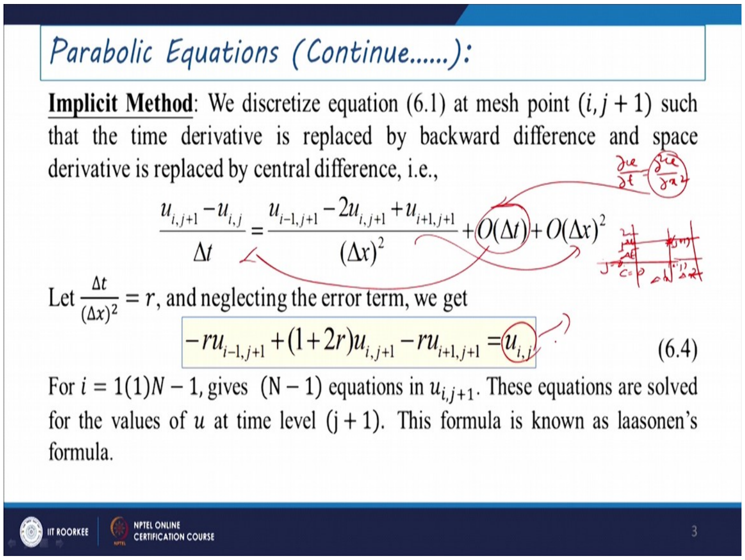

So, in the implicit method first we will just discretize this equation suppose if the equation is in the form like del u by del t this equals to del square u by del x square then we can just replace this derivative like at the point ij plus 1 such that time derivative is replaced by backward difference and space derivative is replaced by central difference. So, if you will just look at these earlier slides.

So, you can just find that in the explicit method especially we are just discretizing this differential equation at the mesh point ij there it iself, but in the implicit method we are just going towards next iteration step in the time to visualize that what it is happening in the next cell.

So, that is why we are just considering this mesh point at ij plus 1 cell such that we can just consider this advance cell considering this previous cycle calculation or the previous step calculation in that solution process. So, if you will just take this time derivative by a backward difference formulation and space derivative by a central difference approximation then this is del u by del t it can be represented as u ij plus 1 minus u ij by del t and if you will just go for this discretization of del square u by del x square term here.

By the central difference approximation at j plus 1 point, so, then we can just discretize this one in the form like ui minus 1 j plus 1 minus 2 u i j plus 1 plus u i plus 1 j plus 1 by del x square and if you will just go for this a harder of her convergence here or this highest order of a approximation that we are just neglecting the terms that is in the form of like first order terms we are just neglecting here for time and second order approximation we are just considering for the central difference approximation here and if you will just consider here r as it in the form of like del t by del x whole square and neglecting the error terms then we can just represent this a system as a linear system of equation as minus r ui minus 1 j plus 1 plus 1 plus 2 r u ij plus 1 minus r ui plus 1 j plus 1.

This equals to uij here since u ij is a known to us if you will just look at the structure you can just find that we are just moving i in the direction of X axis here that is as a starting as a suppose like 0 1 2 and j is also starting from 0 1 2 likewise and if we are just moving towards x direction we are just taking the space length as a del x here in each of this grid intervals and if we are just moving in the time direction we are just considering here each interval spacing is del t here. So, that is why if we will just a consider this space direction at i j th level suppose then we can just consider.

This as the j plus 1 th level for time here and we are just considering this calculation at the next time level oh or at t plus del t level if we are just starting t equals to 0 then t 1 equals to like t 0 plus del t that is nothing, but del t there in that level especially we are just signifying this grid as a j plus 1 th level there. So, that is why if we want to find out these values at the unknown level we have to consider these values which should be known to us at the known level. So, that is why we are just considering uij on the right hand side of this equation since this values are known to us. So, if you will just go for like I equals to 1 2 of to n minus 1 it will just a furnish like n minus 1 set of equations in uij plus 1 here and these equations are solved for the values of u at each time level of j plus 1 there.

So, this formula is known as lassonens formula and if you will just see this a molecule here equation is in the form of like a.

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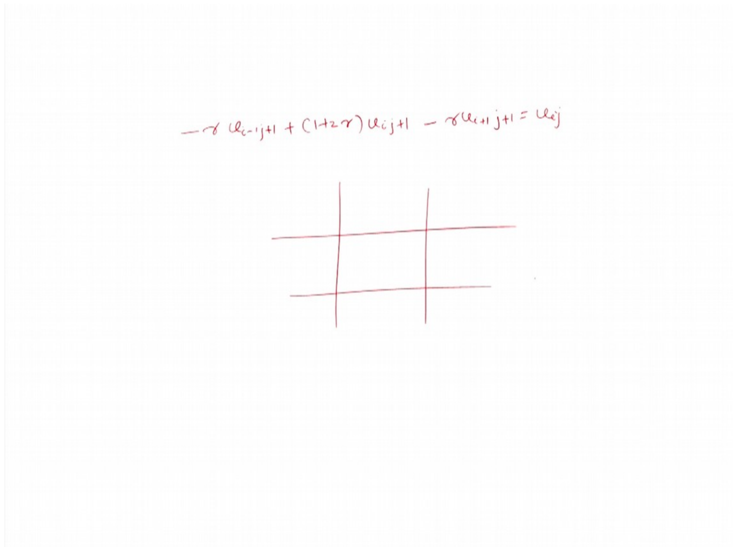

Minus r u I minus 1 j plus 1 plus 1 plus 2 r uij plus 1 minus r ui plus 1 j plus 1 this equals to uij here. So, if you will just see that this cell we are just formulating here at i th level and I plus 1 th level and jth level and j plus 1 th level we can just see that.

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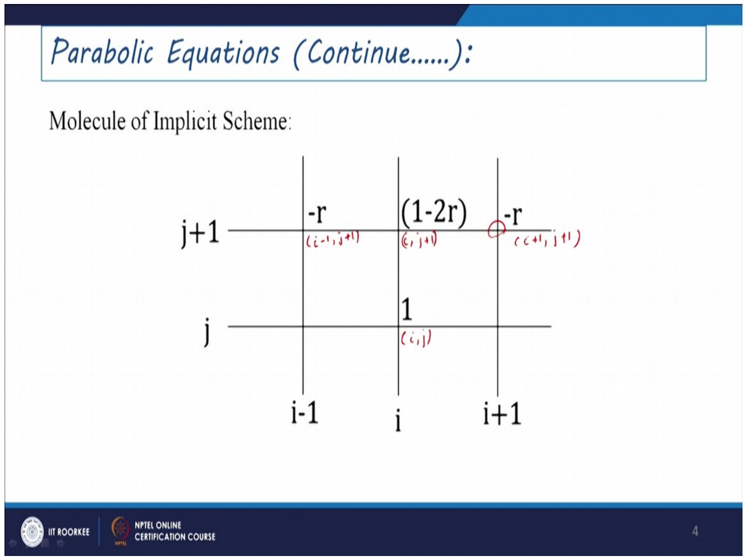

First coefficient this is just a taking as a minus r u i minus 1 j plus 1 and then 1 minus 2 r means at j plus 1 th level if you will.

Just see this takes the coefficients of you ij plus 1 there and if you will just see this last coefficient here that is nothing, but the coefficient of i plus 1 j plus 1 here since we are just considering these a levels this 1 as i plus 1 j plus 1 and this level as i j here and this level as like i j plus 1 and this level as like i minus 1 j plus 1 here. So, all these coefficients if you can just visualize a the previous slide you can just find that these are these coefficients associated throughout this equation.

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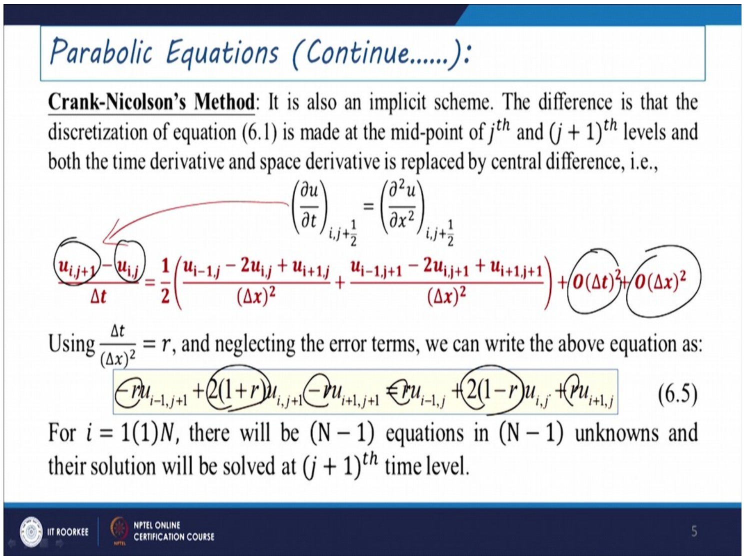

In crank nicholsons method this is like a average scale method especially it is said to be since in the earlier method.

If you will just go for explicit method we are just considering this values at uij level and in the next level we are just considering this implicit scheme first one that is in the form of like this computation is going at the node point that is at ij plus 1 th level and if you will just take the average of for these 2 grid points level that will just to generate a crank Nicholson's method here the advantage of for this crank Nicholson method is that here we are just considering both explicit and a implicit scheme together. So, that is why this convergence will be more accurate compared to explicit scheme or implicit scheme there. So, in this method especially as I have told that the difference is that the discretization equation is made at the midpoint of j th and j plus 1 th levels and both the time derivative and space derivative is replaced by central difference approximations here.

Since the central difference approximations are gives the higher order convergence compared to the other methods or the compared to the other difference schemes. So, if you will just take del u by del t in a like a central difference scheme here since we are just a discretizing this equation at the average point that is at ij plus half here. So, that is why we can just take this one as a j plus half plus half since we are just considering like a neighbourhood point of j plus half. So, that is why one side it will be j plus 1 other side it will be like j it will be present. So, that is why you can just consider this one as ij plus half plus half minus u ij j plus half minus half there. So, that is why.

This is just coming as a this equation as like u ij plus 1 minus u i j by del t and if you will just take the average of this space derivative term in the right hand side we can just find that this is nothing, but the average of the central difference schemes at ij th level and ij plus 1 th level. So, if you will just take the central difference scheme of del square u by del x square at ij th level we can just write this equation is in the form of u i minus 1 j minus 2 u 2 u ij plus ui plus 1 j by del x square and if you will just take this space derivative at ij plus 1 point then we can just write this a derivative term as ui minus 1 j plus 1 minus 2 ui j plus 1 plus ui plus 1 j plus 1 by del x square and this order of approximation.

If you will just see the central difference approximation for first order derivative that is considered as the order of a delta t here and if you are just considering this order of approximation for a central difference scheme then we can just find that one as the order of a del x square here and as usual a as we have used in the earlier slide. So, if you will just consider here r equals to del t by del x whole square and neglecting the error terms then we can just rewrite the above equation as in the form like since we are just evaluating at j plus 1 th level values or the unknown level values we are just evaluating. So, that is why we can just keep the terms associated with j plus 1 terms in the left hand side and the values.

Which is a known to us like j th level values it can be kept on the right hand side of the equation. So, that is why we are just keeping these terms as the minus r ui minus 1 j plus 1 plus 2 into 1 plus r uij plus 1 minus r ui plus 1 j plus 1 this equals 2 r u I minus 1 j plus 2 into 1 minus r since if you will just see the coefficients here we can just observe everything there. So, this equation set will vary from I equals to 1 to m. So, hence there will be n minus 1 equations with n minus 1 unknowns and their solution will be solved at j plus 1 th level since a j th level all values we are known to us or at the beginning of this calculation.

So, these values will be taken from this boundary conditions and it can be implemented for the further calculations. So, if you will just you see this molecule of this crank Nicolson scheme here we can just.

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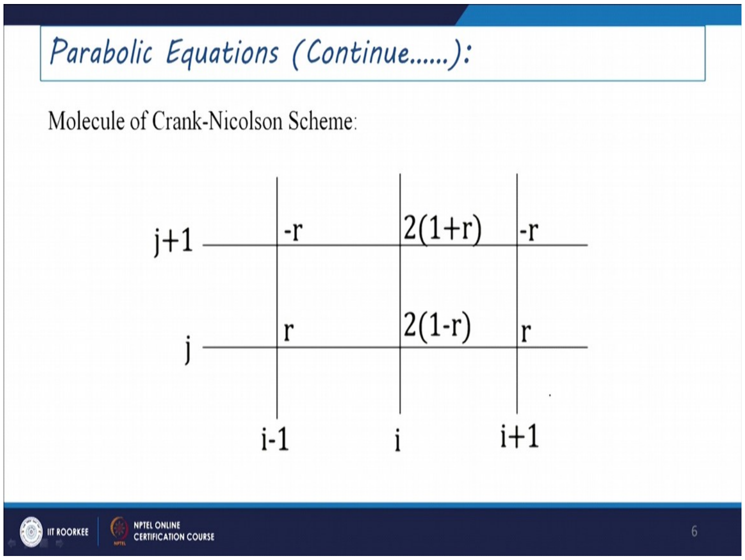

Find that at z plus 1 th level all the coefficients they are just taken as like if you will just see here r del t by del x square means we will have this coefficients here minus r 2 into 1 plus r this is minus r here and if you will just see like a j th level calculation we can just find this is as plus r here 2 into 1 minus r here r here. So, the same 1 it is just written at j th level it is just written as r 2 into 1 minus r and j plus 1 the level it is written as minus r 2 into 1 plus r and minus r.

Here, So, if you will just go for this a solution of this problem based on certain conditions that is prescribed a initially and these conditions are prescribed at the boundaries then directly we can just obtain the solution. So, for this we have considered a practical example.

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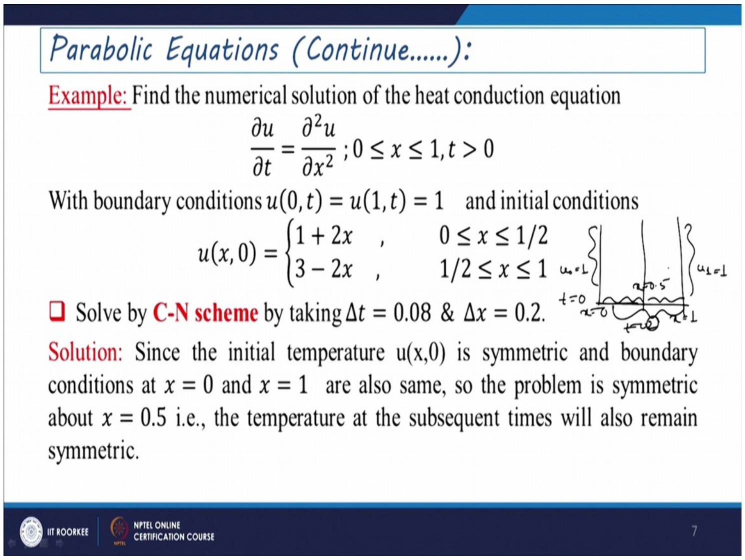

Where the symmetricity is occurring at the center line and the problem statement is written here as a del u by del t equals to del square u by del x square where x lies between 0 to 1 and time especially it is greater than 0 with the boundary conditions that is prescribed at x equals to 0 and at x equals to 1 as 1 here and the initial condition that is just a given as a 1 plus 2 x when x lies between 0 to half if especially if you will just see in a domain if we.

Your boundary x boundary is starting at 0 and the x boundary is ending at x equals to 1 here and at z th level means we can just say that at t equals to 0 all these boundary conditions city has been prescribed to us and along these boundaries this condition is also prescribed to us if you will just see here u 0 is as special it is to consider as a 1 here and u 1 is considered as 1 here also and for this boundary if you will just see at t equals to 0 these values are given as a 1 plus 2 x if you at the end point or if you will just see at x equals to half this is nothing, but we are just getting as the value as 2 here and if you will also come as x equals to half for this equation we are just also getting this as a 2 here also. So, that is why there existed the symmetry at the center line of this equation here.

So, the problem is symmetric about we can just say at x equals to 0 point 5 here and subsequently we can just calculate this discretized to domain at the first domain here and also in the last section of this domain.

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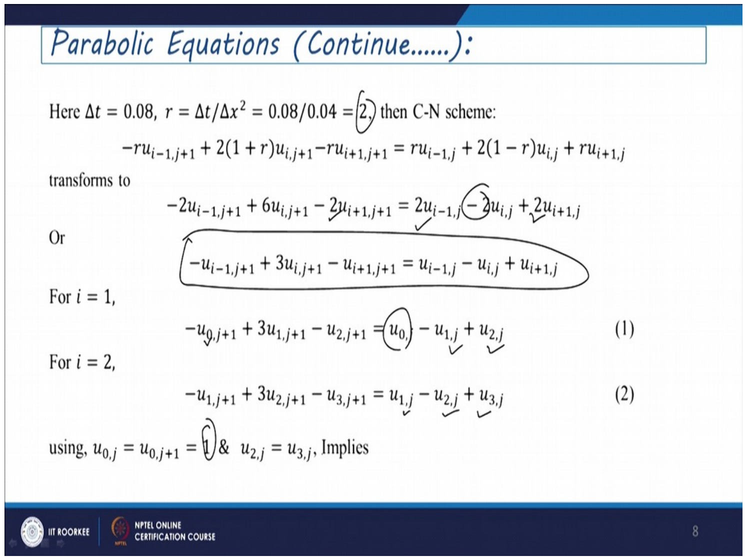

So, if you will just go for this calculation here since a del t is given as a 0 point 0 8 here and r can be defined since a del x is a prescribed as 0 point 2. So, that is why we can just write r equals to del t by del x square that is nothing, but 0 point 0 8 by 0 point 0 4. So, the value is coming as 2 here and if you will just use this Crank Nicolson scheme then this scheme can be written as minus r ui minus 1 j plus 1 plus 2 into 1 plus r u ij plus 1 minus r ui plus 1 j plus 1 this equals to r ui minus 1 j plus 2 into 1 minus r u ij plus r ui plus 1 j. So, if you will just put.

All these coefficients that is in the form of r here that is as minus 2 then this will be 6 then this will be like minus 2 then this will be like 2 then 2 into 2 sorry one minus 2 here. So, that is why this is minus 2 is coming here. So, then this is 2 here. So, for or if we can just take common 2 from both the sides we and cancel then we can just obtain this equation in this form here. So, for I equals to 1 if you will just put we are at the boundary that is u 0 j plus 1 plus 3 u 1 j plus 1 minus u 2 j plus 1 again we are at the boundary u 0 j minus u 1 j plus u 2 j here. So, these boundary conditions it is a known from this given conditions and that can be separated so, for I equals to 2 if you I will just put here.

Then this will just a compute like the values minus u 1 j plus 1 plus 3 u 2 j plus 1 minus u 3 j plus 1 this equals to u 1 j minus u 2 j plus u 3 j here since already from the boundary conditions we can just use u 0 j equals to u 0 j plus 1 since it is along the first boundary there. So, always is remain 1 there and this implies that u 2 j equals to u 3 j and which just impacts this equations as in the form like 3 u 1 j plus 1 minus u 2.

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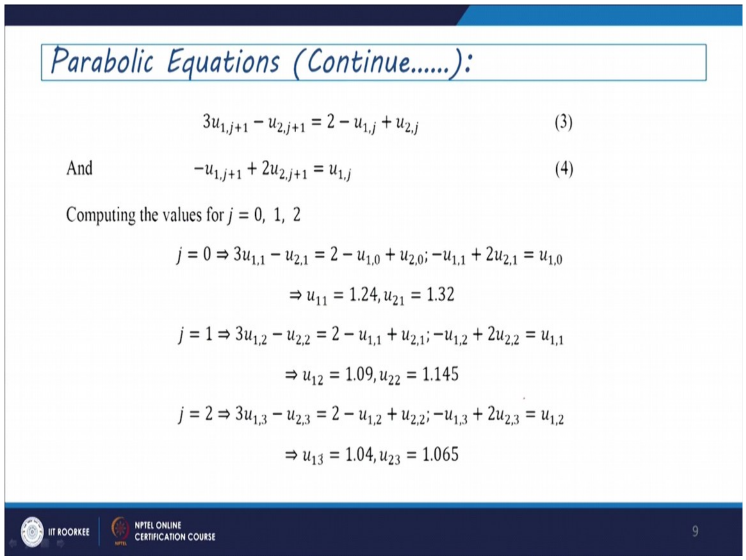

J plus 1 this equals to 2 minus u 1 j plus u 2 j here and we are just obtaining this second equation as in the form of minus u 1 j plus 1 plus 2 u 2 j plus 1 this equals u 1 j. So, if you will just go for this computation.

For different j values here like j equals to 0 1 2 for different time levels then we can just obtain this equation as a 3 u 1 minus u 2 1 as a 2 minus u 1 0 plus u 2 0 here and the second one we can just get from this as a like minus u 1 1 plus 2 u 2 1 this equals to u 1 0 here and if you will just a separate this u 1 1 term that can be computed as a 1 point 2 4 here and u 2 1 as 1 point 3 2 here similarly if you will just put j equals to 1 here then we can just find 3 u 1 2 minus u 2 2 this equals 2 minus u 1 1 plus u 2 1.

And the other one we can just get as minus u 1 2 plus 2 u 2 2 this is equals to u 1 1 if you will just solve together these 2 equations here then we can just get u 1 2 as 1 point 0 9 and u 2 2equals to 1 point 1 4 5 here. So, for j equals to 2 if you will just put then we can just get these 2 equations are in the form like 3 u 1 3 minus u 2 3 this equals to 2 minus u 1 2 plus u 2 2 here and the other equation is minus u 1 3 plus 2 u 2 3 this equals to u 1 2 and if you will just combine these 2 equations then we can obtain the values as a u 1 3 equals to 1 point 0 4 u 2 3 equals to 1 point 0 6 5 here. So, further if you will just see that the values obtained in the explicit approach especially that is just coming as a 1 point 2 0.

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Parabolic Equations (Continue......):

t x	0	0.2	0.4	0.6	0.8	1.0
0.00	1.0	1.4	1.8	1.8	1.4	1.0
0.08	1.0	1.24	1.32	1.32	1.24	1.0
		[1.20]	[1.325]			
0.16	1.0	1.090	1.145	1.145	1.090	1.0
		[1.086]	[1.139]			
0.24	1.0	1.040	1.065	1.065	1.040	1.0
		[1.037]	[1.060]			

Here and 1 point 3 2 5 for this case here and if you are just going for this implicit approach we are just obtaining these values as the scale as a 1 point 2 4 here and 1 point 3 2 here and similarly at a different levels we have just a computed these values in an explicit approach also in implicit approach and the differences you can just find from this table here then we will just go for this a compatibility stability and convergence of these methods suppose if you are just a representing u as the exact solution of the partial differential equation and u d is the exact solution of the discretized equation since here always we are just neglecting some higher order terms.

That is why we are just writing that 1 as a order of s or the order of del x square or order of a del t.

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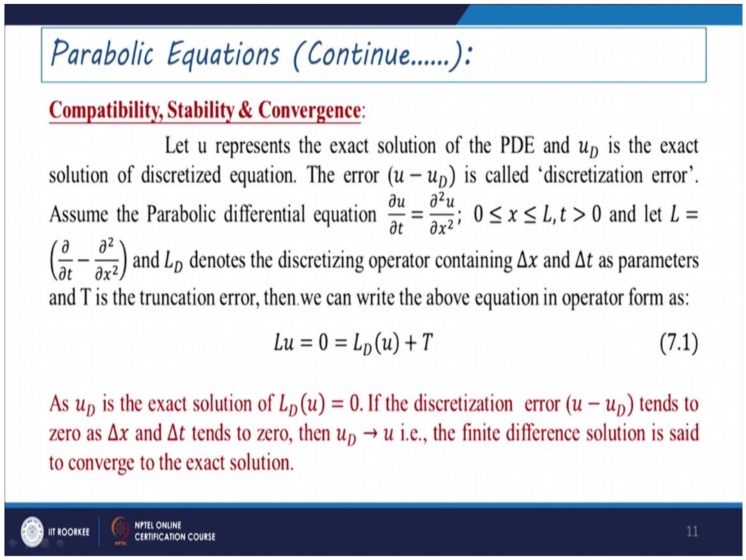

So, this error like if the exact value is a different from this approximated value we can just write this as the error term that is in the form of u minus u d this is called the discretization error here assume this parabolic differential equation that is in the form of del u by del t equals to del square u by del x square which is a written as like x lies between 0 to l t greater than 0 and let l is the operator which is defined as a del by del t minus del square by del x square and suppose this l t denotes the discretizing operator containing del x and del t as the parameters and t is the truncation error here then we can just rewrite the above operator as in the form of like l u equals to 0 always we are just writing.

That is nothing, but the discretized solution plus the truncation error there. So, that is why we are just writing here l u equals to 0 that represents as the solution in the discretized operator form plus the truncation error and especially we can just say that if ud is the exact solution of ld u equals to 0 if the discretization error u minus ud tends to 0 as del x and del t tends to 0 then ud is approaching towards u that is the finite difference solution is said to converge to the exact solution. So, if the finite difference approximation that is ld u tends to lu as a del x and delta t tends to 0 here.

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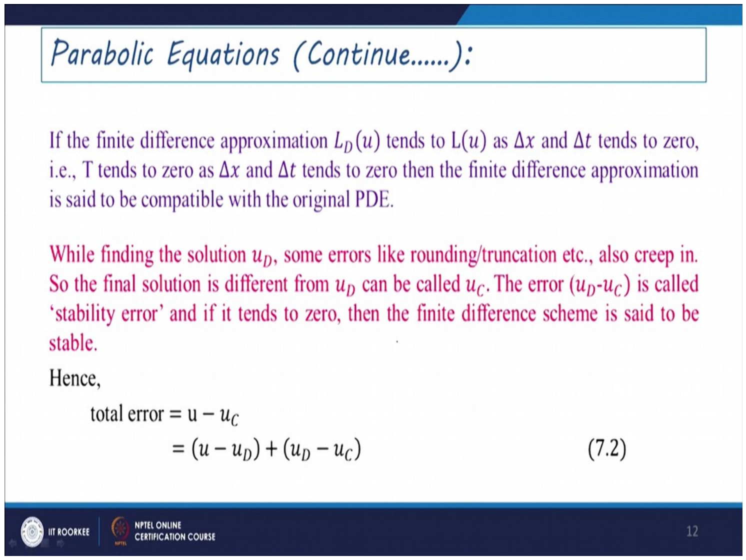

Then t tends to 0 as a del x and del t tends to 0 then the finite difference approximation is said to be compatible with the original ordinary differential equation or partial differential equation. So, while finding the solution for u d. So, some errors like rounding and truncation error we are just associating in each of these equations.

So, that may be in algebraic form or may be in a differential form that is associated there. So, especially whenever we are just taking like finite difference form always we will have like a order of a delta t or order of a del x square or order of del x is present and this is we will also keep on increasing whenever we are just moving for a different steps there. So, the final solution will be different from this discretized solution at the earlier one and if we can just find.

The final solution it may be differed from u d and it can be called as u c here then this error associated with this propagation error that is especially it can be written as u d minus u c it is called the stability error and if it tends to 0 then the finite difference scheme is said to be stable and total error we can just write it as u minus u c that is nothing, but u minus u d plus u d minus u c. So, first we are just considering the discretization error then this propagated error of course, and if you will just go for this compatibility of this explicit scheme for taylor series expansion here.

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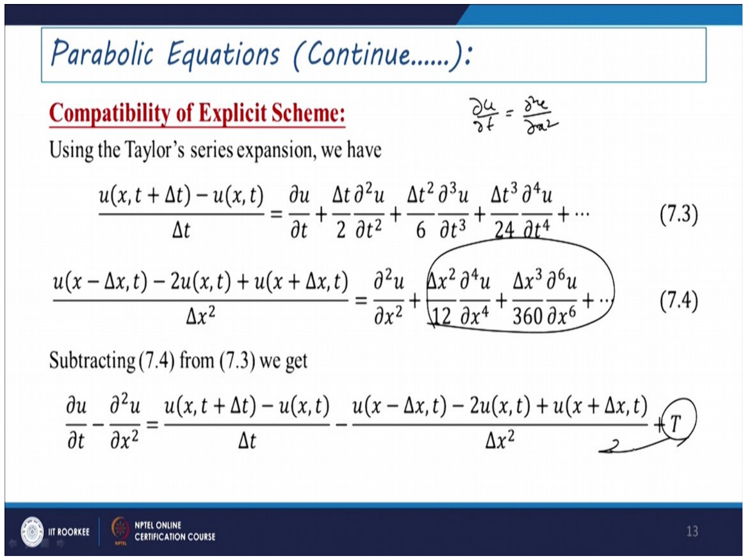

First we are just considering here del u by del t term since the equation is written.

Especially in the form of like del u by del t equals to del square u by del x square. So, that is why if you will just consider this first term only del u by del t here. So, this difference can be written in the form of u of x t plus del t minus u of xt by del t that is nothing, but del u by del t equals to r del t by 2 del square u by del t square plus del square t del t whole square by 6 del q u by del t q plus del t q by 24 since it is 4 factorial like therefore, u by del t to the power 4 here and if you will just to take the central difference approximation then this added truncation error this is of this form here and if you will just subtract these 2 terms then we can just find that the discretized error or the truncation error are associated with this expansion here.

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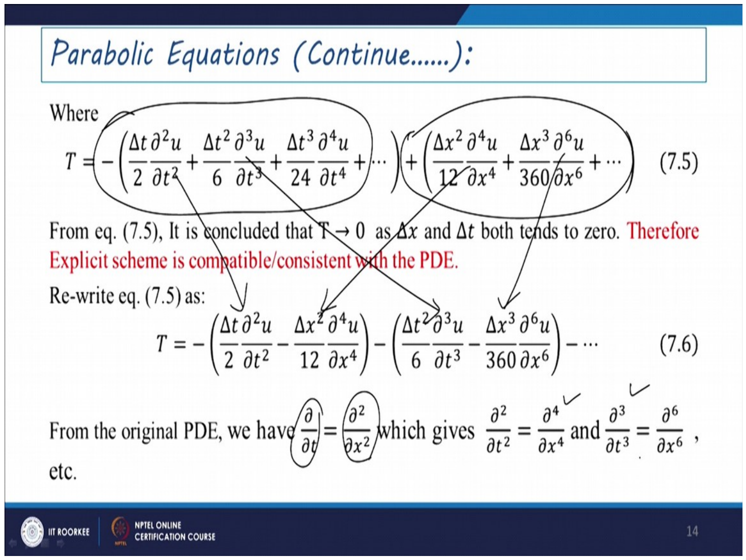

If you will just write this truncation error that is the difference of a first this a time expansion that is nothing, but minus of a del t by 2 del square u by del x square sorry del t square plus del t square by 6 del q u by del t q plus del t q by 24 del 4 u by del t to the power 4 plus up to rest of the terms since we are just taking difference and we again we are just taking to the left hand side. So, that is why this term is positive here and this term is negative and from this equation we can just conclude that t tends to 0 whenever the del x and del t both tends to 0 since this is the multiplier factor it is associated with both these terms here therefore, explicit scheme is compatible or consistent with this partial differential equation and if you will just the rewrite.

This equation 7 point 5 again we can just find this difference as in the form of like minus del t by 2 del square u by del t square and first associated term from this one that is minus del x square by 12 del to the power 4 u by del x to the power 4 then the second term if I will just write here minus of del t square by 6 del q u by del t q minus del x q del to the power 6 u by 360 del x to the power 6 from this original partial differential equation we have del by del t since this operators can be written in equal form since it is operated on the variable there which gives the del square by del t square as a del to the power 4 by del x to the power 4 and del q by del t q it is just giving you del to the power 6 by del x to the power 6 since these are operators especially.

We can just write f square equals to f into f there. So, that is why we have just represented in this form and if you will just consider del t by del x square equals to r here then equation 7 point 6 if you will just see here then it can be reduced in the form like.

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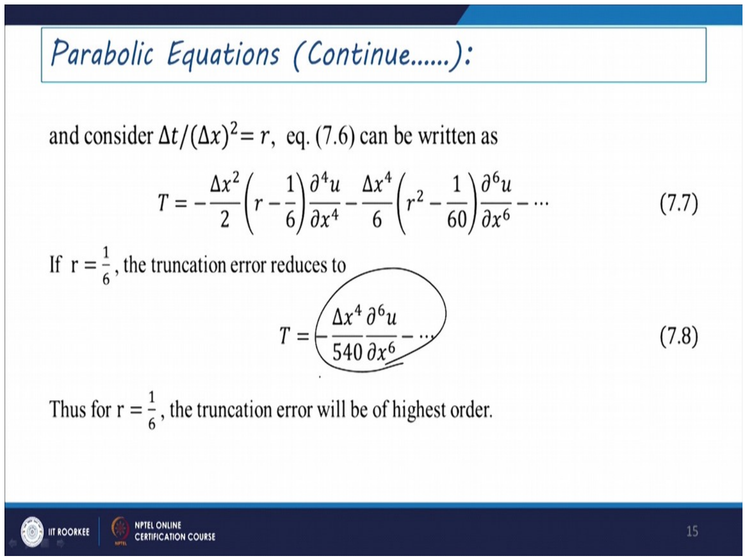

Minus of a del x square by 2 r minus 1 by 6 del square sorry del to the power 4 u by del x to the power 4 minus del x to the power 4 by 6 into r square minus 1 by 60 delta where 6 u by del x 2 the power 6 minus the rest of the terms if you will just consider r equals to 1 by 6 here then this truncation error reduces to like minus of a del to the power x 4 by 560 del 6 u by del is x to the power 6 minus rest of the terms for r equals to 1 by 6 the truncation error will be of highest order here.

Thank you for listen this lecture.