Numerical Methods: Finite Difference Approach
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Lecture - 06
Solution of parabolic equations

Welcome to the lecture series on numerical methods, finite difference approach and in this lecture. In the last class, we have discussed that how we can just approximate this differential approximations in a finite difference form with these polynomials and in this lecture first we will just go for this likely one sided approximations. How we can just treat the polynomial in one directional sense considering different points. That we will just to discuss.

Then we will just go for the solution of parabolic equations. In this form we can just discuss the different approaches like explicit approach, implicit approach or crank Nicholson approach to find the solution f for parabolic equations.

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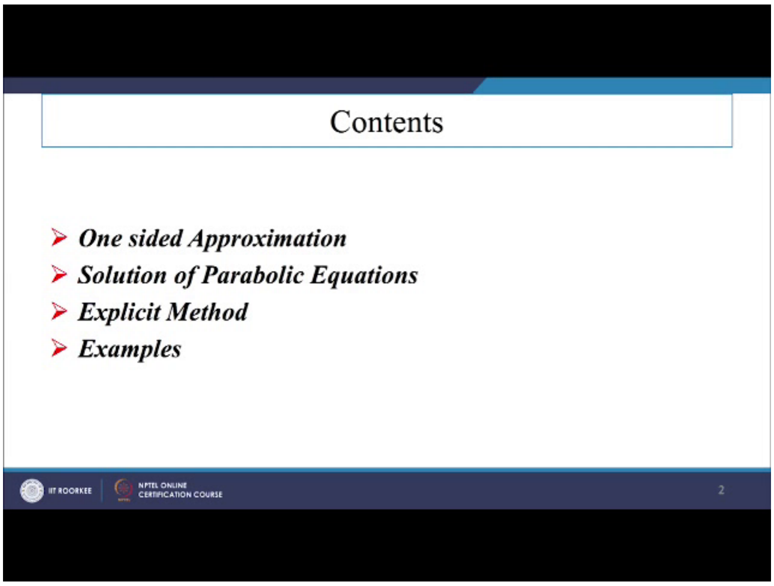

First we will just go for this explicit method.

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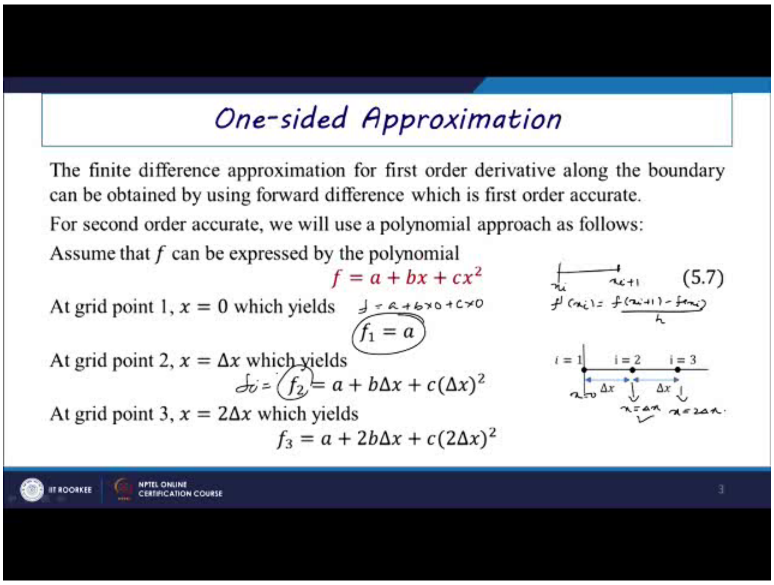

And if you just go for this one sided approximation, the finite difference approximation for first order derivatives along the boundary can be obtained by using forward difference which is your first order accurate. Especially if you will just see like a forward difference approximation if you are just using at a single point suppose. So, if you just see. That we are just writing in the form like if you will have 2 points. Especially if you are just considering like xi and xi plus 1 here, then especially we are just writing this function as a like f dash of xi as f of xi plus 1 minus f of xi divided by h here and if we want to fit this polynomial with this approximation here. Especially we can just assume a function as in the form of like f of x equals to a plus bx plus cx square here. A second degree polynomial or we can just say it is a quadratic polynomial here and at each of this grid point suppose if you are just starting the boundary I equal to 1, then we will just consider consecutive 2 points.

That is suppose I equals to 2 and I equals to 3 here and each are spaced at equal distances like del x distances says. Then at each grid point one for suppose at first grid point we can just consider x equals to 0 here. Then at the I equals to 2 we can just consider this is as the del x here and I equal to 3 we can just consider x equals 2 del x here; so, at the first grid point if you will just approximate this function that is for x equal to 0 here. So, we can just provide you as f equals to a plus b into 0 plus c into 0 here. Especially if you just to see that at point I equals to 1 here. So, f1 equals to a here and for grid point 2. Suppose if you will just see here x equals to consider as a del x here and if you will just approximate this point with this function here, that function can be represented as a since we are just a putting this subscript I as fi here.

So, I equals to 2. That is why we can just write it as a plus b delta x plus c del x whole square and at grid point 3 suppose if you will just find this function that can be written in the form of like f 3. This equals to a plus 2 b delta x plus c 2 del x whole square here. So, if you will just go for this is solving off of these expressions like 3 equations, we will have 3 unknowns there. If you will just solve these equations, then from the first equation we can just get this coefficient a as f1 here and b if you will just calculate.

That can be represented as a minus 3 f1 plus 4 f2.

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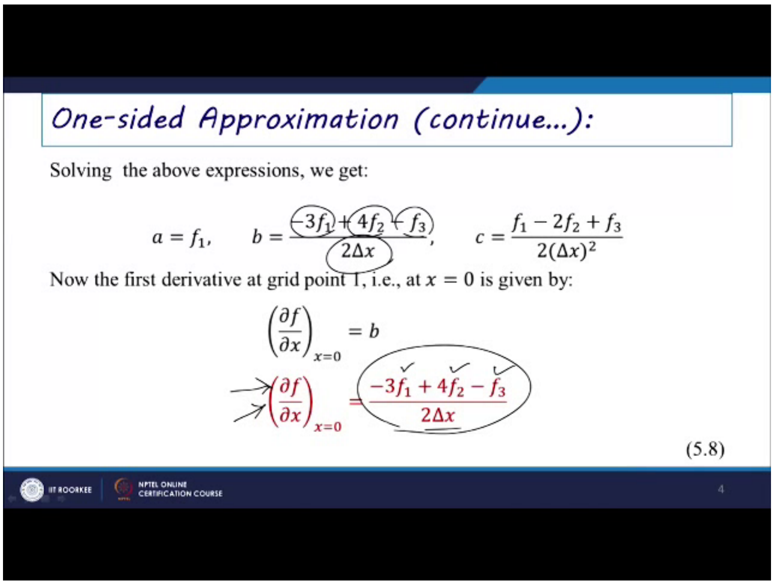

Minus f3 by 2 del x and the c especially we can just find it as a f1 minus 2 f2 plus f3 by 2 del x whole square here. So, why we are interested in this like one way of transformation is that or one sided approximation is that, sometimes in a physical problems if suppose some physical behaviour is occurring inside the system, which is considers the property at the boundary, but this property is a transmitting towards the inner points, then we have to consider inner 2 points considering with the first point to find the flow properties or to find the physical behaviour of the system. So, that is why higher order approximation it is needed to get more accurate solutions.

So, that is why we are just considering this a formulation as in the form of like a plus bx plus cx square. For the polynomial approximation to approximate in these 3 different points at a particular time. Now if you will just approximate this first order derivative here, then we can just write del f by del x that is nothing but b. So, b especially if you will just see it is nothing but a minus 3 f1 plus 4 f2 minus f3 by 2 del x here. If you will just see here. 3 different points I equals to 1, I equal 2 and I equals to 3 all are involved for this like a first order differentiation here. Even if sometimes if you will just to see this boundary conditions, we are just rotating as this normal boundary conditions that is a normal derivative equals to certain value or 0.

If you are just considering, then this is a normal derivative sense we are just considering the first order derivative there and if we want to include all these fluid properties or like physical behaviour properties there. So, then we have to consider like 3 different points at a particular instant or in a particular subject. Then at a time we can just write this formulation as in this form here.

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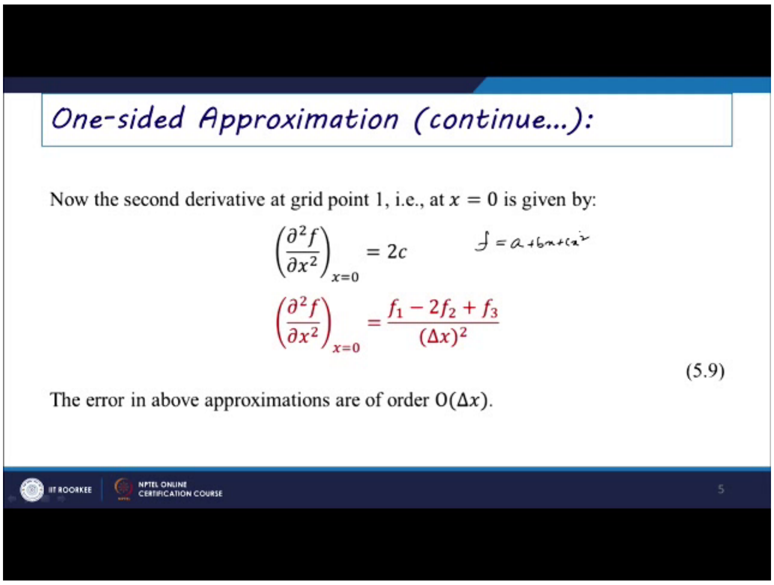

So, if you will just go for like second order derivative here though we can just differentiate that one this a function that is we defined as in the form of like if you will just see a plus bx plus cx square there. So, that is why it will just to give you 2c there. 2c means c has a value here.

That is in the form of if you will just see f one minus 2 f2 plus f3 by 2 del x square here. So, 2 can cancel it out. That is why it can be written in the form of like a1 minus 2 f2 plus f3 by del x whole square here. This is a approximations can be used for first order or second order differentiation in a one way approach or in a one sided approximation. We can just write. Especially this is very useful whenever we are just considering any physical problem and which can be used along the boundary levels. After then we will just to go for this methods for solving parabolic equations and in the beginning of this lecture we have discussed this classification of partial differential equations when we can just say that the differential equation is like parabolic in nature, high parabolic in nature or elliptic in nature.

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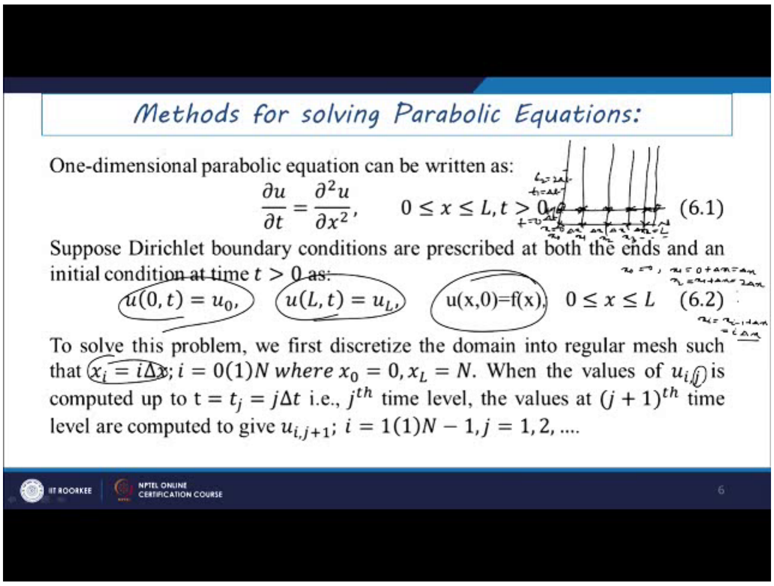

So, if you will just see that if we will be consider like one dimensional parabolic equation, it can be written as like del u by del t equals to del square u by del x square. Where you will have this boundary for this problem. That boundary is defined at x equals to 0 and x equals to l here and this is a time always it will start from 0 and this computation will you go for the t greater than 0 suppose along the boundary if we the boundary conditions are prescribed for both the ends either at the initial level. So, initial level means we can just consider that as like t equals to 0 here that is nothing, but u of x 0 which is defined as fx here. That is the initial condition.

And if you will just go for this boundary for particular times or the with variation of time also we can just consider that the boundary will you remain same or it will not get change with respect to the complication. Since, we will just do this computation fixing these boundary conditions. So, that is why these boundary conditions are given as like u of 0 t as u 0 that is nothing but a dirichlet condition here and u of lt as ul here.

So, to solve this problem we first to discretize the domain into regular mesh such that x I equals to I del x here. If you will just refine a domain as in the form of like you will have 2 boundaries here. That is a1 is defined at x equals to 0. Otherwise it defined as x equals to l here.

And initial time level t equals to 0. All the values are supplied here and as time proceeds, then we can just go like t equals to 1 here. For t equals to 1, we have to consider like a boundary condition. Whatever existing here and here we will just to get the next point here or we will just get the next point here we will just get the next point here we will just get the next point here. So, that is why first we how to discretize this domain into regular mesh or regular intervals. For this we will just divide this domain into like regular intervals here and in this regular interval each of this interval can be defined as a del x, del x, del x or some space length as h or.

You can just define in your way. So, if you will just discretize this domain in this form then you will have these points like a starting point if you will just consider as 0 here, then the first point will be x1, second point x2, third point x3 likewise it will just continued and each of it is a distances it can be computed. Since x0 we will be started at like x0 equals to 0 here. Suppose then x1 can be written as a0 plus delta x that is nothing but delta x. Then x 2 can be written as like x1 plus delta x that is nothing but 2 delta x. So, likewise if you will just to define here then finally, we can just find this xi any arbitrary point. We can just write as xi minus 1 plus delta x here and especially if you will just you see.

This is nothing, but I delta x. So, if we can just define this domain by putting all these a nodal points, that as in the form of xi is here. Which are placed at equal distances and which can be written as like xi equals to I delta xi equals to 0 one up to n. Where x0 equals to 0 and xl equals to n. If we are just considering here, we will have this values of uij, but is j represent here the time level and I represents the coordinates for x direction here. So, we can just compute these values at each of the increment of time levels. That is nothing but p equals to tj here. Which can be written as like if del t is the space of time here. For the first interval we can just write t1 equals to del t here and t2 we can just write 2 del t here.

So, likewise we can just proceed and; obviously, j has a no restriction. So, that is why you can just write j equals to 1 to up to a convergence state level. That where we can just find a steady solution there or up to that level that we will have a converse solution. Based on this problem if you will just go for this like explicit method here. First we will just to discretize this.

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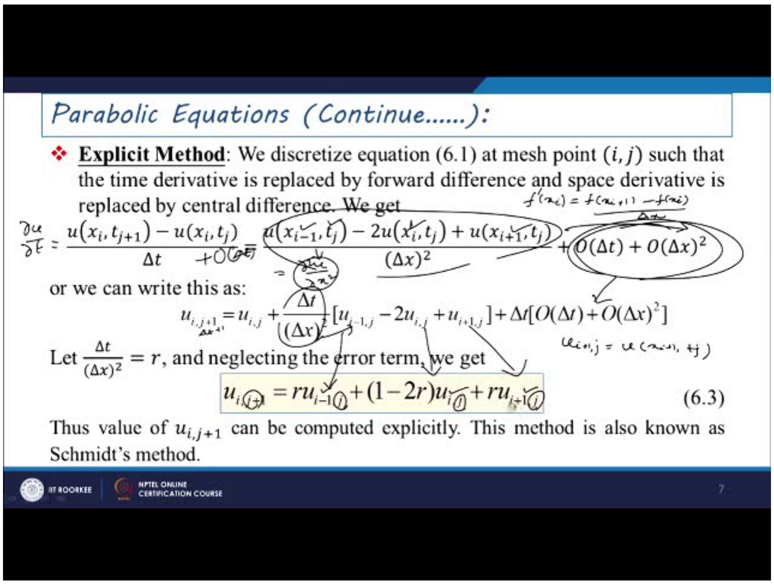

Equation at mesh points. That is at is a point suppose such that the time derivative is replaced by a forward differential and the space derivative is replaced by a central difference approximation here. If you will just go for this a time derivative by a forward difference approximation here we can just write. Since forward difference approximation.

Always it is just written in the form like f of f t of xi as f of xi plus 1 minus f of xi divided by delta x and at the time level since we are just considering this forward time level here. That is you written as a del u by del t here. So, del u by del t this can be written as like u of xi means a your I direction we are just doing this derivative in the j direction here itself. You will just see del u by del t means time space. It is just to getting changed. So, that is why we are just writing I is fixed here. u of xi tj plus 1 minus u of xi tj by del t here. Which can be written as like del square u by del x square.

This is nothing but del square u by del x square here. Where we can just consider time level is fixed. Since this variation is just coming or this partial derivative we are just considering this has a variation in the x axis itself. So, x axis means we can just consider this a central difference scheme at 3 different points in the x direction. That can be considered this point as I minus 1, I and I plus 1 here. Hence, this central difference approximation can be written as u of xi minus 1j minus 2 u of xi tj plus u of xi plus 1 tj by del x square plus. This order of approximation if you will just see this left hand side order of approximation is a order of a delta t here and if you will just see this central difference approximation it is this order of approximation.

That is in the form of order of delta x square. So, that is why in a combined form both the sides we are just putting it as order of a delta t plus order of 4 del x square here and once it is written in this form, then we can just write this formulation as in the form of u ij plus 1. Since, in the next time level we are just computing if you just see here. This level we are just computing this values if you will just say j plus 1th level, this means that t equals to 0th level. All the values has been supplied and we want to compute these values at the 1th level here. Since, a boundary values it is given. These 2 values it is provided to us. Then we will just compute this.

Next level values. So, that is why u I of j plus 1 this is nothing but uij plus del t by del x square. Since, a del t is a multiplied here over. del t by del x square into u I minus one j minus tu uij plus ui plus 1 j. Especially we are just writing u I plus 1 j means that is nothing but u of x I plus 1 tj here and then if you will just multiply by del t. Also del t can be multiplied also this order of approximation here. So, that is why this is written as a del t into order del t plus order of a del x whole square here. Let del t by del x square suppose if you will just consider it is r suppose if you will just consider this as r here and neglecting all the error terms we can just to get this formulation as.

Since the u ij plus 1. It is just written, then we are just writing this a del t by del x square is equal to r here. r into ui minus 1 j and then 1 I plus 1, ij is there. So, that is why 1 minus 2 r into uij. Then next is a r ui plus 1 j. Where this a higher order terms is you neglected here and thus the value of uij plus 1 can be computed explicitly. Since we are just involving this, if you will just see here that is I plus1 j value is a involved here I j values involved I minus 1 j value is involved here, but all these time levels if you will just see these all are at the 0th level for the 1th level calculation here this means that.

If you will just see this slide here. This means that this all values here known to us at 0 th level for this domain here. So, based on these values we can just compute all the values in this level here. This method is a specifically known as smeeds method and people are broadly using this explicit method for their computation of different differential equations in a 1 dimensional sense.

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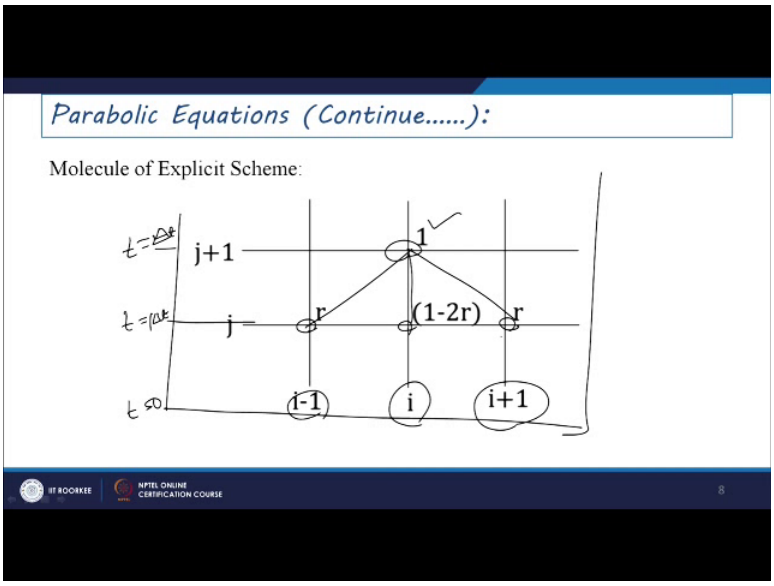

Or in 2 dimensional sense and if you will just formulate this molecule for this system here then our system is written in the form of like you ij plus 1 rui minus 1 j plus 1 minus 2 r plus r here . So, since we are just defining this domain as in the form of like a, if a this is the domain here then we will have these points.

Like I minus 1 I and I plus 1 here and j th level it is just starting like since it equals to 0 it is a starting then t equals to 1. Then t equals to 2 or we can just write t equals to 0. This is the del t, this is 2 delta t likewise it is just moving. So, that is why at a first level if you just see here these coefficients are written as a I minus 1I and I plus 1 here and suppose if you will just consider like j equals to 0 suppose here 0th level. So, 0th level means this all values were known to us and we are just computing this level value here. So, this is called the molecule. Here molecule means we can just consider these 3 values at a particular time to get.

This values here this means that this level of computation requires at this 3 different points with the coefficients to get the values if it is asked the question like.

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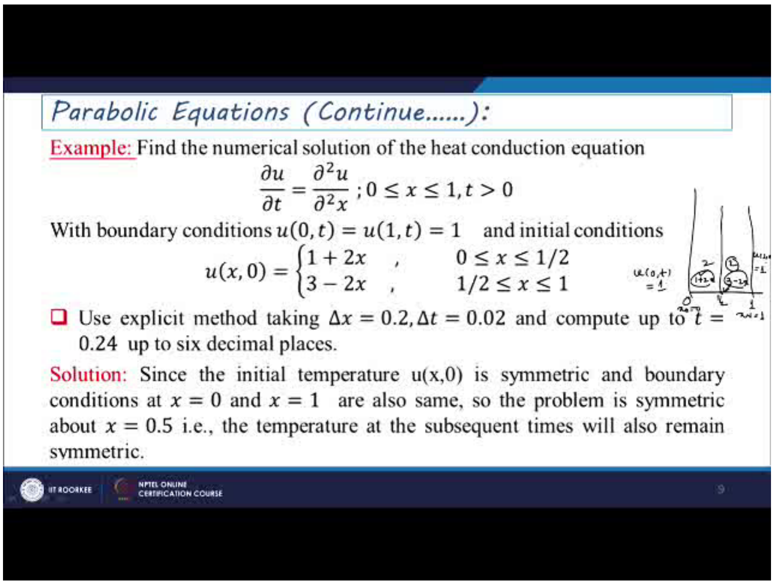

Del u by del t equal to del square u by del x square x lies between 0 to 1 and t greater than 0 with the boundary condition co of 0 t equals to u of 1 t equals to 1 here. With these initial conditions as u of x0 equals to 1 plus 2x. Along this half of this domain and 3 minus 2x as other half of this domain if you will just see here that the domain is a defined in this form that your range is going from 0 to 1 here like if you will just define x 0 equals to 0 here and xn equals to 1 here.

And one particular level it is just considered as like a half here. So, where if you just see the boundary condition, it is just given as a1 plus 2x. For this region and this region it is just written as 3 minus 2x and at this point if you will just see this is nothing but if will put x here as half here, then you can just to this is getting as a 2 here and especially if you will just put this condition also here you can just find that 3 minus 2 into half here. This is nothing but 2 here. That is why this represents a symmetricity along the center line there and the boundary condition it is defined as u of 0t. That is nothing but along the first boundary it is defined.

As 1 and if the last boundary it is defined as 1 of t equals to 1 here also. So, for this if you will just used this space length has a 0.2. Since it is a given and your time scale it is just given as 0.02. Then we can compute this calculation of 2 like t equals to 0.24. So, up to 6 decimal places it is just asked. The problem is a asymmetric at x equals to 0.5 if you will just to put.

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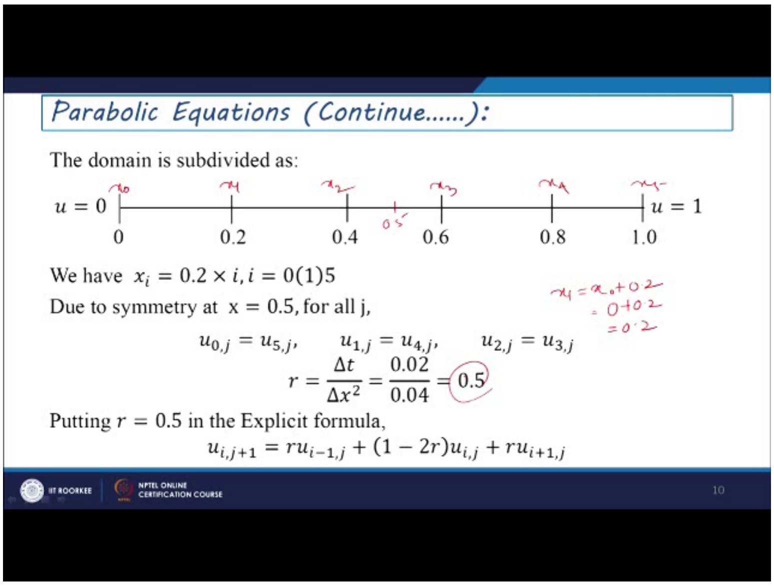

Now, the domain is a divided into 5 sub intervals if you will just see here. So, you will start as u equals to 0 at x equals to 0 and it will ended like a final point. That is a x equals to 1.0.

You will have u equals to 1 there. Since the grid spacing is defined as 0.2 here. So, the total domain that is 0 to 1 will be sub divided into 5 sub intervals each of length like 0.2. Hence, if you will just consider here like x0 as the 0.0 here, then your starting point like x1, this can be written as a x0 plus 0.2; that you will just give you as x0 is 0 here 0 plus 0.2 that is nothing but 0.2 here. That is the first point here. If your starting point is like x0 here, then x1, x2, x3, x4 and final point is like x5 here. That is why we have just 2 defined xi equals to 0.2 into I here. I is varying from 0 to 5 here due to symmetry as we have.

Told that if you will just consider like ax equals to 0.5 here, then we can just find for all j as u of 0 j equals to u of 5 j. Since the symmetry is occurring at 0.5 here. So, that is why we can just consider u of 0 j equals to u of 5 j here. Then u of 1 j equals to u of 4 j. Then u of 2 j equals to u of 3 j. So, likewise we can just to find the values and if you will just see this r value as here r you can be defined as the del t by del x is square which can be written as a 0.02 by 0.04. Since a del x is a chosen as a 0.2 here and del t as a 0.02. That is why this a r value can be find out as a 0.5 here. If you will just use this explicit formula for r equals to 0.5.

Then the complete formula it is just written as u I of j plus 1. This can be written as r ui minus 1 j plus 1 minus 2 r uij plus r ui plus 1 j here.

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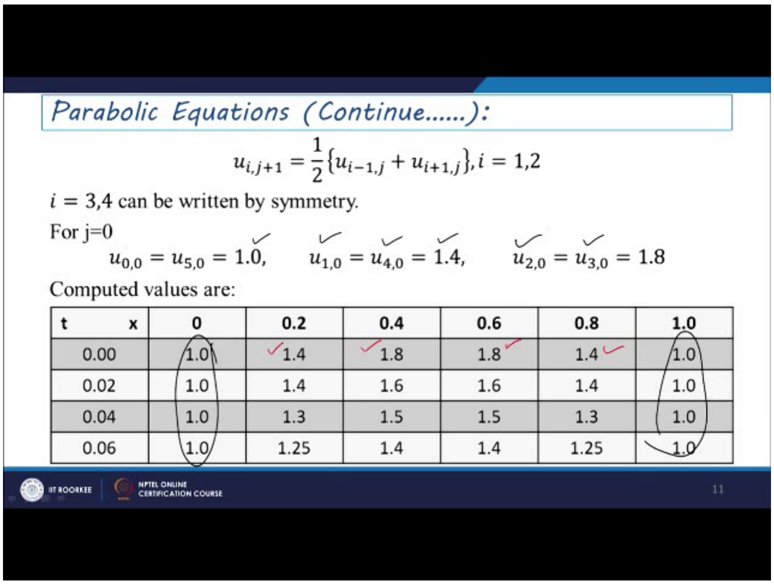

This expression in the form of like uij plus 1; this as like half of ui minus 1 j plus uij plus 1 I equals to 1 to 2 here. Then we will have like a this boundary conditions. So, that is defined as u of 0 as u of 5 0 that is nothing, but 1 here and u of 1 0 that is the u of 4 0. That is 1.4 here and u of 2 0 u of 3 0 that is nothing but 1.8 here. So, it will just start this computation, then we can just find these values are in this form.

Like 0.00, 0.02, 0.04, 0.06 these are all like time increments we can just get and your space increments. You can just get it as a 0, 0.2, 0.4, 0.6 and 0.8 and last point is 1.0 here and if you will just go for this computation of these values by putting this molecules here. So, then you can just find this solution as like first point it is a boundary values. It is just given like a x0. It is just satisfied all these are boundary conditions. So, that is just defined in the form of like a u of 0 t and last boundary you can just also find.

The same value that is 1 also there so that is why we can just write this as like a 1.0, 1.0, 1.0, 1.0 here and this is also 1.0, 1.0, 1.0 here and if you will just see this further calculation. It can just take this boundary conditions there. So, first value we are just getting 1.4. Then second one we are just getting 1.8. Third we can just if you will just to use the symmetricity here 0.2. It remains the same for 0.8 here and 0.4 is a remains same for 0.6. Likewise if you will just computed this is the complete table that we can just to get all the solutions.

Thank you for listen this lecture.